LE MATEMATICHE
Vol. XLVIII (1993) – Fasc. I, pp. 163-177

A CLASS OF ELLIPTIC OPERATORS IN $\mathbb{R}^3$ IN NON DIVERGENCE FORM WITH MEASURABLE COEFFICIENTS

ORAZIO ARENA - PAOLO MANSELLI

In an open cylinder of $\mathbb{R}^3$ a linear uniformly elliptic operator in non-divergence form, with coefficients time independent but measurable only, is investigated.

Existence and uniqueness results in suitable Sobolev spaces for the Dirichlet problem are obtained.

1. Introduction.

A linear uniformly elliptic operator in non-divergence form in $\mathbb{R}^3$ is studied, with coefficients measurable but depending on two variables only, and existence and uniqueness results for the Dirichlet problem are obtained in suitable Sobolev spaces.

The interest of our investigation comes from the fact that, while the theory of smooth nonlinear elliptic operators had great advances in recent times, the theory of linear non-smooth elliptic operators in non-divergence form still does not have a natural class of solutions for which the uniqueness theorem holds.

Actually, "viscosity" solutions (see [5] where a large bibliography can be found), "good" solutions (see [1], [2]) seem to be good candidates for natural solutions of these equations.

Entrato in Redazione il 20 settembre 1993.

The authors are supported by Italian M.U.R.S.T. Research Project 40%.

In the latter sense, M.C. Cerutti, L. Escauriaza and E.B. Fabes [1], [2] proved a uniqueness theorem for elliptic equations with coefficients continuous except for a countable set having at most one cluster point. The case of a finite number of discontinuities was due to L. Caffarelli (see [2]). In the same context we point out the work by N. V. Krylov [8] on the so-called weak uniqueness problem for elliptic equations.

L. Escauriaza [6] has obtained a uniqueness theorem for elliptic operators in $\mathbb{R}^n$ with coefficients depending on $(n-1)$ variables, continuous in these variables except by a countable set of points with countable closure.

Our results are partially overlapping with these above mentioned: the coefficients of our operators in $\mathbb{R}^3$ depend on two variables but no continuity assumption is required.

The plan of the paper is the following. First we look for solutions periodic with respect to the variable which does not appear in the operator's coefficients. Then such a requirement is removed and the Dirichlet problem is studied in the general setting.

Let us now introduce a few notations.

Let $B = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}$ to be the unit ball in $\mathbb{R}^2$. Let a^{ij}, b^j measurable functions in B and consider the operators:

$$S = a^{ij} D_{ij} \ , \ \ L_1 = b^j D_j$$

where $D_j = \partial/\partial x_j$, $D_{ij} = \partial^2/\partial x_i \partial x_j$ $(i, j = 1, 2)$ and summation convention is assumed.

Let now $R = \{(x, t) \in \mathbb{R}^3 : x \in B,\ 0 < t < 1\}$ an open cylinder in $\mathbb{R}^3$ and $\partial_0 R = \partial B \times [0, 1]$ its lateral boundary.

In R we will study a second order linear uniformly elliptic operator of the form:

$$(1) \qquad \begin{aligned} Lu = a^{ij}(x) D_{ij} u(x, t) + b^j(x) D_j u_t(x, t) + u_{tt}(x, t) = \\ = Su + L_1 u_t + u_{tt}, \end{aligned}$$

where $u_t = \partial u/\partial t$.

The uniform ellipticity condition is : there exists a positive constant α such that

$$(2) \qquad \alpha \le a^{ij}(x)\xi_i\xi_j + b^j(x)\xi_j\eta + \eta^2 \le 1/\alpha, \quad \xi^2 + \eta^2 = 1, \quad x \in B.$$

The Sobolev spaces that will be mainly used in B are:

$$W^{2,p}(B) = \{u : u \in L^p(B),\ D^2 u \in L^p(B)\},$$

where $D^2u = (D_{ij}u)$ and

$$W^{2,p}_{\gamma_0}(B) = \{u : u \in W^{2,p}(B),\ u_{|\partial B} = 0\}.$$

Analogously in R will be used $W^{2,p}(R)$ and $W^2_{\gamma_0}(R)$; we will need also $W^{2,p}_{\mathrm{loc}}(R \cup \partial_0 R) = \{u \in W^{2,p}(B \times (a,b)),\ 0 < a < b < 1\}$. Moreover $W^{2,p}_{\#}(R)$ will be the closure in $W^{2,p}(R)$ norm of

$$\{u \in C^\infty(\bar{B} \times \mathbb{R}),\ u \quad \text{periodic of period } 1, \quad u_{|\partial B \times \mathbb{R}} = 0\}.$$

In all the inequalities troughout the paper C_α will be a constant, not necessarily the same, depending on α only, $C_{\alpha,p}$ will depend on α, p only and so on.

2. A priori bounds and existence theorems in $W^{2,p}_{\#}(R)$.

Lemma 1. *Let a^{ij}, b^j $(i, j = 1, 2)$ satisfying (2), $S = a^{ij}D_{ij}$, $L_1 = b^jD_j$. Let $\nu \in \mathscr{Z}$, $U \in W^{2,2}_{\gamma_0}(B)$ (complex valued) and let*

$$SU + i\nu L_1 U - \nu^2 U = f + ig. \tag{3}$$

Let $z \in W^{2,2}_{\gamma_0}(B)$ be the solution of

$$Sz - a\nu^2 z = -(f^2 + g^2)^{1/2}. \tag{4}$$

Then:

$$|U| \leq z \quad \text{in } B. \tag{5}$$

Proof. Let $u^{(1)} = \operatorname{Re} U$, $u^{(2)} = \operatorname{Im} U$. Then $u^{(j)} \in W^{2,2}_{\gamma_0}(B)$ $(j = 1, 2)$ and

$$Su^{(1)} - \nu L_1 u^{(2)} - \nu^2 u^{(1)} = f, \tag{6}$$

$$Su^{(2)} + \nu L_1 u^{(1)} - \nu^2 u^{(2)} = g. \tag{7}$$

Let now $w = |U|$. By evaluating $D_i w$ and $D_{ij}w$ in the open set $w > 0$ we have a.e. in B:

$$\begin{aligned} w^3 Sw = w^2\Big[u^{(1)}Su^{(1)} + u^{(2)}Su^{(2)} + a^{ij}D_iu^{(1)}D_ju^{(1)} &+ \\ + a^{ij}D_iu^{(2)}D_ju^{(2)}\Big] &+ \\ -\Big[(u^{(1)})^2 a^{ij}D_iu^{(1)}D_ju^{(1)} + 2u^{(1)}u^{(2)}a^{ij}D_iu^{(1)}D_ju^{(2)} &+ \\ + (u^{(2)})^2 a^{ij}D_iu^{(2)}D_ju^{(2)}\Big]&. \end{aligned}$$

Therefore:

$$w^3 Sw = w^2\Big[u^{(1)}Su^{(1)} + u^{(2)}Su^{(2)}\Big] + (u^{(2)})^2 a^{ij} D_i u^{(1)} D_j u^{(1)} +$$

$$+ (u^{(1)})^2 a^{ij} D_i u^{(2)} D_j u^{(2)} - 2u^{(1)}u^{(2)} a^{ij} D_i u^{(1)} D_j u^{(2)}.$$

Namely we have that:

$$\begin{aligned} w^3 Sw = {} & w^2\Big[u^{(1)}Su^{(1)} + u^{(2)}Su^{(2)}\Big] + \\ & + a^{ij}\Big[u^{(2)}D_i u^{(1)} - u^{(1)}D_i u^{(2)}\Big]\Big[u^{(2)}D_j u^{(1)} - u^{(1)}D_j u^{(2)}\Big]. \end{aligned}$$

Now using (6), (7) and writing for short $\xi_i = u^{(2)}D_i u^{(1)} - u^{(1)}D_i u^{(2)}$, we get:

$$\begin{aligned} w^3 Sw = {} & w^2\Big[u^{(1)}f + u^{(2)}g\Big] + \nu^2 w^4 + \\ & + \nu w^2 b^j\Big[u^{(1)}D_j u^{(2)} - u^{(2)}D_j u^{(1)}\Big] + a^{ij}\xi_i\xi_j. \end{aligned}$$

Let $\eta = \nu w^2$. Then:

$$w^3 Sw = w^2\Big[u^{(1)}f + u^{(2)}g\Big] + \eta^2 + b^j\xi_j\eta + a^{ij}\xi_i\xi_j.$$

Taking into account (2), it follows that:

$$w^3 Sw \geq w^2\Big[u^{(1)}f + u^{(2)}g\Big] + \nu^2 w^4.$$

Thus:

$$Sw - \alpha\nu^2 w \geq \frac{u^{(1)}}{w}f + \frac{u^{(2)}}{w}g \geq -(f^2+g^2)^{1/2}.$$

Let us now show that $w < z$. First observe that $z > 0$ in B, by Alexandrov-Pucci's maximum principle. On the other hand the function $w - z$ is zero on the boundary of B and not greater than zero where $w = 0$. Since in the region where $w > 0$ one has:

$$S(w-z) - \alpha^2\nu^2(w-z) \geq 0,$$

again by the maximum principle the function $w - z$ cannot have a positive maximum there. Therefore $w \leq z$, i.e. (5) holds.

We are now able to prove the following a priori estimate.

Theorem 1. *Let $u \in W^{2,2}_{\#}(R)$. Then:*

$$\|u\|_{W^{2,2}_{\#}(R)} \le C_\alpha \|Lu\|_{L^2(R)}. \tag{8}$$

Proof. Let $f = Lu$. Let us expand u and f in Fourier series with respect to the t variable:

$$u(x,t) = \sum_{-\infty}^{+\infty}{}_{\nu}\, u_\nu(x)\, e^{i2\pi\nu t}$$

$$f(x,t) = \sum_{-\infty}^{+\infty}{}_{\nu}\, f_\nu(x)\, e^{i2\pi\nu t}.$$

The functions u_ν ($\nu \in \mathbb{R}$) are in $W^{2,2}_{\gamma_0}(B)$ and we have:

$$Su_\nu + i\nu L_1 u_\nu - \nu^2 u_\nu = f_\nu \qquad \text{a.e. in } B. \tag{9}$$

As u_ν satisfies the hypothesis of Lemma 1, then

$$|u_\nu| \le |z_\nu|,$$

where $z_\nu \in W^{2,2}_{\gamma_0}(B)$ is the unique solution of the equation:

$$Sz_\nu - a^2\nu^2 z = -|f_\nu|.$$

Let us recall (see e.g. M. Chicco [3]) that:

$$|\nu^2 z_\nu\|_{L^2(B)} \le C_\alpha \|f_\nu\|_{L^2(B)} \qquad \nu \in \mathscr{Z}.$$

Thus:

$$\|\nu^2 u_\nu\|_{L^2(B)} \le C_\alpha \|f_\nu\|_{L^2(B)} \qquad \nu \in \mathscr{Z}. \tag{10}$$

Then, owing to the Plancherel theorem:

$$\|u_{tt}\|_{L^2(R)} \le C_\alpha \|f\|_{L^2(R)}. \tag{11}$$

From Talenti's [11] a priori bound, we derive:

$$\|D^2 u_\nu\|_{L^2(B)} \le C_\alpha \left(\|f_\nu\|_{L^2(B)} + |\nu|\, \|L_1 u_\nu\|_{L^2(B)} + \nu^2 \|u_\nu\|_{L^2(B)} \right).$$

By interpolation on $|\nu|\,||L_1 u_\nu||_{L^2(B)}$ and using again (10), one has:

$$||D^2 u_\nu||_{L^2(B)} \leq C_\alpha ||f_\nu||_{L^2(B)}$$

and by Plancherel theorem:

$$||D_{ij} u||_{L^2(R)} \leq C_\alpha ||f||_{L^2(R)}.$$

This inequality and (11) give:

$$||\Delta u||_{L^2(R)} \leq C_\alpha ||f||_{L^2(R)}. \tag{12}$$

As u is periodic in the t variable, then:

$$||u||_{L^2(R)} \leq C_\alpha ||u_{tt}||_{L^2(R)}.$$

From the last two inequalities and (11) the thesis of the Theorem follows.

Let us now prove a similar apriori bound in L^p, for p in *a* neighborhood of 2.

Theorem 2. *There exist $p_1(\alpha) < 2 < p_2(\alpha)$, such that for any u of class $W^{2,p}_{\#}(R)$ with $p \in [p_1(\alpha), p_2(\alpha)]$ the following estimate holds:*

$$||u||_{W^{2,p}_{\#}(R)} \leq C_\alpha ||Lu||_{L^p(R)}. \tag{13}$$

Proof. We will use the complex interpolation result of R. Coifman and others [4].

(i) Let $1 < p < \infty$. There exists $\eta_0^1(\alpha, p)$ such that if we take $|\eta| \leq \eta_0^1(\alpha, p)$, p_1 between 2 and p, then:

$$||\Delta u||_{L^{p_1}(R)} \leq C_{\alpha,p} ||(\Delta + i\eta(L - \Delta))u||_{L^{p_1}(R)}, \tag{14}$$

for every $u \in W^{2,p_1}(R)$.

This fact follows from the inequality (see e.g. [7]):

$$||D^2 u||_{L^{p_1}(R)} \leq C_p ||\Delta u||_{L^{p_1}(R)}$$

and small perturbation techniques.

(ii) There exists $\eta_0^{(2)}(\alpha)$ such that if $\eta \leq \eta_0^{(2)}(\alpha)$ and $0 \leq \xi \leq 1$ then:

$$\|\Delta u\|_{L^2(R)} \leq C_\alpha \left\| \left\{ [1 - (\xi + i\,\eta)]\Delta + (\xi + i\,\eta)L \right\} u \right\|_{L^2(R)} \tag{15}$$

for every $u \in W_{\#}^{2,2}(R)$.

In fact, for every $\xi \in [0, 1]$, the operator $(1 - \xi)\Delta + \xi L$ is of the form (1) with coefficients satisfying (2) and therefore, by Theorem 1, for any $u \in W_{\#}^{2,2}(R)$ one has:

$$\|\Delta u\|_{L^2(R)} \leq C_\alpha \left\| \left\{ (1 - \xi)\Delta + \xi L \right\} u \right\|_{L^2(R)}.$$

Thus a perturbation argument gives (15).

Now set $\eta_0(\alpha, p) = \min\left\{\eta_0^1(\alpha, p),\ \eta_0^2(\alpha)\right\}$ and let

$$Q = \left\{\xi + i\eta,\ 0 \leq \xi \leq 1,\ |\eta| \leq \eta_0(\alpha, p)\right\}.$$

If $z = \xi + i\eta \in Q$ and $f \in L^\infty(R)$ is a simple function, extended periodically in t to $B \times \mathbb{R}$, then the equation:

$$L_z u = \left\{(1 - z)\Delta + zL\right\} u = f \tag{16}$$

has a unique solution in $W_{\#}^{2,2}(R)$; notice that if $z = i\eta$ then $L_{i\eta}$ has a unique solution in $W_{\#}^{2,p_1}(R)$, p_1 between 2 and p.

Let us define: $T_z f := \Delta u$. Then $T_z f$ is a measurable function for every $z \in Q$. If $\|T_z\|_p$ denotes the norm of T_z as an operator in L^p, then

$$\|T_z\|_2 \leq C_{\alpha,p}\ ,\ \ \|T_{i\eta}\|_{p_1} \leq C_{\alpha,p}.$$

(iii) Let us prove now that, if f and g are simple functions, then:

$$\int_R (T_z f) g\, dx dt$$

is an holomorphic function inside Q.

Let z_0 an interior point in Q and let $v = \sum_{k=0}^{\infty} u_k (z - z_0)^k$ where $u_k \in W_{\#}^{2,2}(R)$ and $L_{z_0} u_0 = f$, $L_{z_0} u_k = (L - \Delta) u_{k-1}$ $(k > 1)$. Notice that, by (15) :

$$\|\Delta u_0\|_{L^2(R)} \leq C_{\alpha,p} \|f\|_{L^2(R)}$$

and

$$\|\Delta u_k\|_{L^2(R)} \leq C_{\alpha,p} \|(L - \Delta) u_{k-1}\|_{L^2(R)} \leq C_{\alpha,p} \|\Delta u_{k-1}\|_{L^2(R)};$$

thus:

$$||\Delta u_k||_{L^2(R)} \leq C_{\alpha,p}^{k+1}||f||_{L^2(R)}.$$

Therefore, if $|z - z_0| < C_{\alpha,p}^{-1}$, then $v \in W_{\#}^{2,2}(R)$ and $L_z v = f$; moreover :

$$\int_R (T_z f) g\, dxdt = \int_R (\Delta v) g\, dxdt = \sum_{k=0}^{\infty} (z - z_0)^k \int_R (\Delta u_k) g\, dxdt.$$

Hence $\int_R (T_z f) g\, dxdt$ is holomorphic in a neighborhood of z_0 and so everywhere inside Q.

Now we are in a position to apply the complex interpolation result [4].

Let T_z be a linear operator, mapping simple functions, defined in R, into measurable functions defined in R. Moreover, let:

$$\omega = \omega(\alpha, p) = \pi(2\eta_0(\alpha, p))^{-1}$$

and

$$b(z) = \frac{1}{2} + \left(\frac{1}{p} - \frac{1}{2}\right) \frac{\sinh \omega(1 - \xi)\ \cos \omega\eta}{\sinh \omega}.$$

Notice, by the way, that $b(z)$ is harmonic in Q,

$$b(0, \eta) = \frac{1}{2} + \left(\frac{1}{p} - \frac{1}{2}\right) \cos \omega\eta\ ,\ b(1, \eta) = \frac{1}{2}\ ,\ b(\xi, \pm\eta_0) = \frac{1}{2}.$$

Let $\mathcal{N}(z)$ denote the norm of T_z as an operator from $L^{1/b(z)}(R)$ into $L^{1/b(z)}(R)$; then $\log \mathcal{N}(z)$ is subharmonic.

Observe that, if $z = i\eta$, then $1/b(z)$ is between 2 and p; if $z = 1 + i\eta$, $\xi \pm i\eta_0(\alpha, p)$ then $1/b(z) = 2$. As a consequence, $\log \mathcal{N}(z)$ is bounded on ∂Q by $C_{\alpha,p}$; thus: $\log \mathcal{N}(\xi + i0) \leq C_{\alpha,p}$.

So we have that:

$$||\Delta u||_{L^{1/b(\xi)}(R)} \leq C_{\alpha,p}||L_\xi u||_{L^{1/b(\xi)}(R)}, \quad 0 \leq \xi \leq 1. \tag{17}$$

Notice that $1/b(\xi)$ is a monotonic function with $1/b(0) = p$, $1/b(1) = 2$.

Now let us take an operator Λ of the form (1), with coefficients satisfying (2), and define:

$$L = \vartheta^{-1}\Lambda - \vartheta^{-1}(1 - \vartheta)\Delta, \quad 0 < \vartheta < 1.$$

L is of the form (1) and satisfies (2) with α replaced by $\alpha/2$, if ϑ is sufficiently near to 1.

Then equation (17) holds with $p = 3$ and $p = 3/2$, and α replaced by $\alpha/2$. If $\xi = \vartheta$, then $L_\xi = \Lambda$, so we get two exponents p_1 and p_2 (both of them depending on $\alpha/2$ only), $p_1 < 2 < p_2$, such that:

$$||\Delta u||_{L^{p_1}(R)} \leq C_{\alpha/2,3/2} \, ||\Lambda u||_{L^{p_1}(R)}$$

$$||\Delta u||_{L^{p_2}(R)} \leq C_{\alpha/2,3} \, ||\Lambda u||_{L^{p_2}(R)}.$$

The thesis follows.

Remark 1. Theorem 1 and Theorem 2 easily give us an existence and uniqueness theorem in R for the equation $Lu = f$ of the type (1), with coefficients satisfying (2), with boundary conditions: $u_{|\partial_0 R} = 0$, $u_{|t=0} = u_{|t=1} = 0$ and $u_{t_{|t=0}} = u_{t_{|t=1}} = 0$.

Moreover, by Sobolev imbedding theorems, we have that, if the function f is of class $L^p(R)$ (p around 2), then:

(i) $u \in W^{1,q}(R)$, $1/q = 1/p - 1/3$ (q is around 6);

(ii) for $p > 3/2$ $u \in C^{0,\sigma}(\bar{R})$, $\sigma = 2 - 3/p$ (σ is around 1/2), and:

$$||Du||_{L^q(R)} \leq C_{\alpha,p} ||f||_{L^p(R)} \tag{18}$$

$$||u||_{C^{0,\sigma}(\bar{R})} \leq C_{\alpha,p} ||f||_{L^p(R)}. \tag{19}$$

Remark 2. If one changes R into $B \times (a, a+1)$ theorems 1 and 2 hold with the same constants and exponents. If we consider $B \times (a, b)$, then theorems 1 and 2 hold, but the costants and exponents may change; however, if $1/2 < b - a < 2$ (say) there exists C_α such that if $p_1(\alpha/4) < p < p_2(\alpha/4)$, then

$$||D^2 u||_{L^p(B\times(a,b))} \leq C_\alpha ||Lu||_{L^p(B\times(a,b))} \qquad \forall u \in W^{2,p}_{\#}(B \times (a, b)).$$

From now on we will redefine: $p_1(\alpha) := \max\{p_1(\alpha/4), 3/2\}$, $p_2(\alpha) := p_2(\alpha/4)$.

3. Uniqueness and existence theorems for the Dirichlet problem with homogeneous boundary conditions.

Theorem 3. *Let $p_1(\alpha) < p < p_2(\alpha)$. Let $u \in W^{2,p}_{\text{loc}}(R \cup \partial_0 R) \cap C^0(\overline{R})$, $u_{|\partial R} = 0$, $Lu \in L^p(R)$. Then the following estimate holds:*

$$\|u\|_{L^\infty(R)} \leq C_{\alpha,p} \|Lu\|_{L^p(R)}. \tag{20}$$

Proof. Let $w \in C^\infty(\bar{R})$, $w_{|\partial_0 R} = 0$. Let us show that

$$\|w\|_{L^\infty(R)} \leq C_{\alpha,p}\Big(\|Lw\|_{L^p(R)} + \|w\|_{L^\infty(\partial R \setminus \partial_0 R)}\Big). \tag{21}$$

For let a sequence of operators :

$$L_n = a_n^{ij} D_{ij} + b_n^j D_j \partial/\partial t + \partial^2/\partial t^2$$

smooth with $a_n^{ij} \to a^{ij}$, $b_n^j \to b^j$ $(i, j = 1, 2)$ a.e. in B. We may assume that the coefficients of L_n satisfy (2). Let us solve the problem:

$$L_n v_n = L_n w, \quad v_n \in W^{2,p}_{\#}(R) \cap C^2(R).$$

If $q = (p + p_1(\alpha))/2$, by Remark 1:

$$\|v_n\|_{L^\infty(R)} \leq C_{\alpha,p} \|L_n v_n\|_{L^q(R)} = C_{\alpha,p} \|L_n w\|_{L^q(R)}. \tag{22}$$

As $L_n(w - v_n) = 0$ in R, by the maximum principle:

$$\|w - v_n\|_{L^\infty(R)} \leq \|w\|_{L^\infty(\partial R \setminus \partial_0 R)} + \|v_n\|_{L^\infty(R)}. \tag{23}$$

Then, from (22) and (23):

$$\|w\|_{L^\infty(R)} \leq \|w\|_{L^\infty(\partial R \setminus \partial_0 R)} + C_{\alpha,p} \|L_n w\|_{L^q(R)}. \tag{24}$$

On the other hand:

$$\|L_n w - Lw\|_{L^q(R)} \leq$$

$$\leq \left\{ \sum_{ij=1}^{2} \|a_n^{ij} - a^{ij}\|_{L^{pq/p-q}(R)} + \sum_{j=1}^{2} \|b_n^j - b^j\|_{L^{pq/p-q}(R)} \right\} \|D^2 w\|_{L^p(R)}.$$

Thus, letting $n \to \infty$ yields $L_n w \to Lw$ in L^q and so the estimate (24) easily implies (21), once we recall also that $q < p$.

Now let $0 < \varepsilon < 1/2$ and $R_\varepsilon = B \times (\varepsilon, 1-\varepsilon)$.

Notice that $u \in W^{2,p}(R_\varepsilon)$; hence there exist $u_n \in C^\infty(\bar{R}_\varepsilon)$, $u_{n|\partial R_\varepsilon} = 0$ such that $u_n \to u$ in $W^{2,p}(R_\varepsilon)$. Then, for u_n, the estimate (21) holds in R_ε (recall also Remark 2). Taking the limit as $n \to \infty$, we have:

$$||u||_{L^\infty(R_\varepsilon)} \leq C_{\alpha,p}\Big(||Lu||_{L^p(R_\varepsilon)} + ||u||_{L^\infty(\partial R_\varepsilon \setminus \partial_0 R_\varepsilon)}\Big).$$

Now, letting $\varepsilon \to 0$ yields $R_\varepsilon \to R$ and:

$$||u||_{L^\infty(R_\varepsilon)} \to ||u||_{L^\infty(R)},$$

$$||Lu||_{L^p(R_\varepsilon)} \to ||Lu||_{L^p(R)},$$

$$||u||_{L^\infty(\partial R_\varepsilon \setminus \partial_0 R_\varepsilon)} \to 0.$$

Hence the estimate (20) follows and the Theorem is proved.

In what follows we need next Lemma 2, involving weighted norms.

Let $w(t) = t(1-t)$. The following result is true.

Lemma 2. *Let $u \in C^2(\bar{R})$, $u_{|\partial R} = 0$. Let $p \in [p_1(\alpha), p_2(\alpha)]$. Then:*

$$(25) \qquad ||w^2 D^2 u||_{L^p(R)} \leq C_{\alpha,p}\Big(||w^2 Lu||_{L^p(R)} + ||u||_{L^p(R)}\Big).$$

Before giving the line of the proof, let us point out for our purposes that standard Sobolev spaces techniques allow to get the following weighted interpolation inequality: *for any $u \in C^2(\bar{R})$, $u_{|\partial R} = 0$, and for any $\varepsilon > 0$:*

$$(26) \qquad ||wDu||_{L^p(R)} \leq \varepsilon ||w^2 D^2 u||_{L^p(R)} + C_{\varepsilon,p}||u||_{L^p(R)}.$$

Proof of Lemma 2. Set $v = w^2 u$. The function v can be extended outside of R into $B \times \mathbb{R}$ as a C^2 function periodic in t, with period 1. By applying Theorem 2 to the function v, writing back the bounds for u and using interpolation inequality (26), one ends up with (25).

We are now able to show the following theorem.

Theorem 4. *Let L of the form (1), with coefficients satisfying (2). Let $p \in [p_1(\alpha), p_2(\alpha)]$. Then, if $f \in L^p(R)$, the problem*

$$Lu = f \qquad \text{in } R \tag{27}$$

$$u_{|\partial R} = 0 \tag{28}$$

has a unique solution $u \in W^{2,p}_{\mathrm{loc}}(R) \cap C^0(\bar{R})$ and:

$$\|u\|_{L^\infty(R)} \leq C_{\alpha,p}\|Lu\|_{L^p(R)}, \tag{29}$$

$$\|w^2 D^2 u\|_{L^p(R)} \leq C_{\alpha,p}\|Lu\|_{L^p(R)}, \tag{30}$$

$$|u(x,t)| \leq C_{\alpha,p}\Big\{\mathrm{dist}((x,t),\partial R)\Big\}^{2-3/p}\|Lu\|_{L^p(R)}. \tag{31}$$

Proof. The uniqueness of the solution follows by Theorem 3. Indeed, if $u_1 \in W^{2,p}_{loc}(R)$ and $u_2 \in W^{2,p}_{loc}(R)$ are two solutions of (27) - (28), then $u_1 - u_2 \in W^{2,p}_{loc}(R) \cap C^0(\bar{R})$ vanishes on ∂R and $L(u_1 - u_2) = 0$ a.e. in R. Thus, Theorem 3 applies and $u_1 - u_2 \equiv 0$.

Let us prove the existence. Let $f_n \in C^\infty_0(R)$ and $f_n \to f$ in $L^p(R)$. Consider a sequence of operators:

$$L_n = a^{ij}_n D_{ij} + b^j D_j \partial/\partial t + \partial^2/\partial t^2$$

smooth and such that $a^{ij}_n \to a^{ij}$, $b^j_n \to b^j$ a.e. in B.
We may assume that the coefficients of L_n satisfy (2).

Let u_n be the solutions of

$$\begin{aligned} L_n u_n &= f_n \qquad \text{in } R \\ u_{n_{|\partial R}} &= 0. \end{aligned}$$

Clearly $u_n \in C^2(\bar{R}) \cap W^{2,p}(R)$ and, by Theorem 3 as well as Lemma 2:

$$\|u_n\|_{L^\infty(R)} \leq C_{\alpha,p}\|L_n u_n\|_{L^p(R)}, \tag{32}$$

(33) $$||w^2 D^2 u_n||_{L^p(R)} \leq C_{\alpha,p} ||L_n u_n||_{L^p(R)}.$$

Let us extend f_n as a periodic (in t) function and solve the problem: $L_n v_n = f_n$, $v_n \in W^{2,p}_{\#}(R)$. Actually, the v_n functions are smooth in $\bar{B} \times \mathbb{R}$. From (19), recall that:

$$||v_n||_{C^{0,\sigma}(\bar{R})} \leq C_{\alpha,p} ||f_n||_{L^p(R)}, \quad \text{where} \quad \sigma = 2 - 3/p.$$

By a result of C. Pucci [10], since $L_n(v_n - u_n) = 0$ in R and $u_{n|\partial R} = 0$, if $(x, t) \in R$ and $(\xi, \tau) \in \partial R$ such that $\operatorname{dist}((x, t), (\xi, \tau)) = \operatorname{dist}((x, t), \partial R)$, then:

$$|v_n(x, t) - u_n(x, t) - v_n(\xi, \tau)| \leq C_{\alpha,p} \Big\{\operatorname{dist}((x, t), \partial R)\Big\}^{\sigma} ||v_n||_{C^{0,\sigma}(\partial R)},$$

and therefore:

$$|u_n(x, t)| \leq |v_n(x, t) - u_n(x, t) - v_n(\xi, \tau)| + |v_n(x, t) - v_n(\xi, \tau)| \leq$$

$$\leq C_{\alpha,p} \Big\{\operatorname{dist}((x, t), \partial R)\Big\}^{\sigma} ||v_n||_{C^{0,\sigma}(\partial R)}.$$

Thus:

(34) $$|u_n(x, t)| \leq C_{\alpha,p} \{\operatorname{dist}((x, t), \partial R)\}^{\sigma} ||L_n u_n||_{L^p(R)}.$$

Observe now that, as $L_n u_n = f_n \to f$ in $L^p(R)$, $||L_n u_n||_{L^p(R)}$ is bounded. As a consequence, by (32), there is a subsequence of solutions (let us call it again u_n) such that $u_n \rightharpoonup u \in W^{2,p}_{\text{loc}}(R \cup \partial_0 R)$, $u_n \to u$ uniformly in every compact subset of $R \cup \partial_0 R$, so $u \in C^0(R)$. Then u satisfies (29), (30), (31) and so $u \in C^0(\bar{R})$, $u = 0$ on ∂R.

On the other hand, if φ is a smooth function with support in $R \cup \partial_0 R$, we have:

$$\int_R (Lu - f)\varphi \, dxdt = \int_R L(u - u_n)\varphi \, dxdt + \int_R (L - L_n)u_n \varphi \, dxdt +$$

$$+ \int_R (f_n - f)\varphi \, dxdt.$$

Letting $n \to \infty$, clearly yields: $Lu = f$ a.e. in R.

The Theorem 4 is so completely proved.

Remark 3. Let $f \in L^3(R)$ and let u to be a "good" solution to the problem $Lu = f$ in R, $u_{|\partial R} = 0$; namely, u is the uniform limit on $\bar{R}$ of the sequence $\{u_n\}$ formed by the solutions to $L_n u_n = f$ in R, $u_n = 0$ on ∂R, where L_n are regularizations of L of the form (1), with coefficients satisfying (2) and converging pointwise to those of L on B.

Then, as in Theorem 4, $u \in W^{2,p}_{\mathrm{loc}}(R)$ (for some $p > 2$), $u \in C^0(\bar{R})$ and $Lu = f$ a.e. in R, $u_{|\partial R} = 0$.

Therefore Theorem 3 give us the uniqueness of *"good" solutions* to the Dirichlet problem for L, with no regularity assumptions on its coefficients a^{ij}, b^j.

REFERENCES

[1] M.C. Cerutti - L. Escauriaza - E.B. Fabes, *Uniqueness for the Dirichlet problem for elliptic operators with discontinuous coeffcients,* Ann. Mat. Pura Appl. (IV), 163 (1993), pp. 161-180.

[2] M.C. Cerutti - L. Escauriaza - E.B. Fabes, *Uniqueness for some diffusions with discontinuous coefficients,* Annals of Probability, 19 n. 2 (1991), pp. 525-537.

[3] M. Chicco, *Equazioni ellittiche del secondo ordine di tipo Cordes con termini di ordine inferiore,* Ann. Mat. Pura Appl. (IV), 85 (1970), pp. 347-356.

[4] R. Coifman - M. Cwikel - R. Rochberg - Y. Sagher - G. Weiss, *The complex method for interpolation of operators acting on families of Banach spaces,* Lecture Notes in Math., 779, Proceedings Un. Maryland 1979, Springer-Verlag, Berlin, 1980.

[5] M.G. Crandall - H. Ishii - P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations,* Bulletin A.M.S., 27 (1992), pp. 1-67.

[6] L. Escauriaza, *Uniqueness in the Dirichlet problem for time independent elliptic operators,* PDE with minimal smoothness and applications, Chicago, (1990), pp. 115-127; IMA vol. Math. Appl. 42, Springer, New York, 1992.

[7] V.I. Judovič, *Somebounds for the solutions of elliptic equations,* Trasl. Am. Math. Soc. (2), 56 (1966), pp. 1-18.

[8] N.V. Krylov, *On one-point weak uniqueness for elliptic equations,* preprint.

[9] C. Pucci, *Limitazioni per soluzioni di equazioni ellittiche,* Ann. Mat. Pura Appl., 74 (1966), pp. 15-30.

[10] C. Pucci, *Regolarità alla frontiera di soluzioni di equazioni ellittiche,* Ann. Mat. Pura Appl., 65 (1964), pp. 311-328.

[11] G. Talenti, *Equazioni lineari ellittiche in due variabili,* Le Matematiche, 21 (1966), pp. 339-376.

Istituto di Matematica,
Facoltà di Architettura,
Università di Firenze,
50100 Firenze (Italy)