LE MATEMATICHE
Vol. XLVIII (1993) – Fasc. I, pp. 179-181

ON THE CAMPANATO NEARNESS CONDITION

SALVATORE LEONARDI

We prove a Lax-Milgram type theorem using the concept of nearness between operators.

1. Introduction.

S. Campanato conjectured the possibility to extend the Lax-Milgram theorem to the complete metric vectorial spaces. In this note we give another definition of nearness between operators (see [1], [2], [3]) and by the mean of it we prove the Lax-Milgram theorem holds in complete metric vectorial spaces.

2. Preliminary lemmata.

We are in the following situation:

$\mathscr{B}$ is a set without a priori structure,
$\mathscr{B}_1$ is a metric space with metric δ,
A, B are operators $\mathscr{B} \to \mathscr{B}_1$.

Lemma 1. (Campanato). *If B is injective then $\mathscr{B}$ is a metric space equipped with the metric:*

$$d_{\mathscr{B}}(u, v) = \delta(B(u), B(v)). \tag{1}$$

Entrato in Redazione il 20 settembre 1993.

Lemma 2. (Campanato). *If $\mathscr{B}_1$ is complete and B is bijective then $\mathscr{B}$ is complete equipped with the metric* (1).

Definition. (Campanato). *A is a small perturbation of B if there exists $k \in]0, 1[$ such that*

$$\delta(A(u), A(v)) \leq k\delta(B(u), B(v)) \quad \forall u, v \in \mathscr{B}.$$

Lemma 3. (Campanato). *If $\mathscr{B}_1$ is complete, B is bijective and A is a small perturbation of B then there exists unique $u \in \mathscr{B}$ such that*

$$A(u) = B(u).$$

3. Lax-Milgram theorem.

Definition. *Let $\mathscr{B}_1$ be a metric vectorial space. We say A is near by B if there exist $\alpha > 0$, $k \in]0, 1[$ such that*

$$\delta(B(u) - \alpha A(u), B(v) - \alpha A(v)) \leq k\delta(B(u) - x, B(v) - x)$$

$\forall u, v \in \mathscr{B}$, $\forall x \in \mathscr{B}_1$.

Theorem. (Lax-Milgram). *If $\mathscr{B}_1$ is a complete metric vectorial space,*

$$B : \mathscr{B} \to \mathscr{B}_1$$

is bijective and A is near by B then A is bijective.

Proof. Let $f \in \mathscr{B}_1$. The problem $A(u) = f$ is equivalent to the following one:

$$B(u) - \alpha f = B(u) - \alpha A(u).$$

Put

$$F(u) = B(u) - \alpha A(u),$$
$$G(u) = B(u) - \alpha f \quad \text{(bijective)},$$

in virtue of Lemma 3 it will be sufficient to prove F is a small perturbation of G. We have:

$$\delta(F(u), F(v)) = \delta(B(u) - \alpha A(u), B(v) - \alpha A(v)) \leq k\delta(B(u) - x, B(v) - x)$$

$\forall u, v \in \mathscr{B}, \forall x \in \mathscr{B}_1$.
Put $x = \alpha f$, we get:

$$\delta(F(u), F(v)) \leq k\delta(B(u) - \alpha f, B(v) - \alpha f) = k\delta(G(u), G(v))$$

$\forall u, v \in \mathscr{B}$. $\square$

Remarks. Our definition of nearness between operators gives back the Campanato's one in the case $\mathscr{B}_1$ normed space. Indeed, provided the further requirement of "invariance under translation" of the metric δ in $\mathscr{B}_1$ (like that one deduced by a norm), the operator A is near by B (see Definition above) if and only if there exist $\alpha > 0$, $k \in]0, 1[$ such that

$$\delta(B(u) - \alpha A(u), B(v) - \alpha A(v)) \leq k\delta(B(u), B(v))$$

$\forall u, v \in \mathscr{B}$.

Acknowledgements.

The author wishes to thank Professor F. Nicolosi for his constant encouragements.

REFERENCES

[1] S. Campanato, *On the condition of nearness between operators,* Ann. Mat. Pura Appl., to appear.

[2] S. Campanato, *Sistemi differenziali del secondo ordine di tipo ellittico,* Quaderno # 1 del Dottorato di Ricerca in Matematica, Università di Catania, 1991.

[3] S. Campanato, *Sulla condizione di vicinanza tra operatori,* Quaderno # 2 del Dottorato di Ricerca in Matematica, Università di Catania, 1993.

Dipartimento di Matematica,
Università di Catania,
Viale A. Doria 6,
95125 Catania (Italy)