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FURTHER CONTRIBUTION TO THE THEORY OF NEAR MAPPINGS

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We generalize the concept of contraction mappings by introducing that of small perturbations and we show for small perturbations of a bijective map, a fixed point theorem of Banach - Caccioppoli.

As a consequence we give a proof of the theorem of Schauder.

In recent years we have developped a method of existence based on the notion of near mappings.

This method has unified several existence methods used in analysis reobtaining and generalizing, the theorems of Schauder, of Lax - Milgram, of Cordes and of the theory of monotone operators ([1], [2]).

An examination of these lead us to identify two basic concepts: the concept of *small perturbations* and that of a *mapping A near to a mapping B*.

Let us consider the following situation:

(1)
$\mathscr{B}$ is a set,
$\mathscr{B}_1$ is a metric space with metric δ,
A and B are two mappings $\mathscr{B} \to \mathscr{B}_1$.

It is well known that

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Lemma 1. *If B is injective then*

$$d_B(u, v) = \delta(B(u), B(v)) \tag{2}$$

is a metric on $\mathscr{B}$ and $\{\mathscr{B}, d_B\}$ is complete if $\mathscr{B}_1$ is complete and $B : \mathscr{B} \to \mathscr{B}_1$ is bijective.

Remark 1. In particular if B is injective and the image $\mathscr{B}^* = B(\mathscr{B})$ is complete ([1]) then $\{\mathscr{B}, d_B\}$ is also complete.

In a similar way one can prove that

Lemma 2. *If B is injective and $\mathscr{B}^*$ is compact then $\{\mathscr{B}, d_B\}$ is also compact.*

Proof. Infact, if $\{u_n\}$ is a sequence in $\mathscr{B}$, then $\{B(u_n)\}$ is a sequence in $\mathscr{B}^*$ which is compact and hence (modulo passing to a subsequence which, for simplicity, is again denoted by the same symbol) $\exists\, \mathscr{U} \in \mathscr{B}^*$ such that

$$\delta(B(u_n), \mathscr{U}) \to 0.$$

Setting $u = B^{-1}\mathscr{U}$ ([2]) we have $u \in \mathscr{B}$ and

$$d_B(u_n, u) = \delta(B(u_n), \mathscr{U}) \to 0.$$

This proves that $\{\mathscr{B}, d_B\}$ is compact.

We now have the following definition.

Definition 1. *We say that A is a small perturbation of B, with constant K, if there exist $K \in (0, 1)$ such that $\forall\, u, v \in \mathscr{B}$ we have*

$$\delta(A(u), A(v)) \le K\delta(B(u), B(v)). \tag{3}$$

In particular, if A is a small perturbation of B and B is injective, then A is continuous on $\{\mathscr{B}, d_B\}$.

The concept of small perturbation generalizes that of contraction because when $\mathscr{B} = \mathscr{B}_1$ is a metric space, to say that $A : \mathscr{B} \to \mathscr{B}_1$ is a contraction means that A is a small perturbation of the identity $I : \mathscr{B} \to \mathscr{B}_1$.

For contractions one proves the classical fixed point theorem (theorem of Banach - Caccioppoli) and the method with which one constructs a sequence of elements $\{u_n\}$ of $\mathscr{B}$ which converges in the metric d_B to the fixed point u is called the method of successive approximation.

This theorem and the method of successive approximation, easily extends to small perturbations of a bijection B also between sets $\mathscr{B} \neq \mathscr{B}_1$.

([1]) It is a complete metric subspace of $\mathscr{B}_1$.

([2]) We note that $B : \mathscr{B} \to \mathscr{B}^*$ is bijective.

Theorem 1. *If A and B are mappings $\mathscr{B} \to \mathscr{B}_1$ and if*

(4) $\mathscr{B}_1$ *is complete*

(5) B *is bijective*

(6) A *is a small perturbation of* B,

then $\exists_1 u \in \mathscr{B}$ *(fixed point) such that*

$$A(u) = B(u). \tag{7}$$

Proof. Infact, by the completeness of $\mathscr{B}_1$ and the bijectivity of B, also $\{\mathscr{B}, d_B\}$ is complete (Lemma 1) and, by the assumption (6), the mapping $B^{-1}A$ is a contraction of $\{\mathscr{B}, d_B\}$ into itself; infact, $\forall u, v \in \mathscr{B}$,

$$\begin{aligned} d_B(B^{-1}A(u), B^{-1}A(v)) &= \\ &= \delta(A(u), A(v)) \leq K\delta(B(u), B(v)) = Kd_B(u, v). \end{aligned} \tag{8}$$

Remark 2. In view of the Remark 1 one can weaken the hypothesis (4), (5). The Theorem 1 remains true if B is injective, the image $\mathscr{B}^* = B(\mathscr{B})$ is complete and $A(\mathscr{B}) \subset B(\mathscr{B})$.

We shall now consider a more particular situation than (1): Suppose that $\mathscr{B}_1$ has a linear structure also. Let us suppose, for simplicity, that

(9) $\mathscr{B}_1$ is a Banach space with the usual metric $\delta(u, v)$,

$$\delta(u, v) = ||u - v||_{\mathscr{B}_1} \tag{10}$$

and the assumptions on $\mathscr{B}$ and A, B remain unaltered.

Having a linear structure on $\mathscr{B}_1$ one can prove a simple generalization of Theorem 1 on small perturbations.

We give the following definition.

Definition 2. *If A and B are two mappings $\mathscr{B} \to \mathscr{B}_1$ we say that A is a perturbation of B of constant K (or also that A is bounded by B with constant K) if $\exists K > 0$ such that, $\forall u, v \in \mathscr{B}$, we have*

$$||A(u) - A(v)||_{\mathscr{B}_1} \leq K||B(u) - B(v)||_{\mathscr{B}_1}. \tag{11}$$

In particular, if A is a perturbation of B and B is injective it is still true that A is continuous on $\{\mathscr{B}, d_B\}$.

If $K \in (0, 1)$, we reobtain the definition of small perturbation of B.

This latter however, as we have already seen, has a sense also when the space $\mathscr{B}_1$ is only a metric space.

Theorem 2. *If A and B are mappings $\mathscr{B} \to \mathscr{B}_1$ and if*

(12) *B is bijective*

(13) *A is a perturbation of B with constant K,*

then $\forall M > K$, $\exists_1 u_M \in \mathscr{B}$ such that

$$A(u_M) = MB(u_M). \tag{14}$$

Proof. Infact from (11), on dividing both sides by M, we obtain that $\frac{A}{M}$ is a small perturbation of B. The assertion then follows from Theorem 1.

Remark 3. As in the case of Theorem 1, if we take into account of the Remark 1, one can weaken the hypothesis of Theorem 2. The Theorem 2 remains true if B is injective and the image $\mathscr{B}^* = B(\mathscr{B})$ is closed and $A(\mathscr{B}) \subset B(\mathscr{B})$.

Suppose now that $\mathscr{B}_1$ is a Banach space and that

$$\mathscr{B}^* = B(\mathscr{B}) \tag{15}$$

is a compact subset of $\mathscr{B}_1$, then, if B is injective, $\{\mathscr{B}, d_B\}$ is also compact (Lemma 2). Let $\{a_n\}$ be a real sequence, such that

$$a_n > 1 \ \forall n \quad \text{and} \quad a_n \to 1.$$

For example, $a_n = 1 + \frac{1}{n}$. If A is a perturbation of B with constant K, $M = a_n K$ is $> K$ and hence, by the Theorem 2 and the Remark 3, we see that $\forall n$ there exists $u_n \in \mathscr{B}$ such that

$$A(u_n) = a_n K B(u_n). \tag{16}$$

But $\mathscr{B}^*$ is compact and hence $\exists \mathscr{U} \in \mathscr{B}^*$ such that $\|B(u_n) - \mathscr{U}\|_{\mathscr{B}_1} \to 0$. Setting $u_\infty = B^{-1}\mathscr{U}$, we have $u_\infty \in \mathscr{B}$ and

$$d_B(u_n, u_\infty) = \|B(u_n) - B(u_\infty)\|_{\mathscr{B}_1} = \|B(u_n) - \mathscr{U}\|_{\mathscr{B}_1} \to 0.$$

Since A is continuous on $\{\mathscr{B}, d_B\}$, we get from (16) on passing to the limits as $n \to \infty$ that

$$A(u_\infty) = K B(u_\infty). \tag{17}$$

We can conclude with the following

Theorem 3. *If A and B are mappings $\mathscr{B} \to \mathscr{B}_1$ and $A(\mathscr{B}) \subset B(\mathscr{B})$, if $\mathscr{B}_1$ is a Banach space and the image $B(\mathscr{B})$ is compact then, if B is injective and A is bounded by B with constant K, $\exists\, u_\infty \in \mathscr{B}$ such that*

$$A(u_\infty) = KB(u_\infty).$$

Obviously, the fixed point u_∞ is not unique: there is one for every choice of the sequence $\{a_n\}$.

This Theorem 3 is a generalization of the classical theorem of Schauder which one reobtains when $K = 1$.

REFERENCES

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