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DUALITY PROBLEM FOR MULTI-OBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS USING THE TWO CONCEPTS (DUAL SPACE AND SUBGRADIENT)

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The problem dual to a multi-objective fractional programming problems is defined by using the concept of dual space of the objective space and using the concept of subgradient. Some assumptions considered in recent works are relaxed in our proposed approach.

1. Introduction.

Recently, some duality results have been obtained for minimax fractional programming problems, involving several ratios in the objective functions [10], [8], and for multi-objective convex programming problems using the concept of ordering convex cones defined on the objective and decision spaces [7].

Crouzeix, et al. in [5] have shown that the minimax fractional program can be solved by solving a minimax nonlinear parametric program. Also, Ibrahim, et al. and Youness in [6], [10] solved the multi-objective fractional programming problems by solving a nonlinear parametric program.

The purpose of this paper is to develop duality for multi-objective fractional program using a hybrid parametric program based on those considered in [6], [10] and [8].

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2. Problem formulation, notations and definitions.

Consider the multi-objective fractional programming problem:

$$(\mathrm{P}_0)\begin{cases}\min f_j(x)/g_j(x) \quad \text{subject to}\\ M=\{x\in\mathbb{R}^n\colon h_r(x)\le 0,\ r=1,2,\ldots,\ell\},\end{cases}$$

where $f_j(x)$, $g_j(x)$, $j = 1,2,\ldots,m$ are real valued functions, convex and concave, respectively, $g_j(x) > 0$, $j = 1,2,\ldots,m$, and $h_r(x)$, $r = 1,2,\ldots,\ell$ are convex real valued functions.

Definition 1. *A point $x^* \in M$ is said to be an efficient solution for problem (P_0), if there is no $x \in M$ such that*

$$f_j(x)/g_j(x) \le f_j(x^*)/g_j(x^*), \quad j=1,2,\ldots,m,$$

with strict inequality for at least one j.

To establish the efficiency conditions and duality we shall make use of problem (P_d) defined in [7] :

$$(\mathrm{P}_\mathrm{d})\begin{cases}\min f_j(x)-d_jg_j(x) \quad \text{subject to}\\ M=\{x\in\mathbb{R}^n\colon h_r(x)\le 0,\ r=1,2,\ldots,\ell\},\end{cases}$$

Where $d_j > 0$, $j = 1,2,\ldots,m$ are auxiliary parameters.

If we assume that $f_j(x) > 0$ for all $x \in M$, $j = 1,2\ldots,m$ then the following results are established:

(i) If x^* is an efficient solution for problem P_0 then there exists $d_j^* > 0$ such that x^* is an efficient for problem $\mathrm{P}_{\mathrm{d}^*}$.

(ii) If there exists (x^*, d^*) such that:
a) x^* is an efficient solution for $\mathrm{P}_{\mathrm{d}^*}$,
b) $f_j(x^*) - d_j^* g_j(x^*) = 0$, $j = 1,2,\ldots,m$,
then x^* is an efficient solution for P_0.

Now, we shall prove the equivalence between the parametric multi-objective programming problem (P_d) and the following parametric multi-objective program :

$$(\mathrm{EP}_\mathrm{d})\begin{cases}\min q_j \quad \text{subject to}\\ M'=\{(x,d,q)\in\mathbb{R}^{n+2m}\ :\ \begin{matrix} f_j(x)-d_jg_j(x)\le q_j, & j=1,2\ldots,m\\ h_r(x)\le 0, & r=1,2,\ldots,\ell\}\end{matrix}\end{cases}$$

It is clear that, if $(x,d,q) \in \mathbb{R}^{n+2m}$ is feasible for (EP_d), then $x \in \mathbb{R}^n$ is feasible for (P_0) and if $x \in \mathbb{R}^n$ is a feasible for (P_0), then there exist d and q in $\mathbb{R}^m$ such that $(x,d,q) \in \mathbb{R}^{n+2m}$ is feasible for (EP_d).

Lemma 1. *If* $(\bar{x}, \bar{d}, \bar{q}) \in M'$ *is* EP$_{\rm d}$*-efficient, then* $\bar{x}$ *is* P$_{\bar{\rm d}}$*-efficient.*

Proof. Since $\bar{q} \in \mathbb{R}^m$ is nondominated point corresponding to $\bar{x} \in \mathbb{R}^n$, $h_r(\bar{x}) \leq 0$, $r = 1, 2, \ldots, \ell$. Then

$$f_j(\bar{x}) - \bar{d}_j g_j(\bar{x}) \leq \bar{q}_j, \quad j = 1, 2, \ldots, m.$$

Let $\bar{x}$ be not efficient solution for (P$_{\bar{\rm d}}$), then there is $\tilde{x} \in M$ such that

$$f_j(\tilde{x}) - \bar{d}_j g_j(\tilde{x}) \leq f_j(\bar{x}) - \bar{d}_j g_j(\bar{x}) \leq \bar{q}_j, \quad j = 1, 2, \ldots, m,$$

with strict inequality for at least one j. Therefore, there is $\tilde{q} \in \mathbb{R}^m$ such that $\tilde{q}_j \leq \bar{q}_j$, $j = 1, 2, \ldots, m$ with strict inequality for at least one j which leads to a contradiction.

Lemma 2. *Let, for given* $d, \bar{x} \in M$ *be an efficient solution for* (P$_{\rm d}$), *then* $(\bar{x}, d, \bar{q}) \in \mathbb{R}^{n+2m}$ *with* $\bar{q}_j = f_j(\bar{x}) - d_j g_j(\bar{x})$ *is* EP$_{\rm d}$*-efficient.*

Proof. Since $\bar{q}_j = f_j(\bar{x}) - d_j g_j(\bar{x})$, $j = 1, 2, \ldots, m$ and $h_r(\bar{x}) \leq 0$, $r = 1, 2, \ldots, \ell$, then $(\bar{x}, d, \bar{q}) \in M'$. Let $(\bar{x}, d, \bar{q})$ be not EP$_{\rm d}$-effcient, then there exists $(\tilde{x}, d, \tilde{q}) \in M'$ such that

$$f_j(\tilde{x}) - d_j g_j(\tilde{x}) \leq \tilde{q}_j \leq \bar{q}_j = f_j(\bar{x}) - d_j g_j(\bar{x}), \quad j = 1, 2, \ldots, m$$

with strict inequality for at least one j, which contradicts the efficiency of $\bar{x}$ for (P$_{\rm d}$). Hence the result.

Remark. Lemma 1, Lemma 2, statements i) and ii) and the assumption that $f_j(x) > 0$ for each $x \in M$ show that $x^* \in M$ is an efficient solution for P$_0$ with corresponding objective value d_j^*, $j = 1, 2 \ldots m \iff (x^*, d^*, q^*)$ is EP$_{\rm d}^*$-efficient with objectives value equal to zero, i.e., $d_j^* = 0$, $j = 1, 2, \ldots, m$.

3. Duality using subgradient concept.

Under the assumptions in the remark, the dual problem of the primal problem P$_0$ depends upon the dual problem of the parametric program (EP$_{\rm d}$) which equivalent to the problem (P$_{\rm d}$) that used to solve the program (P$_0$). There are many ways of formulating the dual program to the program (P$_{\rm d}$) in the case of single objective with differentiablity of the objective and the constraint functions [7], [8]. Also, [8] formulated a dual program of multi-objective programming problem by using the notion of subdifferentiability. In this paper we introduce a modifications in the presented works by researchers in [7], [8] to deal with our main problem and to relax some assumptions imposed.

Definition 2. *Let $f(x)$ be a real valued convex function defined on $\mathbb{R}^n$. A vector $\nu \in \mathbb{R}^n$ is called a subgradient of f at x_0 if*

$$f(x) - f(x_0) \geq \nu(x - x_0) \qquad \forall x \in \mathbb{R}^n.$$

Let us denote to the set all subgradients of f at x_0 by $\partial f(x_0)$.

Now, we formulate the dual program ($\mathrm{DEP_d}$) *to the program* ($\mathrm{EP_d}$) *as follows:*

$$(\mathrm{DEP_d}) \begin{cases} \max f_j(u) - d_j g_j(u) \quad \text{subject to} \\ N = \{u \in \mathbb{R}^n : \begin{array}{l} 0 \in \partial\{f_j(u) - d_j g_j(u)\}, \\ 0 \in \partial h_r(u), \quad j = 1, 2, \ldots, m, \; r = 1, 2, \ldots, \ell\}. \end{array} \end{cases}$$

Theorem 1. (Weak duality theorem) *For a given $d \in \mathbb{R}^m_+$, let $(\hat{x}, d, \hat{q}) \in \mathbb{R}^{n+2m}$ be a feasible point of* ($\mathrm{EP_d}$). *Then for $\hat{u}, \in N$,*

$$\hat{q}_j \geq f_j(\hat{u}) - d_j g_j(\hat{u}).$$

Proof. From the feasibility of $(\hat{x}, d, \hat{q})$ for ($\mathrm{EP_d}$), we have

$$f_j(\hat{x}) - d_j g_j(\hat{x}) \leq \hat{q}_j, \quad j = 1, 2, \ldots, m.$$

Since $\{f_j(x) - d_j g_j(x)\}$ are convex functions, then from Definition 2 we have

$$\{f_j(\hat{x}) - d_j g_j(\hat{x})\} - \{f_j(\hat{u}) - d_j g_j(\hat{u})\} \geq \nu_j(\hat{x} - \hat{u}),$$

for all $\nu_j \in \partial\{f_j(\hat{u}) - d_j g_j(\hat{u})\}$. So,

$$\hat{q}_j \geq f_j(\hat{u}) - d_j g_j(\hat{u}), \quad j = 1, 2, \ldots, m$$

since, $\hat{u} \in N$, i.e. $0 \in \partial\{f_j(\hat{u}) - d_j g_j(\hat{u})\}$.

Theorem 2. (Strong duality theorem) *If, for given $d \in \mathbb{R}^m_+$, $(\bar{x}, d, \bar{q}) \in M'$ and $\bar{q}$ is efficient for problem* ($\mathrm{EP_d}$) *with $\bar{q}_j = f_j(\bar{x}) - d_j g_j(\bar{x})$. Then $\bar{x}$ is an efficient solution for* ($\mathrm{DEP_d}$).

Proof. Since $(\bar{x}, d, \bar{q}) \in M'$, then

$$f_j(\bar{x}) - d_j g_j(\bar{x}) \leq \bar{q}_j, \quad j = 1, 2, \ldots, m.$$

But, for $u \in N$, we have

$$(*) \qquad f_j(u) - d_j g_j(u) \leq \bar{q}_j$$

and, from definition (2).

$$(**) \qquad h_r(u) \leq 0;$$

which implies $(u, d, \bar{q}) \in M'$.

Now, we shall prove that $\bar{x} \in N$. For this purpose, let $\bar{x} \notin N$, then

$$\{f_j(u) - d_j g_j(u)\} \geq \{f_j(\bar{x}) - d_j g_j(\bar{x})\} + \nu_j(u - \bar{x}),$$

$$\nu_j \neq 0, \quad j = 1, 2, \ldots, m, \quad u \neq \bar{x}.$$

Thus there are two cases:

(i) $\nu_j(u - \bar{x}) > 0$, $j = 1, 2, \ldots, m$ and hence

$$f_j(u) - d_j g_j(u) > f_j(\bar{x}) - d_j g_j(\bar{x}) = \bar{q}_j$$

and this contradicts the result $(*)$.

(ii) $\nu_j(u - \bar{x}) < 0$, $j = 1, 2, \ldots, m$ and hence $\sum_{j=1}^{m} w_j(\nu_j(u - \bar{x})) < 0$, $w_j \geq 0$. But, from the weighting problem, we have $\bar{x}$ is a minimizer point of $\sum_{j=1}^{m} w_j\{f_j(x) - d_j g_j(x)\}$ on M. But in [7] there is a theorem states that $\bar{x}$ is a minimizer point of $f(x)$ on S if and only if $\nu(x - \bar{x}) \geq 0$ for each $x \in S$ and $\nu \in \partial f(\bar{x})$. Therefore $\sum_{j=1}^{m} w_j(\nu_j(u - \bar{x})) \geq 0$. Thus $< \nu_j, (u - \bar{x}) < 0$ is impossible.

Finally, we shall prove that $\bar{x}$ is an efficient solution for problem (DEP_d). Let us suppose $\bar{x}$ not efficient solution for (DEP_d), then there is $\bar{u} \in N$ such that:

$$f_j(\bar{u}) - d_j g_j(\bar{u}) \geq f_j(\bar{x}) - d_j g_j(\bar{x}), \quad j = 1, 2, \ldots, m$$

with strict inequality for at least one j, which implies $(\bar{u}, d, \bar{q}) \notin M'$ contradicts $(**)$. Hence $\bar{x}$ is an efficient solution of (DEP_d).

Theorem 3. (Converse duality theorem) *For given d, let $u^* \in N$ be an efficient solution of* (DEP$_\mathrm{d}$), *then there exists $q^* \in \mathbb{R}^m$ with $q_j^* = f_j(u^*) - d_j g_j(u^*)$ such that (u^*, d, q^*) is a feasible point of* (EP$_\mathrm{d}$) *and q^* is nondomainated.*

Proof. Let $u^* \in N$, then for any $(x, d, q) \in M'$, we have

$$h_r(x) \geq h_r(u^*) + \nu_r(x - u^*) \qquad \forall\, \nu_r \in \partial h_r(u^*),\ r = 1, 2, \ldots, \ell$$

and hence

$$h_r(u^*) \leq 0, \qquad r = 1, 2, \ldots, \ell.$$

Since $q_j^* = f_j(u^*) - d_j g_j(u^*)$, then (u^*, d, q^*) is a feasible point for (EP$_\mathrm{d}$), and q^* is nondominated since if it does not nondominated then there is $(\tilde{x}, d, \tilde{q}) \in M'$ such that $\tilde{q}_j \leq q_j^* = f_j(u^*) - d_j g_j(u^*)$ which contradicts Theorem 1.

4. Duality using the dual space concept.

In order to establish the duality for problem (P$_0$) with another approach, (the dual space of the objective space), we shall make use of problem (EP$_\mathrm{d}$) and seek the following notations and definitions:

Let X be a vector space over $\mathbb{R}$, and Y and Z are topological vector spaces over $\mathbb{R}$ partially ordered by the convex cones D_Y and D_Z, respectively. Let M be a nonempty convex subset of X, and let $(f - dg) : M \to Y$ and $h : M \to Z$, where $(f - dg)$ and h are convex m and ℓ, vector valued functions, respectively, as considered in problem (P$_0$).

Definition 3. ([7]) *Let S be a nonempty subset of Y. An element $\bar{y} \in Y$ is called a minimal element of S, if $\big(\{\bar{y}\} - D_Y\big) \cap S = \{\bar{y}\}$, and $\bar{y}$ is called a maximal element of S, if $\big(\{\bar{y}\} + D_Y\big) \cap S = \{\bar{y}\}$.*

Definition 4. *A space of all linear maps t from Y into Y is called the dual space of Y and denoted by Y^*.*

Definition 5. ([7]) *The ordering cone of the topological dual space Y^* of Y is given by*

$$D_{Y^*} = \big\{t \in Y^* : t(y) \geq 0 \quad \forall\, y \in D_Y\big\}$$

and the quasi-interior of D_{Y^*} given by

$$\bar{D}_{Y^*} = \big\{t \in Y^* : t(y) > 0 \quad \forall\, y \in D_Y \setminus \{0\}\big\}.$$

Definition 6. ([7]) *Let S be a nonempty subset of Y. An element $\bar{y} \in Y$ is callled a properly minimal element of S, if $\bar{y} \in S$ and there exists a $\bar{t} \in \bar{D}_{Y^*}$ with*

$$\bar{t}(\bar{y}) \leq \bar{t}(y) \qquad \forall y \in S.$$

Note that, every properly minimal element of a set is also a minimal element of this set, but the converse is not true in general. A properly maximal element of a set can be defined in a similar way.

Now, we reformulate the problem ($\mathrm{EP_d}$) as follows:

$$V = \{q \in Y : -h(x) \in D_z, \ -(f(x) - dg(x) - q) \in D_Y\}.$$

This is the primal problem. The dual problem is formulated similar to the dual problem in [7] as follows:

$$W = \{f(u) - dg(u) \in Y, \ u \in \mathbb{R}^n : \exists t^* \in \bar{D}_{Y^*}, \ v^* \in D_{Z^*} \quad \text{with} \\ t^*(q) + v^*h(x) \geq t^*(f(u) - dg(u)) \qquad \forall x \in M\}$$

The proof of the following theorem and two lemmas are similar to the proof of Theorem 2.1 and Lemma 2.2 in [7].

Theorem 4. (Weak duality theorem) *For given $d \in \mathbb{R}^m_+$, let V be a nonempty and $f(\bar{u}) - dg(\bar{u}) \in W$, $\bar{u} \in \mathbb{R}^n$, then there exists a $\bar{t} \in \bar{D}_{Y^*}$ with*

$$\bar{t}(f(\bar{u}) - dg(\bar{u})) \leq \bar{t}(q) \quad \forall q \in V.$$

Lemma 3. *For given $d \in \mathbb{R}^+$, let $q \in V$ and $f(u) - dg(u) \in W$, then*

$$f(u) - dg(u) - q \notin D_Y \setminus \{0\}.$$

Lemma 4. *If $f(\bar{u}) - dg(\bar{u}) \in W$, for given $d \in \mathbb{R}^m_+$, $\bar{u} \in \mathbb{R}^n$ and $\bar{q} \in V$ such that $f(\bar{u}) - dg(\bar{u}) = \bar{q}$, then $f(\bar{u}) - dg(\bar{u})$ is a maximal element of W and $\bar{q}$ is a minimal element of V.*

The proof of the following theorem (strong duality theorem) does not depend on the stability assumption imposed by J. Jahn in [7].

Theorem 5. (Strong duality theorem) *If $\bar{q} \in V$ is a proper minimal element of V with $t^* \in \bar{D}_{Y^*}$ such that $t^*(\bar{q}) \leq t^*(q)$ for each $q \in V$, then $\bar{q}$ is a maximal element of W.*

Proof. Let $\bar{q} \in V$ be a proper minimal element of V with $t^* \in \bar{D}_{Y^*}$ such that

$$t^*(\bar{q}) \leq t^*(q) \quad \forall q \in V.$$

Consequently, there is some $\bar{x} \in M$ with $\bar{q} = f(\bar{x}) - dg(\bar{x})$, and

$$t^*(f(\bar{x}) - dg(\bar{x})) \leq t^*(q) \quad \forall q \in V.$$

Without loss of generality, we can write

$$t^*(f(\bar{x}) - dg(\bar{x})) \leq t^*(f(x) - dg(x)) \quad \forall x \in M$$

or the system

$$\begin{cases} t^*(f(x) - dg(x)) - t^*(f(\bar{x}) - dg(\bar{x})) < 0 \\ v^*h(x) \leq 0 \end{cases}$$

has no solution $x \in M$. Therefore, from generalized Gourdan theorem [9], there exists $(p, k) \geq 0$ such that

$$p[t^*(f(x) - dg(x)) - t^*(f(\bar{x}) - dg(\bar{x})] + kv^*h(x) \geq 0$$

or

$$pt^*(f(\bar{x}) - dg(\bar{x})) \leq pt^*(f(x) - dg(x)) + kv^*h(x) \quad \forall x \in M.$$

But pt^* and kv^* are, again, linear maps from Y into Y and from Z into Z, respectively. Hence $f(\bar{x}) - dg(\bar{x}) \in W$ and from Lemma 4 it is a maximal element of W.

Theorem 6. (Converse duality theorem) *If $V \cap W \neq \phi$, then every proper maximal element $f(\bar{u}) - dg(\bar{u}) \in W$, for given $d \in \mathbb{R}^m_+$, is a properly minimal element of V.*

Proof. Since $f(\bar{u}) - dg(\bar{u})$ is a proper maximal element of W, then there is $t^* \in \bar{D}_{Y^*}$ such that

$$t^*(f(\bar{u}) - dg(\bar{u})) \geq t^*(f(u) - dg(u)) \, \forall f(u) - dg(u) \in W, \quad u \in \mathbb{R}^n.$$

Let $f(\bar{u}) - dg(\bar{u})$ does not belong to V, then

$$-[(f(x) - dg(x)) - (f(\bar{u}) - dg(\bar{u}))] \notin D_Y \quad \forall x \in M$$

or

$$t^*(f(\bar{u}) - dg(\bar{u})) < t^*(f(x) - dg(x)) \quad \forall t^* \in \bar{D}_{Y^*}, \; x \in M,$$

which leads to a contradiction since $V \cap W \neq \phi$. Therefore $f(\bar{u}) - dg(\bar{u}) \in V$. If $f(\bar{u}) - dg(\bar{u})$ does not proper minimal element of V, then there is $\bar{q} \in V$ such that

$$t^*(\bar{q}) < t^*(f(\bar{u}) - dg(\bar{u}) \quad \forall t^* \in \bar{D}_{Y^*}$$

which contradicts the weak duality theorem. Therefore $f(\bar{u}) - dg(\bar{u})$ is a proper minimal element of V.

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