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L_1 OF A VECTOR MEASURE

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Let (Ω, Σ) be a measurable space, X a real Banach space and $\nu : \Omega \to X$ a countably additive vector measure.

We define a Σ-measurable function $f : \Omega \to R$ to be *weakly ν-integrable* if it is $x^*\nu$-integrable for each $x^* \in X^*$. We show that the space w-$L_1(\nu)$, the space of all weakly ν-integrable functions, is a Banach space containing $L_1(\nu)$ as a closed linear subspace (the space $L_1(\nu)$ was defined by I. Kluvanek and G. Knowles [5], for measures taking values in a locally convex topological vector spaces X, and studied in details by G.P. Curbera [2] and [3], for Banach space valued measures).

We give necessary and sufficient conditions for $L_1(\nu)$ to equal w-$L_1(\nu)$. Also we show that in certain cases, ν-integrability (resp. weak ν-integrability) can be viewed in terms of integrability in the sense of Pettis (resp. Dunford). Finally, we show that when ν is of bounded variation, we can approximate ν by measures μ, (in variation norm), where $L_1(\nu)$ is order isomorphic to an abstract L-space.

1. Introduction.

Assume (Ω, Σ) is a measurable space, (X, τ) a locally convex linear topological vector space and $\nu : \Sigma \to X$ a vector measure. In this setting, Lewis [6] defines a real-valued Σ-measurable function f to be *ν-integrable* if

(1) f is $x^*\nu$-integrable for each $x^* \in X^*$, and

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(2) for every $E \in \Sigma$ there exists an element of X denoted by $\int_E f\,d\nu$ such that $x^* \int_E f\,d\nu = \int_E f\,d(x^*\nu)$ holds for each $x^* \in X^*$.

Lewis shows that whenever (X, τ) is sequentially complete then f is ν-integrable *if and only if*

(1′) there is a sequence (f_n) of simple Σ-measurable functions which conveges pointwise to f, and

(2′) $(\int_E f_n\,d\nu)$ is Cauchy for each $E \in \Sigma$.

In particular, Lewis's integral coincides with that of Bartle, Dunford and Schwartz in their setting, i.e. for Banach-valued measures [1].

Adopting Lewis's integral, Kluvaneck and Knowles [5] define the analogue of the Lebesgue space of integrable functions.

A ν-integrable function f is said to be ν-null if its indefinite integral is (identically) the zero vector measure, and two ν-integrable functions f and g are said to be ν-*equivalent* or to be equal ν-almost everywhere (ν-a.e.) if the indefinite integral of $|f - g|$ is ν-null. A set $E \in \Sigma$ is said to be ν-null if its characteristic function is ν-null.

Every τ-continuous semi-norm p on X defines a semi-norm on the space $L(\nu)$ of all ν-integrable functions via the application

$$f \mapsto p(\nu)(f) \stackrel{\text{def}}{=} \sup\left\{\int_\Omega |f|\,d|x^*\nu| : x^* \in U_P^\circ\right\}.$$

Where U_P° is the polar of the set $U_P = \{x \in X : p(x) \leq 1\}$. The above semi-norms turn $L(\nu)$ into a locally convex linear lattice. The quotient space of $L(\nu)$ modulo the subspace of all ν-null functions is denoted by $L_1(\nu)$.

For (X, τ) is sequentially complete, Kluvanek and Knowles show that ν-essentially bounded functions are ν-integrable, $L_\infty(\nu) \subset L_1(\nu)$, and that convergence theorems of the type of Beppo Levi and Lebesgue hold.

G. Curbera [2] shows that when ν is Banach-valued, the space $L_1(\nu)$, defined by Kluvanek and Knowles, is an order continuous Banach lattice with weak unit. In [3] he studies a priori conditions on the vector measure in order to guarantee that the resulting L_1 is an abstract L-space.

The purpose of this note is to show how, in case of Banach-valued measures, Lewis's integral can be presented in terms of operators. Introducing integrable functions this way suggests a natural extension of the space $L_1(\nu)$ to a Banach space we have chosen to call w-$L_1(\nu)$. The element of w-$L_1(\nu)$ appear briefly in [6], are said to have *generalized integral*. We show that for certain measures ν the space $L_1(\nu)$ (resp. w-$L_1(\nu)$) is isomorphic to a subspace of Pettis (resp. Dunford) integrable functions.

2. Notation and terminology.

Throughout this paper X denotes a Banach space and X^* its dual. The unit ball of X (resp. X^*) is denoted by B_X (resp. B_{X^*}) and the natural image of X in $(X^*)^* = X^{**}$ is denoted by $\widehat{X}$.

The variation of a real-valued and countably additive measure λ is denoted by the symbol $|\lambda|$. If ν is an X-valued vector measure and

$$\lim_{|\lambda|(E)\to 0} \nu(E) = 0$$

we say that ν is *λ-continuous* and write $\nu \ll \lambda$. In that case, λ is called a *control measure* for ν. By a theorem of Rybakov [4], Theorem IX.2.2, there exists x^* in X^* such that $\nu \ll |x^*\nu|$. We then call $|x^*\nu|$ a *Rybakov control measure* for ν.

Let $(\Omega, \Sigma, \lambda)$ be a finite measure space. A function $g : \Omega \longrightarrow X$ is called *weakly λ-measurable* if for each x^* in X^* the real-valued function x^*g is λ-measurable. g is said to be *strongly λ-measurable* if it is weakly λ-measurable and λ-essentially separably valued; that is, if there exists a set $E \in \Sigma$ with $\lambda(E) = 0$ and such that $g(\Omega - E)$ is a (norm) separable subset of X.

g is said to be *determined* by a subspace D of X (with respect to a probability measure λ) if for every x^* in X^*,

$$x^*|_D = 0 \quad \text{implies} \quad x^*g = 0; \qquad \lambda\text{-a.e.}.$$

A weakly λ-measurable function g is called *Dunford integrable* (with respect to λ) if x^*g in $L_1(\lambda)$ for every x^* in X^*. In that case, the operator $X^* \to L_1(\lambda)$, $x^* \mapsto x^*g$ is bounded and thus, for each E in Σ the mapping

$$x^* \mapsto \int_E x^*g \, d\lambda,$$

defines an element of X^{**} and is called the *Dunford integral* of g over E. We denote the Dunford integral of g over E by $D\text{-}\int_E g \, d\lambda$. The function g is said to be *Pettis integrable* if $D\text{-}\int_E g d\lambda$ is in $\widehat{X}$ for all E in Σ.

If g is strongly λ-measurable there exists a sequence (ϕ_n) of simple functions such that $\|g(\omega) - \phi_n(\omega)\|$ tends to zero a.e.-λ. If the sequence $(\|g - \phi_n\|)$ converges to zero in $L_1(\lambda)$ the function g is called *Bochner integrable*. In that case, the sequence $(\int_E \phi_n \, d\lambda)$ is Cauchy in X for all E in Σ and its limit is the Bochner integral of g over, E, $(B)\text{-}(\int_E g d\lambda)$.

The symbol $L_1(\lambda, X)$ denotes the vector space of all (equivalence classes of) Bochner integrable functions. When equipped with the norm

$$\|g\|_1 = \int_\Omega \|g\| \, d\lambda$$

then $L_1(\lambda, X)$ becomes a Banach space.

The symbol $P_1(\lambda, X)$ denotes the vector space of all (weak equivalence classes of) Pettis integrable functions $g : \Omega \longrightarrow X$. For such function g define

$$\|g\|_P = \sup_{x^* \in B_{X^*}} \int_\Omega |x^* g| \, d\lambda .$$

Then $(P_1(\lambda, X), \|\cdot\|_P)$ is a normed linear space; not necessarily a Banach space.

If g is Pettis integrable its indefinite integral, $\psi_g : E \mapsto$ (Pettis)-$\int_E g \, d\lambda$ is a countably additive vector measure and the function g is called a Pettis density for ψ_g with respect to λ. In general, the indefinite integral of a Dunford integrable function h is countable additive *iff* the set $\{x^* h : x^* \in B_{X^*}\}$ is a relatively weakly compact subset of $L_1(\lambda)$, and Pettis integrable functions are known to have this property.

3. Integration.

Let $(\Omega, \Sigma, \lambda)$ be a complete probability space and let ν be a λ-continuous vector measure taking values in X.

Lemma 1. *The mapping* $S : X^* \to L_1(\lambda)$, $x^* \mapsto S(x^*) = \dfrac{d(x^*\nu)}{d\lambda}$ *is bounded. Moreover, it is weak*-to-weak continuous.*

Proof. For any $g \in L_\infty(\lambda)$ and $x^* \in X^*$,

$$\begin{aligned} \left| \int_\Omega g S(x^*) \, d\lambda \right| &\le \|g\|_\infty \cdot \int_\Omega |S(x^*)| \, d\lambda \\ &= \|g\|_\infty \cdot |x^* \nu|(\Omega) \\ &= \|g\|_\infty \cdot \left| \frac{x^*}{\|x^*\|} \nu \right| (\Omega) \cdot \|x^*\| \\ &\le \|g\|_\infty \cdot \|\nu\|(\Omega) \cdot \|x^*\|. \end{aligned}$$

Hence, $\|S\|$ is bounded by $\|\nu\|(\Omega)$.

To prove weak*-to-weak continuity, assume (x_α^*) is a net in B_{X^*} that converges weak* to zero. Then $(x_\alpha^* \nu)$ converges setwise to zero; that is, $x_\alpha^* \nu(E) \to 0$ for each $E \in \Sigma$. Then $S(x_\alpha^*)$ is a net in $L_1(\lambda)$ bounded by $\|\nu\|(\Omega)$ and

$$\int_E \phi S(x_\alpha^*) \, d\lambda \to 0$$

for all $E \in \Sigma$ and all simple functions ϕ.

Fix $h \in L_\infty(\lambda)$ and let $\varepsilon > 0$. Choose a simple function ϕ such that $\|h - \phi\| < \varepsilon$. Then find α_0 such that $|\int_E \phi S(x_\alpha^*)\, d\lambda| \leq \varepsilon$ for all $\alpha \geq \alpha_0$. Then, for $\alpha \geq \alpha_0$,

$$\begin{aligned}\left|\int hS(x_\alpha^*)\, d\lambda\right| &\leq \left|\int (h-\phi)S(x_\alpha^*)\, d\lambda\right| + \left|\int \phi S(x_\alpha^*)\, d\lambda\right| \\ &\leq \varepsilon \cdot \|\nu\|(\Omega) + \varepsilon \\ &= \varepsilon \cdot (\|\nu\|(\Omega) + 1). \qquad \square\end{aligned}$$

Proposition 2. *If $f : \Omega \to R$ is λ-measurable and $f \in L_1(x^*\nu)$ for all $x^* \in X^*$, then the operator*

$$T_f : X^* \to L_1(\lambda),\ x^* \mapsto f\,\frac{d(x^*\nu)}{d\lambda}$$

is bounded.

Proof. Indeed, if $x_n^* \to x^*$ and $T_f x_n^* \to h_{x^*}$, then for some subsequence $(x_{n_j}^*)$ of (x_n^*),

$$(*) \qquad \frac{d(x_{n_j}^*\nu)}{d\lambda} \to \frac{d(x^*\nu)}{d\lambda} \qquad \lambda\text{-a.e.}$$

by Lemma 1, and

$$(**) \qquad f\,\frac{d(x_{n_j}^*\nu)}{d\lambda} = T_f x_{n_j}^* \to h_{x^*} \qquad \lambda\text{-a.e.}.$$

But $(*)$ certainly implies that

$$f\,\frac{d(x_{n_j}^*\nu)}{d\lambda} \to f\,\frac{d(x^*\nu)}{d\lambda} = T_f x^* \qquad \lambda\text{-a.e.}$$

which, in view of $(**)$, shows that $T_f x^* = h_{x^*}$. An appeal to Banach's closed graph theorem shows that T_f is continuous. $\square$

Corollary 3. (compare [4], Lemma II.3.1) *If f is as in Proposition 2 then for each $E \in \Sigma$ there exists an element $x_E^{**} \in X^{**}$ such that*

$$x_E^{**}(x^*) = \int_E f\, d(x^*\nu)$$

for all $x^* \in X^*$.

Proof. Let T_f be as in Proposition 2. If $I_E : L_1(\lambda) \to R$ denotes integration over $E \in \Sigma$, $I_E(h) = \int_E h\,d\lambda$, then $I_E \circ T_f : X^* \to R$ is an element of X^{**},

$$I_E \circ T_f(x^*) = \int_E f \frac{d(x^*\nu)}{d\lambda}\,d\lambda = \int_E f\,d(x^*\nu).$$

Denote $I_E \circ T_f$ by x_E^{**}. □

In view of Corollary 3 we extend Lewis's definition of integrability as follows (compare [4], Definition II. 3.2):

A Σ-measurable function $f : \Omega \to R$ is said to be *weakly ν-integrable* if f is $x^*\nu$-integrable for all $x^* \in X^*$. In that case, the *weak ν-integral* of f over a set $E \in \Sigma$ is an element $s_E^{**} \in X^{**}$ such that

$$x_E^{**}(x^*) = \int_E f\,d(x^*\nu)$$

for all $x^* \in X^*$ and we write $w\text{-}\int_E f d\nu$ to denote the element x_E^{**}. In the case $w\text{-}\int_E f d\nu$ is in $\widehat{X} \subset X^{**}$ for all $E \in \Sigma$, then f is called *ν-integrable* and we write $\int_E f\,d\nu$ instead of $w\text{-}\int_E f\,d\nu$ to denote the υ-integral of f over $E \in \Sigma$.

The following theorem characterizes ν-integrability in terms of the operator T_f of Proposition 2.

Theorem 4. *Assume ν and λ are as before, and f and T_f as in Proposition 2. The following statements are equivalent:*

(a) *f is ν-integrable.*
(b) *T_f is weak*-to-weak continuous.*

Proof. (a) ⇒ (b) Assume f is ν-integrable and fix $E \in \Sigma$. For any $x^* \in X^*$,

$$T_f^*(\chi_E)(x^*) = \int_E T_f x^*\,d\lambda = \int_E f \frac{d(x^*\nu)}{d\lambda}\,d\lambda = \int_E f\,d(x^*\nu),$$

i.e. $T_f^*(\chi_E) = \int_E f d\lambda \in \widehat{X}$ for all $E \in \Sigma$. Hence $T_f^*(\phi) \in \widehat{X}$ for all simple functions ϕ. Since the simple functions are dense in $L_\infty(\lambda)$, $T_f^*(L_\infty(\lambda)) \subset \widehat{X}$. Consequentely T_f is weak*-to -weak continuous.

(b) ⇒ (a) If T is weak*-to-weak continuous then $T^*(\chi_E)$ is in $\widehat{X}$, but $T^*(\chi_E) = w\text{-}\int_E f d\nu$. Hence $w\text{-}\int_E f\,d\nu \in \widehat{X}$ and f therefore ν-integrable. □

Remark. Characterizing ν-integrability in terms of the operator T_f as above provides us with a very simple proof of the following known result.

Proposition 5. *Let ν and λ be as above*

(i) *If $f \in L_\infty(\lambda)$ then f is ν-integrable.*

(ii) *If f is ν-integrable, g is λ-measurable and $|g| \leq |f|$ almost everywhere, then g is ν-integrable.*

Proof. (i) If $f \in L_\infty(\lambda)$ then f corresponds to a bounded (and hence, weakly continuous) linear functional on $L_1(\lambda)$. The mapping

$$\int f \frac{d(x^*\nu)}{d\lambda}\, d\lambda$$

is a composition,

$$\int \frac{d(x^*\nu)}{d\lambda}\, d\lambda \mapsto \int f \frac{d(x^*\nu)}{d\lambda}\, d\lambda\,,$$

a weak*-to weak continuous mapping followed by a weakly continuous mapping.

(ii) There exists a set E_0 of measure zero such that $|f(w)| \leq |g(w)|$ for all $w \in X - E_0$. Define a function h as follows: $h(w) = g(w)/f(w)$ if $w \notin E_0$ and $f(w) \neq 0$ and define h to be zero otherwise. Then $h \in L_\infty(\lambda)$ and as f in (i), h defines a bounded (and hence weakly continuous) linear functional on $L_1(\lambda)$. The mapping

$$\int g \frac{d(x^*\nu)}{d\lambda}\, d\lambda$$

is a composition

$$\int f \frac{d(x^*\nu)}{d\lambda}\, d\lambda \mapsto \int hf \frac{d(x^*\nu)}{d\lambda}\, d\lambda\,,$$

a weak*-to weak continuous mapping (by ν-integrability of f) followed by a weakly continuous mapping. □

Corollary 6. *T_f is weak*-to-weak continuous if and only if T_f is weakly compact.*

Proof. Necessity is clear. We prove sufficiency.

Since $\{f\, d(x^*\nu)/d\lambda : x^* \in B_{X^*}\}$ is a relatively weakly compact subset of $L_1(\lambda)$, it is uniformly integrable with respect to λ; that is,

$$\lim_{\lambda(E)\to 0} \sup_{x^* \in B_{X^*}} \int_E \left| f \frac{d(x^*\nu)}{d\lambda} \right| d\lambda = 0\,.$$

Uniform integrability of $\{f\, d(x^*\nu)/d\mu : x^* \in B_{X^*}\}$, in turn, implies that the indefinite integral of f, ν_f is countably additive.

Let E be any set in Σ. We want to show that $\nu_f(E) \in X$. For integers $n = 1, 2, 3 \ldots$ let $F_n = \{w \in \Omega : n - 1 \leq |f(w)| < n\}$. Then (F_n) is a pairwise disjoint sequence in Σ and $\Omega = \cap F_n$. By countable additivity, $\nu_f(E) = \Sigma \nu_f(E \cap F_n)$. But $\{\nu_f(A \cap F_n) : A \in \Sigma\} \subset \widehat{X}$ for all n. Hence $\nu_f(E) \in \widehat{X}$. □

We now proceed to illustrate a relation between the above integral and integrals of vector valued functions, the Pettis, the Dunford and the Bochner integral. We will need the following characterization of Pettis integrable functions.

Proposition 7. ([9]) *Assume g is Dunford integrable with respect to a probability measure λ. The following statements are equivalent:*

(a) *g is Pettis integrable.*
(b) *g is determined by a subspace D of X and the operator $X^* \to L_1(\lambda)$, $x^* \mapsto x^*g$ is $\sigma(X^*, D)$-to-weak continuous.*
(c) *g is determined by a subspace D of X which is weakly compactly generated and the set $\{x^*g : x^* \in B_{X^*}\}$ is relatively weakly compact.*

Proposition 8. *Assume ν has Pettis density g with respect to a probability measure λ. Then for any real-valued λ-measurable function f,*

(a) *f is weakly ν-integrable if and only if $f \cdot g$ is Dunford integrable,*
(b) *f is ν-integrable if and only if $\{x^*(f \cdot g) : x^* \in B_{X^*}\} \subset L_1(\mu)$ is relatively weakly compact if and only if $f \cdot g$ is Pettis integrable.*

Proof. If ν has Pettis density g with respect to λ then $\nu \ll \lambda$ and for any x^* in X^*,

$$\frac{d(x^*\nu)}{d\lambda} = x^*g,$$

and hence,

$$f\frac{d(x^*\nu)}{d\lambda} = f \cdot (x^*g) = x^*(f \cdot g)$$

for any λ-measurable function f. It follows that f is weakly ν-integrable if and only if $f \cdot g$ is Dunford integrable, proving (a).

Since g is Pettis integrable it is determined by a weakly compactly generated subspace D of X. Clearly, every multiple $f \cdot g$ is determined by the same space D and is therefore Pettis integrable if and only if $\{x^*(f \cdot g) = f\, d(x^*\nu)/d\lambda : x^* \in B_{X^*}\}$ is relatively weakly compact (by Proposition 7) if and only if f is ν-integrable (by Lemma 6). □

4. The space $L_1(\nu)$.

When the linear topological vector space X is a Banach space, the topology on $L_1(\nu)$, as defined by Kluvanek and Knowles, becomes a norm topology; it is generated by the single (semi-)norm $\|\cdot\|_\nu$, where

$$\|f\|_\nu = \sup\left\{\int |f|\, d|x^*\nu| : x^* \in B_{X^*}\right\}$$

Note that $\|f\|_\nu = \|T_f\|$ where T_f is as in Proposition 2. Extend this norm to include the weakly ν-integrable functions by defining

$$\|f\|_\nu = \|T_f\|\,.$$

If we define two such functions f and g to be *weakly* ν-equivalent if the indefinite integral of $|f-g|$ is the zero vector measure we get a linear space of equivalence classes that we will be denote by w-$L_1(\nu)$.

Theorem 9. *$(w\text{-}L_1(\nu), \|\cdot\|_\nu)$ is a Banach space containing $L_1(\nu)$ as a closed linear subspace.*

Proof. Let (f_n) be a $\|\cdot\|_\nu$-Cauchy sequence in w-$L_1(\nu)$. Then (f_n) is a Cauchy sequence in each of the spaces $L_1(|x^*\nu|)$, $x^* \in X^*$. Let $\lambda = |x_0^*\nu|$ be a Rybakov control measure for $\|\nu\|$ and let

$$f_0 = \lim_n f_n \quad \text{in} \quad L_1(\lambda)\,.$$

Find a subsequence (f_{nj}) a set E_0 with $\lambda(E_0) = 0$ such that

$$f_{n_j}(w) \to f(w) \quad \text{for all} \quad w \notin E_0\,.$$

Fix any $x^* \in X^*$. If

$$f_{x^*} = \lim_n f_n \quad \text{in} \quad L_1(|x^*\nu|),$$

then

$$f_{x^*} = \lim_i f_{n_i}$$

and we can find a subsequence $(f_{n_{i_j}})$ of f_{n_i} and a set E_{x^*} with $|x^*\nu|(E_{x^*}) = 0$ such that

$$f_{n_{i_j}}(w) \to f_{x^*}(w) \quad \text{for all} \quad w \notin E_{x^*}.$$

The set $E_0 \cup E_{x^*}$ is of $|x^*\nu|$-measure zero, and off this set the following statements hold

$$f_{n_{i_j}}(w) \to f_{x^*}(w) \quad \text{and} \quad f_{n_{i_j}}(w) \to f_0(w)\,.$$

Thus

$$\lim_n f_n = f_{x^*} = f_0 \quad \text{in} \quad L_1(|x^*\nu|)\,.$$

Since x^* was arbitrary, it follows that $f_0 \in L_1(|x^*\nu|)$ for all $x^* \in X^*$ and hence, $f_0 \in w - L_1(\nu)$. Evidently,

$$\lim_n \|f_0 - f_n\| = 0.$$

To show that $L_1(\nu)$ is a closed subspace of w-$L_1(\nu)$, assume each f_n is an element of $L_1(\nu)$. Let ν_n be the indefinite integral of f_n and ν_0 the indefinite integral of f_0. Then (ν_n) is a sequence of X-valued measures and since

$$\|\nu_n(E) - \nu_0(E)\| \le \|f_n - f_0\|_\nu \to 0$$

holds for all $E \in \Sigma$, it follows that ν_0 is X-valued and hence, f_0 is ν-integrable. □

In [2] it is shown that $L_1(\nu)$ is an order continuous Banach lattice, and weakly sequentially complete whenever the Banach space in which ν takes its range has no copy of c_0. The space w-$L_1(\nu)$ is a σ-complete Banach lattice but in general, not order continuous. In fact, order continuity of w-$L_1(\nu)$ coincides with weak sequential completeness of w-$L_1(\nu)$ as shown in the following theorem which generalizes [2], Theorem 3.

Theorem 10. *The following statements are equivalent:*
(a) w-$L_1(\nu)$ *is order continuous.*
(b) w-$L_1(\nu) = L_1(\nu)$.
(c) $L_1(\nu)$ *is weakly sequentially complete.*
(d) w-$L_1(\nu)$ *is weakly sequentially complete.*

Proof. (a) $\Rightarrow$ (b). Assume w-$L_1(\nu)$ is order continuous and let $f \in w$-$L_1(\nu)$. We can assume $f \ge 0$. Find an increasing sequence (f_n) of simple functions such that

$$0 \le f_n \le f_{n+1} \le \cdots \le f$$

and

$$f_n \to f \qquad \text{a.e.}.$$

Then (f_n) is order bounded and by order continuity, converges in norm. Evidently the limit is f. But the f_n's are simple and therefore, belong to $L_1(\nu)$ which is closed. Hence, $f = \lim_n f_n \in L_1(\nu)$.

(b) $\Rightarrow$ (c). This is basically Curbera's argument. We prove that a norm bounded increasing sequence in $L_1(\nu)$ converges in norm since in Banach lattices it is equivalent to weak sequential completeness ([5], Theorem 1.c.4). To that end, let (f_n) be norm bounded and increasing. We can assume the f_n's are all nonnegative. For any $x^* \in X^*$, (f_n) is a norm-bounded, nonnegative and increasing sequence in $L_1(|x^*\nu|)$ and therefore converges (in $L_1(|x^*\nu|)$) to some $f_{x^*} \in L_1(|x^*\nu|)$. If $\lambda = |x_0^*\nu|$ is a Rybakov control measure for ν, let f_0 be the limit of (f_n) in $L_1(\lambda)$. As in the proof of Theorem 9, $f_0 \in L_1(|x^*\nu|)$ for all $x^* \in X^*$ and $f_0 = f_{x^*}$ $|x^*\nu|$-a.e.. Hence the sequence (f_n) converges in each of the spaces $L_1(|x^*\nu|)$ to f_0. Then $f_0 \in w - L_1(\nu)$. But $w - L_1(\nu) = L_1(\nu)$, so $f_0 \in L_1(\nu)$ which is order continuous. Being order bounded and increasing, the sequence (f_n) converges in norm to f_0.

(c) $\Rightarrow$ (b). Assume $L_1(\nu)$ is weakly sequentially complete. Then every norm bounded increasing sequence converges in norm. Let $f \in w\text{-}L_1(\nu)$. We can assume f is nonnegative. Find a sequence (f_n) of simple functions such that

$$0 \leq f_n \leq f_{n+1} \leq \cdots \leq f$$

and

$$f_n \to f \qquad \text{a.e..}$$

Then (f_n) is norm bounded ($\|f_n\| \leq \|f\|$ for all n) and increasing in $L_1(\nu)$, and by weak sequential completeness of $L_1(\nu)$, the sequence converges in norm. The limit must be f, which implies that f is integrable. Since f was an arbitrary element of $L_1(\nu)$, it follows that $w\text{-}L_1(\nu) - L_1(\nu)$.

(b) $\Rightarrow$ (a). This is clear.

(a) $\Rightarrow$ (d). If $w\text{-}L_1(\nu)$ is order continuous then $w\text{-}L_1(\nu) = L_1(\nu)$ and hence, $w\text{-}L_1(\nu)$ is weakly sequentially complete.

(d) $\Rightarrow$ (c). Since $L_1(\nu)$ is closed. $\square$

In [3] Curbera proves that $L_1(\nu)$ is (order isomorphic to) an abstract L-space *if and only if* every element f in $L_1(\nu)$ belongs to $L_1(|\nu|)$ in which case the two spaces are order isomorphic. In [7], Theorem 4.2, Lewis characterizes those elements of $L_1(\nu)$ that belong to $L_1(|\nu|)$ as those whose indefinite integrals are of bounded variation; that is, an element f in $L_1(\nu)$ belongs to $L_1(|\nu|)$ if and only if the measure

$$\nu_f : \Sigma \to X, \ E \mapsto \int_E f \, d\nu$$

is of bounded variation.

Lemma 11. *Let ν be a vector measure and λ a probability measure such that $\nu \ll \lambda$. The following two statements are equivalent:*

(a) *ν is of bounded variation.*

(b) *The set $\{d(x^*\nu)/d\lambda : x^* \in B_{X^*}\}$ is an order bounded subset of $L_1(\lambda)$.*

In that case,

$$|\nu|(E) = \int_E h\, d\lambda,$$

where $h = \text{lub}\,\{|d(x^*\nu)/d\lambda| : x^* \in B_{X^*}\}$

Proof. (b) ⇒ (a). Assuming the set $\{d(x^*\nu)/d\lambda : x^* \in B_{X^*}\}$ is an order bounded subset of $L_1(\lambda)$, let $h = \text{lub}\,\{|d(x^*\nu)/d\lambda| : x^* \in B_{X^*}\}$. If E any element of Σ, then for any $x^* \in B_{X^*}$,

$$|x^*\nu(E)| \le |x^*\nu|(E) = \int_E \left|\frac{d(x^*\nu)}{d\lambda}\right| d\lambda \le \int_E h\, d\lambda,$$

and hence,

$$\|\nu(E)\| = \sup_{x^* \in B_{X^*}} |x^*\nu(E)| \le \int_E h\, d\lambda.$$

Let π be any finite partition of Ω into measurable sets. Then

$$\sum_{A\in\pi} \|\nu(A)\| \le \sum_{A\in\pi} \int_A h\, d\lambda = \int_\Omega h\, d\lambda,$$

and consequently

$$|\nu|(\Omega) = \sup_\pi \sum_{A\in\pi} \|\nu(A)\| \le \int_\Omega h\, d\lambda.$$

(a) ⇒ (b). View the measure ν as a measure into X^{**}. Since the measure is of finite variation, a direct consequence of a representation theorem of A. Ionescu Tulcea and C. Ionescu Tulcea [10] provides us with an X^{**}-valued function f such that

(i) $f(\cdot)x^*$ belongs to $L_1(\lambda)$ for all x^* in X^*.
(ii) For any E in Σ and any x^* in X^*,

$$x^*\nu(E) = \int_E f x^*\, d\lambda.$$

The function f is called a *weak*-density* for ν with respect to λ. By [9], Lemma 2.6, there exists a countable partition π of Ω into measurable sets such that for any E in π the set $\{(f(\cdot)x^*)_{\chi E} : x^* \in B_{X^*}\}$ is a bounded subset of $L_\infty(\lambda)$. Denote by κ_E the least upper bound of $\{(f(\cdot)x^*)\chi_E : x^* \in B_{X^*}\}$ and let $\kappa = \sum_\pi \kappa_E$. For any $A \in \Sigma$ and $E \in \pi$

$$|x^*\nu(E\cap A)| \le \int_{E\cap A} |f x^*|\, d\lambda \le \int_{E\cap A} \kappa\, d\lambda.$$

It follows that

$$|\nu|(E \cap A) = \int_{E\cap A} \frac{d|\nu|}{d\lambda}\, d\lambda \leq \int_{E\cap A} \kappa\, d\lambda$$

and consequently,

$$\frac{d|\nu|}{d\lambda}(\omega) \leq \kappa(\omega) \qquad \text{a.e.-}\lambda\,.$$

On the other hand we have that for x^* in B_{X^*}

$$|x^*\nu|(E) = \int_E |fx^*|\, d\lambda \leq |\nu|(E) = \int_E \frac{d|\nu|}{d\lambda}\, d\lambda\,.$$

Hence, $|f(\omega)x^*| \leq \frac{d|\nu|}{d\lambda}(\omega)$ a.e.-λ, which implies that $\kappa(\omega) \leq \frac{d|\nu|}{d\lambda}(\omega)$ a.e.-λ. Hence κ and $\frac{d|\nu|}{d\lambda}$ are equal almost everywhere. Evidently $h = \kappa$. □

Corollary 12. *Assume ν is of bounded variation, λ a finite measure and $\nu \ll \lambda$. Then a ν-integrable function f belongs to $L_1(|\nu|)$ if and only if the set $\{f\, d(x^*\nu)/d\lambda : x^* \in B_{X^*}\}$ is an order bounded subset of $L_1(\lambda)$. In that case,*

$$|\nu_f|(E) = \int_E |f|\, d|\nu|\,.$$

Proof. f belongs to $L_1(|\nu|)$ if and only if the measure ν_f is of bounded variation by [7], Theorem 4.2, if and only if $\{d(x^*\nu_f)/d\lambda : x^* \in B_{X^*}\}$ is an order bounded subset of $L_1(\lambda)$ by Lemma 11. For any x^* in X^*,

$$x^*\nu_f(E) = \int_E f \frac{d(x^*\nu)}{d\lambda}\, d\lambda\,.$$

It follows that $d(x^*\nu_f)/d\lambda = f\, d(x^*\nu)/d\lambda$ and the validity of the first claim follows.

Let h be the function

$$h = \text{lub}\left\{\left|\frac{d(x^*\nu)}{d\lambda}\right| : x^* \in B_{X^*}\right\} = \frac{d|\nu|}{d\lambda}\,.$$

Since

$$\text{lub}\left\{\left|\frac{d(x^*\nu_f)}{d\lambda}\right| : x^* \in B_{X^*}\right\} = \text{lub}\left\{\left|f\, \frac{d(x^*\nu)}{d\lambda}\right| : x^* \in B_{x^*}\right\} = |f| \cdot h$$

it follows that for any $E \in \Sigma$,

$$|\nu_f|(E) = \int_E |f| \cdot h \, d\lambda = \int_E |f| \, d|\nu| . \qquad \square$$

Proposition 13. *Assume ν has a Bochner integrable density g with respect to λ. The correspondence*

$$U : f \mapsto f \cdot g$$

is an isometry mapping $w\text{-}L_1(\nu)$ into the space of Dunford integrable functions. Furthermore

$$U(L_1(\nu)) = U(w\text{-}L_1(\nu)) \cap P(\lambda, X)$$

and

$$U(L_1(|\nu|)) = U(w\text{-}L_1(\nu)) \cap L_1(\lambda, X)$$

Proof. The equation

$$\|f\|_\nu = \sup_{x^* \in B_{X^*}} \int \left| f \frac{d(x^*\nu)}{d\lambda} \right| d\lambda = \sup_{x^* \in B_{x^*}} \int |f(x^*g)| \, d\lambda = \|f \cdot g\|_P$$

together with Theorem 4 prove the validity of the first two claims.

The equations

$$\|g\| = \frac{d|\nu|}{d\lambda} \qquad |f| \cdot \|g\| = \frac{d|\nu_f|}{d\lambda}$$

together with Corollary 12 prove the last claim. $\square$

Remark. When the vector measure ν is represented by a Bochner integrable function g as above the spaces $w\text{-}L_1(\nu)$, $L_1(\nu)$ and $L_1(|\nu|)$ correspond to multiples of $f \cdot g$ of g that are Dunford-Pettis, and Bochner-integrable. Consequently, $L_1(\nu)$ is order isomorphic to an abstract L-space if and only if $f \cdot g$ is Bochner integrable whenever it is Pettis integrable. It follows from Theorem 10 that if $L_1(\nu)$ is order isomorphic to an abstract L-space then $L_1(\nu)$ can not be a proper subspace of $w\text{-}L_1(\nu)$ and consequently, in the setting of the above proposition, $L_1(\nu)$ is order isomorphic to an abstract L-space if and only if $f \cdot g$ is Bochner integrable whenever it is Dunford integrable.

If the function g is only Pettis integrable to begin with it still holds, that f is weakly ν-intgrable (resp. ν-integrable) if and only if $f \cdot g$ is Dunford (resp. Pettis) integrable, but to give a precise description of $L_1(|\nu|)$ is not possible.

Let's assume the measure λ is a Rybakov control measure for ν; that is $\lambda = |z_0^*|$ for some $z_0^* \in B_{X^*}$. Further, assume that the measure ν is X^{**}-valued and find an X^{**}-valued weak*-density g for ν with respect to $\lambda = |z_0^*\nu|$. Since every ν-integrable function f must belong to $L_1(|x^*\nu|)$ for all $x^* \in X^*$, every element of $L_1(\nu)$ belongs to $L_1(\lambda)$. On the other hand, if the set $\{x^*g : x^* \in B_{X^*}\}$ is not only order bounded in $L_1(\lambda)$ but also bounded in $L_\infty(\lambda)$ then every function in $L_1(\lambda)$ is ν-integrable.

If an X^{**}-valued function g is such that the set $\{x^*g : x^* \in B_{X^*}\}$ is a bounded subset of $L_\infty(\lambda)$ we say that g is weak*-bounded.

Lemma 14. *If ν has a weak*-bounded weak*-density g with respect to a Rybakov control measure λ, then $L_1(\nu)$ is order isomorphic to an abstract L-space.*

Proof. Let g be a weak*-bounded weak*-density for ν with respect to a Rybakov control measure λ and let S be the operator $S : x^* \mapsto d(x^*\nu)/d\lambda$ as in Lemma 1. By statement (i) in the proof of Lemma 11, $Sx^* = gx^*$. Since S is weak*-to weak continuous, the adjoint, S^* maps $L_\infty(\lambda)$ into X. Since g is weak*-bounded S^* extends to a bounded operator defined on $L_1(\lambda)$. This means that every element of $L_1(\lambda)$ is weakly ν-integrable. By the density of $L_\infty(\lambda)$ in $L_1(\lambda)$, $S^*(L_1(\lambda))$ is in X which implies that the elements of $L_1(\lambda)$ are ν-integrable as well.

Clearly, every ν-integrable funcion f belongs to $L_1(\lambda)$ and hence, $L_1(\nu)$ is order isomorphic to $L_1(\lambda)$. $\square$

Theorem 15. *Let ν be a vector measure of bounded variation. For every $\varepsilon > 0$ there exists a vector measure μ such that for every $E \in \Sigma$*

$$\|\nu(E) - \mu(E)\| \leq |\nu - \mu|(E) \leq \varepsilon$$

and $L_1(\mu)$ is order isomorphic to an abtract L-space.

Proof. Let λ be a Rybakov control measure for ν and let g be a weak*-density for ν with respect to λ. As in the proof of Lemma 11, find a countable partition $\{A_1, A_2, \ldots, A_n, \ldots\}$ of Ω such that for each n, the function $g \cdot \chi_{A_n}$ is weak*-bounded. Let κ_n denote the least upper bound of $\{g(\cdot)x^*)\chi_{A_n} : x^* \in B_{X^*}\}$ and let $\kappa = \sum_n \kappa_n$. Then $\kappa = d|\nu|/d\lambda$.

Let $\varepsilon > 0$ be given. Find n_0 such that

$$|\nu|(\Omega - \bigcup_{n \leq n_0} A_n) < \varepsilon .$$

Let

$$A_0 = \bigcup_{n \leq n_0} A_n \quad \text{and} \quad g_0 = g \cdot \chi_{A_0} .$$

If we let μ be the measure whose weak*-density is g_0 then
(i) $\mu(E) = \nu(E \cap A_0)$ for all $E \in \Sigma$
(ii) $d|\mu|/d\lambda = \kappa \cdot \chi_{A_0}$ and consequently

$$\|\mu(E) - \nu(E)\| \leq |\mu - \nu|(E)$$

for all $E \in \Sigma$.

Since the weak*-density for μ, g_0 is weak*-bounded the result now follows from Lemma 14. □

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