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ABOUT THE NUMBER OF MAXIMAL SUBSYSTEMS IN $S(2, 4, v)$

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It's of great interest, to find systems $S(2, 4, v)$ with subsystems $S(2, 4, r = \frac{1}{3}(v-1))$- known as maximal subsystems. Up to now there exist only a few results-mainly due to Resmini and Shen.
In this paper we prove a new partial result :
Exactly for all $v \in \{121, 40\} + 108\mathbb{N}$ there exists a system $S(2, 4, v)$ with exactly two, but also a system with exactly four maximal subsystems.

1. What the reader should know.

At first we notify some well-known facts.
Let V be a set of elements, the points with $|V| = v$ and B a collection of k-subsets, the lines ($k \in \mathbb{N} \setminus \{1\}$, $k < v$), such that any t-subset ($t \in \mathbb{N}$, $t \leq k$) is in exactly one line. Then (V, B) is called a Steiner system. In short $S(t, k, v)$. In this paper we consider only the special systems $S(2, 4, v)$.

We call $SN = \{4, 13\} + 12\mathbb{N}_0$ the set of admissible numbers or the set of Steiner numbers. It is known, that $v \in SN$ is a necessary and sufficient condition for the existence of systems $S(2, 4, v)$.

We further need some counting propositions. Each $S(2, 4, v)$ contains exactly $\frac{1}{12}v(v-1) = b = |B|$ lines and each point exactly $\frac{1}{3}(v-1) = r$ lines.

Here is an example.
There exists exactly one system with $v = 13$ points, the projective plane $PG(2, 3) = S(2, 4, 13)$.

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$$
\begin{aligned}
V =& \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\} \\
B =& \{\{13, 3, 5, 9\}, \{13, 1, 6, 8\}, \{13, 2, 4, 7\}, \\
& \{12, 1, 2, 5\}, \{12, 4, 8, 9\}, \{12, 3, 6, 7\}, \\
& \{11, 1, 7, 9\}, \{11, 4, 5, 6\}, \{11, 2, 3, 8\}, \\
& \{10, 5, 7, 8\}, \{10, 2, 6, 9\}, \{10, 1, 3, 4\}, \\
& \{10, 11, 12, 13\}\}.
\end{aligned}
$$

There is a lot of unsolved problems concerning the systems $S(2, 4, v)$. We consider only one of them.

2. What we like to known-the problem.

2.1 Definition: subsystem.

A system $S(2, 4, w)$ with point set W and line set C is called a subsystem of $S(2, 4, v)$ if $W \subseteq V$ and $C \subseteq B$. Then we have $|W| = w \leq v$.

Sometimes we say that $S(2, 4, w)$ is embedded in $S(2, 4, v)$.

2.2 A theorem concerning subsystems.

If a $S(2, 4, w)$ is embedded in $S(2, 4, v)$ with $v, w \in SN$, $w < v$, then necessary we have $w \leq \frac{1}{3}(v - 1)$.

After a lot of preparatory work (Brower-Lenz (1981), Wei-Zhu (1989)) Rees-Stinson in 1989 finally succeeded to prove the converse [2], [4], [7], [8], [10], [11].

For any $v, w \in SN$ with $w \leq \frac{1}{3}(v - 1)$ there exists a $S(2, 4, v)$ with a subsystem $S(2, 4, w)$.

In this paper we are only interested in subsystems with maximal cardinality $w = r = \frac{1}{3}(v - 1)$.

2.3 The problem.

For every $v \in SN$ let $N(v)$ be the set of all numbers $h \in \mathbb{N}_0$ such that there exist $S(2, 4, v)$ with exactly h subsystems $S(2, 4, r)$.

We are looking for $N(v)$, we try to determine this set-that's the problem. For Steiner triple systems $S(2, 3, v)$ we succeeded to solve the analogous problem completely [6], [13]. There we know the total spectrum. In respect of systems $S(2, 4, v)$ we unfortunately obtained only some poor partial results.

3. What is already known.

3.1 Resmini (1981) [9], **Zeitler (1990)** [12].

a) Exactly for all

$$v \in SN' = \{13, 40\} + 36\mathbb{N}_0$$

there exists a system with *at least one* $S(2, 4, r)$.

b) This means that for all the remaining

$$v \in SN'' = \{16, 25, 28, 37\} + 36\mathbb{N}_0$$

never exists a $S(2, 4, v)$ with $S(2, 4, r)$. In other words $N(v \in SN'') = \{0\}$.

3.2 Shen (1992) (communication by letter).

Exactly for all $v \in SN' \setminus \{13\}$ there exist systems $S(2, 4, v)$ with *exactly one* $S(2, 4, r)$. In other words $1 \in N(v \in SN' \setminus \{13\})$.

3.3 $PG(n, 3), n \in \mathbb{N} \setminus \{1\}$.

It is well-known, that a projective space of dimension $n \in \mathbb{N} \setminus \{1\}$ of order 3 contains exactly $v = \frac{1}{2}(3^{n+1} - 1)$ points and exactly just as many subspaces of dimension $n - 1$ (hyperplanes) with exactly $r = \frac{1}{3}(v - 1) = \frac{1}{2}(3^n - 1)$ points each. All the numbers $\frac{1}{2}(3^n - 1), n \in \mathbb{N} \setminus \{1\}$ are Steiner numbers.

Proof. Complete induction.

$n = 2m$

$$m = 1 \Rightarrow \frac{1}{2}(3^2 - 1) = 4 \in SN\,.$$

If $\frac{1}{2}(3^{2m} - 1) = 4 + 12\lambda$, then we have

$$\frac{1}{2}(9 \cdot 3^{2m} - 1) = \frac{1}{2}(9(9 + 24\lambda) - 1) = 40 + 108\lambda \in SN\,.$$

$n = 2m + 1$

$$m = 1 \Rightarrow \frac{1}{2}(3^3 - 1) = 13 \in SN\,.$$

If $\frac{1}{2}(3^{2m+1} - 1) = 13 + 12\lambda$, then we have

$$\frac{1}{2}(9 \cdot 3^{2m+1} - 1) = \frac{1}{2}(9(27 + 24\lambda) - 1) = 121 + 108\lambda \in SN\,.$$

Together with 3.2 now we can write $(v \neq 13)$:

$$\{1, \frac{1}{2}(3^{n+1} - 1)\} \subset N(v = \frac{1}{2}(3^{n+1} - 1)).$$

4. Some definitions and lemmas (needed for proofs).

4.1 Maximal subsystems and hyperovals.

Definition 4.1.1. A point set $H \subset V$ is (in analogy to Steiner triple systems) here called a hyperoval if for each line $g \in B$ we have $|H \cap g| \in \{0, 3\}$ and if the cardinality of H is $2r + 1$. So we obtain only two classes of lines in respect of H: 0-secants and 3-secants.

Lemma 4.1.2. [12] *If in a $S(2, 4, v)$ a hyperoval H exists then the complementary subset $\overline{H} = V \setminus H$ together with all the 0-secants of H is a subsystem $S(2, 4, r)$. If on the other side the subsystem $S(2, 4, r)$ exists, the complement of its point set is a hyperoval.*

(We mention, that the hyperoval of our lemma may be interpreted as a resolvable Steiner triple system of order $2r + 1$. Resmini and Shen need this remark for their proofs.)

4.2 Intersection of subsystems.

Lemma 4.2.1. *Let S_1, S_2 be two subsystems of any order with $|S_1 \cap S_2| \geq 2$ then the intersection $S_1 \cap S_2$ is also a subsystem.*

Proof. Two points in $S_1 \cap S_2$ provide exactly one line . This line is totally in S_1 and totally in S_2 too – therefore also totally in $S_1 \cap S_2$.

Lemma 4.2.2. *Let S_1, S_2 be two subsystems of any order with $S_1 \subsetneqq S_2$ then we have $|S_1| \leq \frac{1}{3}(|S_2| - 1)$.*

Proof. We count all the lines through $P \in S_2 \setminus S_1$.
Each point $Q \in S_1$ provides together with P exactly one line. The missing points of this line are elements of $S_2 \setminus S_1$. There exist exactly $|S_1|$ lines of this kind. In S_2 there are exactly $\frac{1}{3}(|S_2| - 1)$ lines throught P. So we obtain

$$|S_1| \leq \frac{1}{3}(|S_2| - 1).$$

Lemma 4.2.3. *Let S_1, S_2 be two subsystems of any order with $S_1 \not\subset S_2$, $S_2 \not\subset S_1$ and $|S_1 \cap S_2| \geq 1$. Using the notation $A = S_1 \cap S_2$ we obtain*

$$|A| \geq \frac{1}{3}(|S_1| + 3|S_2| - v).$$

Proof. We count the points of all the lines through $P \in S_2 \setminus S_1$.
Point P together with $Q \in S_1 \setminus A$ yields exactly one line. The missing points of this line are elements of $\overline{(S_1 \cup S_2)}$. The number of points in all these lines – P fixed and not counted – is $3|S_1 \setminus A| = 3(|S_1| - |A|)$. Joining the points of S_2, we obtain $v \geq 3(|S_1| - |A|) + |S_2|$ and finally

$$|A| \geq \frac{1}{3}(|S_2| + 3|S_1| - v)\,.$$

5. Some new results.

We now give some new, but unimportant results – only propositions.

5.1 An "If-Then"-proposition.

If in a $S(2,4,v)$ there exist two subsystems S_I, S_{II} of maximal order r, then we have:

a) $A = S_I \cap S_{II} \neq \emptyset$,
b) *A is a subsystem or one point,*
c) $|A| = \frac{1}{9}(v-4)$,
d) *the necessary condition for this situation is $v \in \{121, 40\} + 108\mathbb{N}_0$ or $v = 13$.*

Proof. a) In case $A = \emptyset$ the point set of S_{II} is subset of the point set of $\bar{S}_I$ and because of 4.1 part of a hyperoval H_I. This is a contradiction because H_I does not contain any line.
b) Due to 4.2.1 A is a subsystem if $|A| > 1$ and naturally a point if $|A| = 1$.
c) Using lemma 4.2.2 we obtain immediately

$$|A| \leq \frac{1}{3}(|S_{II}| - 1) = \frac{1}{9}(v-4)\,,$$

because of $A \subset S_{II}$.
Lemma 4.2.3. yields

$$|A| \geq \frac{1}{3}(|S_{II}| + 3|S_I| - v) = \frac{1}{9}(v-4)\,.$$

With both results it follows $|A| = \frac{1}{9}(v-4)$.
d) If A is a subsystem then we necessary have

$$|A| = \frac{1}{9}(v-4) \in \{13, 4\} + 12\mathbb{N}_0$$

and therefore

$$v \in \{121, 40\} + 108\mathbb{N}_0\,.$$

In the special case $|A| = \frac{1}{9}(v-4) = 1$ we obtain $v = 13$. The only $S(2,4,13)$ is the projective plane $PG(2,3)$.

5.2 One more "If-Then"-proposition. – "From 3 make 4".

If in a $S(2,4,v)$ besides S_1, S_{II} with $A = S_I \cap S_{II}$ there exists a further system S_{III} of order r with $A \subset S_{III}$, then:

a) *there exists still one more system S_{IV} of order r with $A \subset S_{IV}$;*
b) *besides S_I, S_{II}, S_{III}, S_{IV} no other system S_x of order r with $A \subset S_x$ exists, $r \neq 4$.*

Proof. a) We claim, that the set $\overline{(S_I \cup S_{II} \cup S_{III})} \cup \overbrace{(S_I \cup S_{II} \cup S_{III})}^{A}$ is a subsystem S_{IV} of order r and therefore $\overline{S_{IV}} = H_{IV}$ a hyperoval. Proving this we have to show either, that S_{IV} is a $S(2,4,r)$ or that H_{IV} is a hyperoval. We prefer to do the latter. First of all we count the points of H_{IV}. With

$$|S_I \setminus A| = \frac{1}{3}(v-1) - \frac{1}{9}(v-4) = \frac{1}{9}(2v+1)$$

we obtain

$$|H_{IV}| = \frac{1}{3}(2v+1) = 2r+1\,.$$

To prove that H_{IV} really is a hyperoval we check systematically all the possible situations of lines in respect to H_{IV}. There remain only two classes of lines in respect of H_{IV}: 0-secants and 3-secants. All other line positions yield contradictions to the fact that a hyperoval never can obtain lines.
b) Now we assume that a system $S_x \neq S_i$ of order $r \neq 4$ with $A \subset S_x$ exists, $i \in \{I, II, III, IV\}$. Due to the proof of a) every point $P \in S_x \setminus A$ must be element of exactly one point set $S_i \setminus A$. Then P, together with the points of A *spans up*, generates as well S_x, as S_i too. Then it follows $S_x = S_i$. This is a contradiction to the assumption.

6. Some more results.

6.1 Questions.

All the results in section 5 are indeed completely unsatisfactory.
Are there perhaps further "If-Then"-propositions? We give an example.
If in a $S(2,4,v)$ there exist two subsystems S_I, S_{II} of order r with $A = S_I \cap S_{II}$ then there exists a system S_{III} of order r with $A \subset S_{III}$ (and due to 5.2 then a further system S_{IV})? If yes, we had "From two make three" – this happens in Steiner triple systems [13].

Much more serious is the argument that only "If-Then"-proposition were given. May systems of this kind really exist? Are there systems $S(2,4,v)$ with 2, 3 or 4 subsystems $S(2,4,r)$?

The following existence propositions give partial answers to these questions.

6.2 An existence proposition.
In the following we have to exclude the case $v = 40$.

Exactly for all $v \in \{121, 148\} + 108\mathbb{N}_0$ *there exist systems* $S(2, 4, v)$ *with exactly four systems* S_I, S_{II}, S_{III}, S_{IV} *of order* r *with*

$$|A| = |S_I \cap S_{II} \cap S_{III} \cap S_{IV}| = \frac{1}{9}(v - 4) = a\,.$$

Other systems of order r *then don't exist.*

Proof. To prove a proposition of this kind we need some constructions.
Here a well-known construction [1], the quadrupling procedure is used.
a) *The starting system.*
We start with a system $S(2, 4, r)$ containing exactly one subsystem $S(2, 4, a)$ with $a = \frac{1}{3}(r - 1)$. The point sets respectively are R and A. Due to 3.2 such systems exist for $r \in \{49, 40\} + 36\mathbb{N}_0$.
b) *Quadrupling (four-leaf-clover).*
Now we connect four isomorphic starting systems S_I, S_{II}, S_{III}, S_{IV} such that $S(2, 4, a)$ is the intersection system.
c) *A new system* $S(2, 4, v = 3r + 1)$.

The points.

Quadrupling yields a point set V with exactly

$$|V| = v = 4(r - a) + a = 4r - 3a = 3r + 1\,.$$

We have $v = 3r + 1 \in \{148, 121\} + 108\mathbb{N}_0$.
It's advantageous to denote all the points in $V \setminus A$ as pairs in the following way $\{1, 2, \ldots, r - a = q\} \times \{I, II, III, IV\}$.

Old lines.

All the lines of our systems shall remain lines in the new system. So we have exactly $\frac{1}{12}a(a - 1)$ lines in A (class α) and exactly $4\frac{1}{12}r(r - 1) - 4\frac{1}{12}a(a - 1)$ lines with exactly one point in A (class β).

New lines.

If we like to obtain a system $S(2, 4, v)$ then

$$\frac{1}{12}v(v - 1) - \frac{1}{3}r(r - 1) + \frac{1}{4}a(a - 1) = (r - a)^2 = q^2$$

lines are still missing. Because of $q > 6$ there exist two mutually orthogonal latin sqares L_1, L_2 of order q. Now we use the corresponding quasigroups $(L_1, \circ)$ and

$(L_2, \square)$. Any point set $\{xI, yII, (x \circ y)III, (x \square y)IV\}$ with $x, y \in \{1, 2, \dots, q\}$ is defined to be a new line. In this way we obtain exactly the q^2 missing lines (class γ). In this way our construction is finished.
It's immediately to see, that any 2-subset is contained in exactly one line and that any line is a 4-subset. Therefore we really have a system $S(2, 4, v)$ with four subsystems of order r, intersecting in a subsystem $S(2, 4, a)$.
Because of 5.2. (b) there are exactly four such systems with subsystem A.
Now let S_x be a subsystem of order r such, that $A \not\subset S_x$. Due to 5.1 then $A' = S_x \cap S_I$ is a subsystem $S(2, 4, a)$ of S_I with $A' \neq A$.
This is a contradiction. Because our starting system contains exactly one subsystem of order a.
In respect of orders different form $v \in \{121, 148\} + 108\mathbb{N}_0$ and $v = 40$ the situation of our Proposition 6.2 can't never occur, because the necessary condition 5.1 (d) is not fulfilled. So, excluding $v = 40$ we have an optimal result and we can write "exactly for all".

6.3 One more existence proposition.

In the following proposition we exclude the case $v = 40$.

Exactly for all $v \in \{121, 148\} + 108\mathbb{N}_0$ *there exist systems* $S(2, 4, v)$ *with exactly two subsystems* S_I, S_{II} *of order* r.

With this proposition we know that the conclusion "From 2 make 3" is wrong. This fact is already proved with the construction of a $S(2, 4, 148)$ with exactly two $S(2, 4, 49)$ given by B. Ganter (communication by letter).

a) *Modification of our quadrupling-construction.*

We modify the construction in such a way that besides the four maximal systems $S_i, i \in \{I, II, III, IV\}$ of order r and the system of order a there exists a system S_x of order 13, a projective plane.
Using the intersection lemmas in 4.2 we see that $S_i \cup S_x$ yields a line g_i. Because S_x is a projective plane any two lines g_i must intersect in one point of A. Let for instance be $g_I \cap g_{II} = \{S\}$ and $g_I \cap g_{II} = \{P\}$ with $P \neq S$. Then we have $S, P \in A$ and $S, P \in g_I$. That's a contradiction to the fact, that $|g_I \cap A| = 1$. Therefore all three and finally all four lines g_i intersect in S. Figure 1 shows the situation.

All the other lines of our projective plane must be of class γ. Comparing figure 1 with the quasigroups $(L_1, \circ)$, $(L_2, \square)$ used for construction we obtain two tables, two orthogonal latin squares M_1, M_2, of order 3.

$g_I : \{S, a, b, c\}$, $g_{II} : \{S, u, v, w\}$, $g_{III} : \{S, x, y, z\}$, $g_{IV} : \{S, d, e, f\}$.

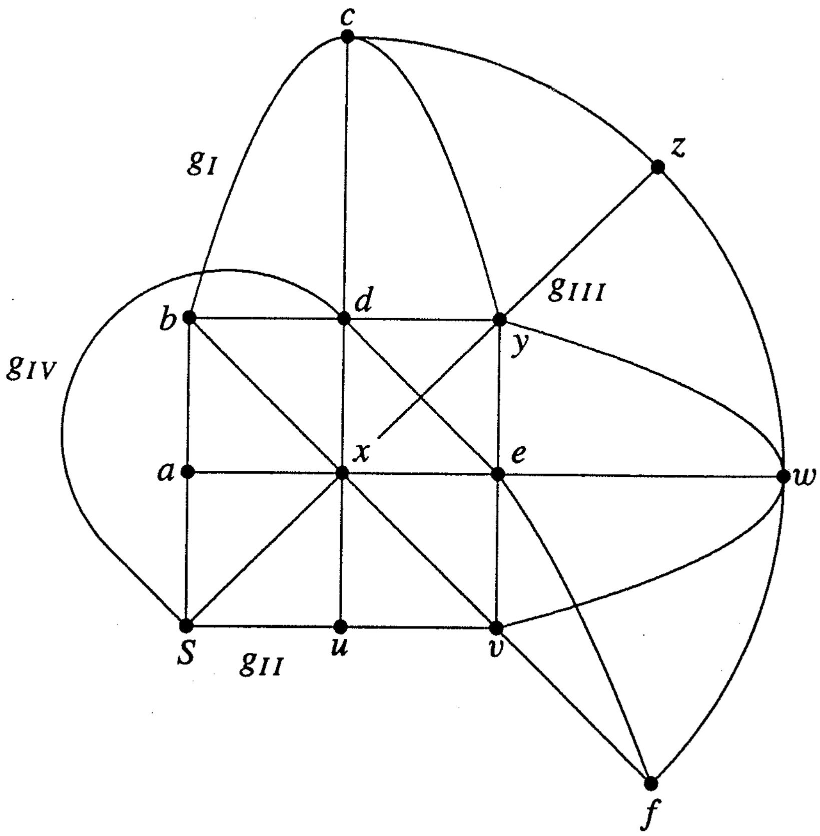

Figure 1

M_1

$\circ$	u	v	w
a	y	z	x
b	z	x	y
c	x	y	z

M_2

$\square$	u	v	w
a	f	d	e
b	e	f	d
c	d	e	f

If the projective plane exists then the orthogonal squares L_1, L_2 must contain the orthogonal subsquares M_1 in L_1 and M_2 in L_2.
K. Heinrich [5] gives a general proof that orthogonal latin squares L_1^*, L_2^* of order q with orthogonal subsquares M_1^*, M_2^* exist, if $q \geq 9$. In our special case

we have $q \in \{33, 27\} + 24\mathbb{N}_0$, therefore $3|q$. Now we use the well-known [5] direct product method for constructing latin squares. Let the orthogonal starting systems A_1, A_2 be of order 3. The "factors" B_1, B_2 are also orthogonal and of order $\frac{1}{3}q \neq 6$. Product constructing yields two orthogonal systems $L_1^* = A_1 \times B_1$, $L_2^* = A_2 \times B_2$ of order q with subsystems of order 3.
We perform the quadrupling procedure once more, but now using the quasigroups $(L_1^*, \circ)$, $(L_2^*, \square)$ for construction. Doing so we finally obtain the system required in the beginning of a).
b) *We kill systems.*
Now we kill, we destroy two systems of order r. We change the points w and z in figure 1. The following lines are deleted:

$$\{w, v, u, S\}\,,\ \{w, e, x, a\}\,,\ \{w, y, d, b\}\,,\ \{z, y, x, S\}\,,\ \{z, d, a, v\}\,,\ \{z, e, u, b\}$$

and they are replaced by

$$\{z, v, u, S\}\,,\ \{z, e, x, a\}\,,\ \{z, y, d, b\}\,,\ \{w, y, x, S\}\,,\ \{w, d, a, v\}\,,\ \{w, e, u, b\}\,.$$

All the other lines in $S(2, 4, v)$ shall be unaltered.
Doing so, the system $S(2, 4, 13)$ as a whole remains a projective plane as before and the systems S_I, S_{IV} are completely unaltered. Only S_{II} and S_{III} are killed as Steiner systems.
Exactly as in the proof of 6.2 we conclude, that – excluding $v = 40$ – the formulation "exactly for all" is correct again.

6.4 Corollary.
The case $v = 40$ is excluded once more.
For all $v \in \{121, 148\} + 108\mathbb{N}_0$ there exist systems $S(2, 4, v)$ with exactly one subsystem $S(2, 4, r)$, but also systems $S(2, 4, v)$ without any maximal subsystem.

If we are looking at 3.2 we see, that the first result in our corollary is not at all optimal, because there exist systems of other orders with exactly one maximal subsystem. Certainly also the second result may be vastly improved.
Proof. Changing points in the following way

$$w \to z \to f \to w$$

or

$$w \to z \to c \to f \to w$$

the systems S_{II}, S_{III}, S_{IV} or S_I, S_{II}, S_{III}, S_{IV} are killed.

6.5 A quick summary.

Using the formulation in 2.3 we can summarize the Propositions 6.2, 6.3 and the Corollary 6.4 in the following way

$$\{0, 1, 2, 4\} \subset N(v \in \{121, 148\} + 108\mathbb{N}_0)\,.$$

That's not at all a very good result. "Es kreißte der Berg und gebar eine Maus" (The mountain was in labour and a mouse was born).

7. The special case $v = 40$.

7.1 Exclusion ?

In the Proposition 6.2 and 6.3 the case $v = 40$ is excluded. What's the reason for?
The quadrupling construction presented in 6.2 also works in case $v = 40$. Then the system A consists only of one line. But the proof by contradiction for the existence of exactly four subsystems $S(2, 4, 13)$ can't be repeated. Because every starting system contains besides A still 12 subsystems $S(2, 4, 4)$.

7.2 "If-then".

Given the system $S(2, 4, 40)$ constructed with the quadrupling procedure in 7.1 with at least four subsystems $S_i, i \in \{I, II, III, IV\}$ of order 13 intersecting in one line A. If another system S_x of order 13 exists then we obtain exactly the situation of section 6.3 a). We need again two orthogonal latin squares L_1, L_2 of order 9 with orthogonal subsquares M_1, M_2 of order 3 each.

Proposition 7.3. *There exists a* $S(2, 4, 40)$ *with exactly four systems* S_i, $i \in \{I, II, III, IV\}$ *of order* 13. *These systems intersect in a line.*

Using the notation in 2.3 we can write $4 \in N(40)$.
Proof. In the nice book [3] written by Denes and Keedwell eight mutually orthogonal latin squares are given which correspond to the so-called Hughes-plane. We take over two of them L_3 and L_5 for further constructing. Testing by trial and error yields that each of these two systems contains exactly one subsystem of order 3, M_3 respectively M_5.

$(L_3, \circ)$

$\circ$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	4	8	9	6	7	5	2	3	1
3	8	1	5	3	4	2	9	6	7
4	3	6	1	9	8	7	4	5	2
5	7	9	8	2	6	1	3	4	5
6	5	3	7	1	9	4	6	2	8
7	6	4	2	8	1	9	5	7	3
8	9	5	6	7	2	3	8	1	4
9	2	7	4	5	3	8	1	9	6

$(L_5, \square)$

$\square$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	6	7	4	8	9	3	1	5	2
3	4	8	2	7	1	9	5	3	6
4	9	5	7	1	2	4	3	6	8
5	8	6	9	3	4	5	2	7	1
6	7	1	5	6	8	2	9	4	3
7	2	9	6	5	3	7	8	1	4
8	3	4	1	2	7	8	6	9	5
9	5	3	8	9	6	1	4	2	7

M_3

$\circ$	3	4	5
1	3	4	5
3	5	3	4
9	4	5	3

M_5

$\square$	2	3	8
1	2	3	8
3	8	2	3
9	3	8	2

Considering the subsquares M_3, M_5 we see, that they are not orthogonal. So the system $S(2,4,40)$, constructed with $(L_3, \circ)$, $(L_5, \square)$ contains exactly four systems S_i, $i \in \{I, II, II, IV\}$ of order 13.

Proposition 7.4. *There exists a* $S(2,4,40)$ *with exactly two subsystems* S_i, $i \in \{I, x\}$ *of order* 13. *These systems intersect in one line*

We can write $2 \in N(40)$. Now the exception $v = 40$ as well in 6.2 as in 6.3 too is eliminated.

Proof. a) Modification of our quadrupling construction

We construct a $S(2,4,40)$ with the quadrupling procedure using two orthogonal latin squares L_5, L_7 out of [3].

$(L_3, \circ)$

$\circ$	1	2	3	4	5	6	7	8	9
1	6	7	4	8	9	3	1	5	2
2	1	2	3	4	5	6	7	8	9
3	4	8	2	7	1	9	5	3	6
4	9	5	7	1	2	4	3	6	8
5	8	6	9	3	4	5	2	7	1
6	7	1	5	6	8	2	9	4	3
7	2	9	6	5	3	7	8	1	4
8	5	3	8	9	6	1	4	2	7
9	3	4	1	2	7	8	6	9	5

$(M_5, \circ)$

$\circ$	2	3	8
2	2	3	8
3	8	2	3
8	3	8	2

$(L_7, \square)$

$\square$	1	2	3	4	5	6	7	8	9
1	3	9	1	5	2	4	8	7	6
2	1	2	3	4	5	6	7	8	9
3	7	3	8	6	9	1	4	2	5
4	8	1	5	2	4	3	6	9	7
5	4	7	6	9	8	2	5	1	3
6	2	4	9	8	6	7	3	5	1
7	9	5	4	3	7	8	1	6	2
8	6	8	2	7	1	5	9	3	4
9	5	6	7	1	3	9	2	4	8

$(M_7, \square)$

$\square$	2	3	8
2	2	3	8
3	3	8	2
8	8	2	3

To simplfy the construction we interchanged the first with the second row and the last with the last but one row in L_5 and in L_7. This is allowed because doing so, the orthogonality is not destroyed. Moreover M_5, M_7 are the only subsquares of order 3 in L_5, L_7 and they are orthogonal. The necessary condition in 6.3 a) is fulfilled. So we obtain a system $S(2, 4, v)$ with exactly 5 subsystems S_x, S_i, $i \in \{I, II, III, IV\}$ of order 13.

Remark. Using the product construction in 6.3 a) we also obtain orthogonal systems of order 9 with 9 orthogonal subsquares of order 3 each. In this way we have a system $S(2, 4, 40)$ with exactly 13 subsystems $S(2, 4, 13)$, $13 \in \mathbb{N}(40)$.

b) Details

My maxim is that mathematics really must be done. Therefore I give the constuction in some detail.

The points:

$\{1, 2, 3, 4, 5, 6, 7, 8, 9\} \times \{I, II, III, IV\}$ and 10, 11, 12, 13.

Starting systems.

The lines:

class: α: $A = \{10, 11, 12, 13\}$

class β:

$\{13, 3i, 5i, 9i, \}$,$\{13, 1i, 6i, 8i\}$,$\{13, 2i, 4i, 7i\}$,
$\{12, 1i, 2i, 5i, \}$,$\{12, 4i, 8i, 9i\}$,$\{12, 3i, 6i, 7i\}$,
$\{11, 1i, 7i, 9i, \}$,$\{11, 4i, 5i, 6i\}$,$\{11, 2i, 3i, 8i\}$,
$\{10, 5i, 7i, 8i, \}$,$\{10, 2i, 6i, 9i\}$,$\{10, 1i, 3i, 4i\}$.
$i \in \{I, II, III, IV\}$

class γ, with $(L_5, \circ)$, $(L_7, \Box)$

$\{xI, yII, x \circ yIII, x \Box yIV\}$ with $x, y \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

The new subsystem S_x.

The points: $2i, 3i, 8i$, 11 with $i \in \{I, II, III, IV\}$.

The lines

class β: $\{11, 2i, 3i, 8i\}$, $i \in \{I, II, III, IV\}$

class γ with $(M_5, \circ)$, $(M_7, \Box)$

$\{2I, 2II, 2III, 2IV\}$,$\{3I, 3II, 8III, 3IV\}$,$\{8I, 2II, 3III, 8IV\}$,
$\{2I, 3II, 3III, 3IV\}$,$\{3I, 3II, 2III, 8IV\}$,$\{8I, 3II, 8III, 2IV\}$,
$\{2I, 8II, 8III, 8IV\}$,$\{3I, 8II, 3III, 2IV\}$,$\{8I, 8II, 2III, 3IV\}$.

In figure 2 you see a picture with some lines of S_x.

c) We kill systems

If we like to obtain our Proposition 7.4 we have to kill subsystems. At first we take over the letters c, z, w, f from figure 1 to figure 2. Then – exactly as in 6.3 b) – we change the points w and z, delete 6 lines and replace these lines by 6 other ones. In this way the systems S_{II}, S_{III} are killed. There remain exactly three maximal subsystems S_I, S_{IV}, S_x.

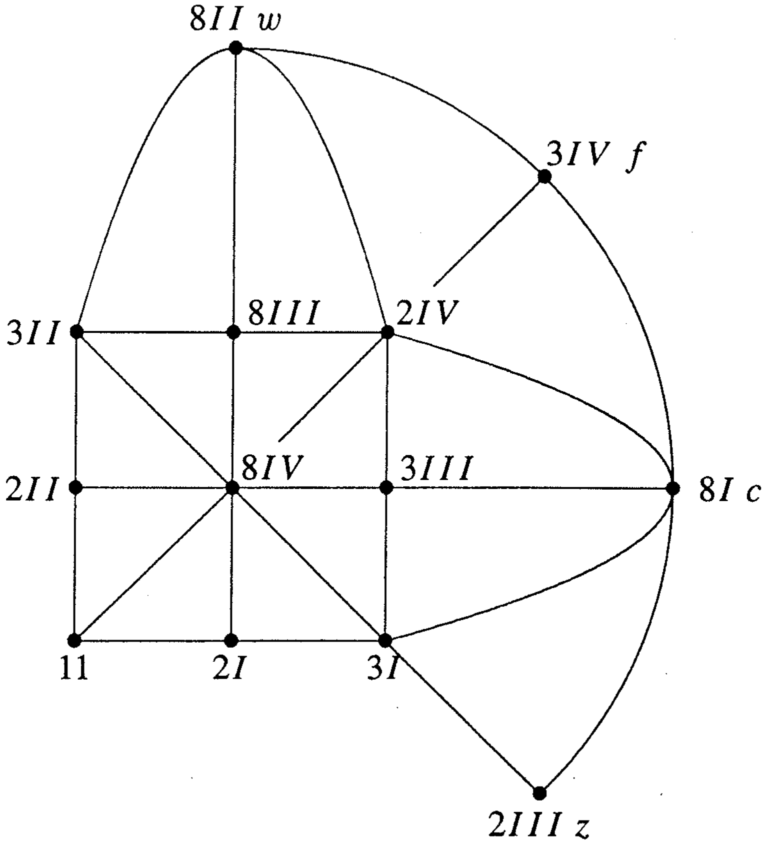

Figure 2

Now we change points in the following way:

$$w \to z \to f \to w$$

or

$$w \to z \to c \to f \to w\,.$$

We obtain exactly two maximal systems S_I, S_x or exactly one S_x.

7.4 A quick summary to the case $v = 40$.

Using the notation of 2.3 and the results 3.2, 3.3 we can together with 7.3 and 7.4 write

$$\{1, 2, 3, 4, 5, 13, 40\} \subset \mathbb{N}(40)$$

(In the cases 3,5,13 the correspoding systems have exactly one point and in the cases 2,4 exactly one line in common.)

8. What remains to do.

8.1 Find the total spectrum $N(v)$!
8.2 Calculate $N(v)$ for small orders v, using the computer! What about $N(40)$?
8.3 Prove the following conjecture:
For all $v \in \{25, 16\} + 12\mathbb{N}_0$ *there exist systems* $S(2, 4, v)$ *without any maximal subsystem* $S(2, 4, r)$.
8.4 Try to find $N(v)$ using algebraic instuments, such as Stein-Quasigroups!

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