LE MATEMATICHE
Vol. XLVIII (1993) – Fasc. II, pp. 253-262

A REMARK ON MINIMIZATION OF AN ENERGY TYPE FUNCTIONAL

ADELE FERONE

We study the problem of the minimization of the energy type functional

$$\Phi(f) = \int_\Omega fGf$$

on a class of function with prescribed distribution function that we denote by $C(f_0)$. We will prove that there exists only one minimizer of the functional considered in the weak closure of $C(f_0)$. We find some conditions that f_0 has to verifie in order to have the minimizer in $C(f_0)$.

1. Introduction.

Let Ω be an open, bounded set of $\mathbb{R}^n$, $n \geq 1$, and let $f \in L^p(\Omega)$, $p \geq 1$: the *distribution function of* f is defined by:

(1.1) $$\mu_f(t) = |\{x \in \Omega : f(x) > t\}|, \qquad \text{for } t \in \mathbb{R}$$

where, here and below, $|A|$ denotes the Lebesgue measure of the set $A \subseteq \mathbb{R}^n$.

Let f and g be measurable functions: f and g are *equimisurable* if they have the same distribution function and this equivalence relation is denoted by $f \sim g$.

Entrato in Redazione il 6 dicembre 1993.

Let $f_0 \in L^p(\Omega)$ and let

$$C(f_0) = \{f \in L^p(\Omega) : f \sim f_0\}. \tag{1.2}$$

Let us consider the following Dirichlet problem:

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

with $\lambda > 0$ and $f(x) \in L^p(\Omega)$, $p > 1$. If Ω is sufficiently smooth, there is only one solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of the problem (1.3) ([9]).

For any $f \in C(f_0)$ let $\Phi(f)$ be the functional

$$\Phi(f) = \frac{1}{2}\int_\Omega (|Du|^2(x) + \lambda u^2(x))\, dx = \frac{1}{2}\int_\Omega f\mathscr{G}f \tag{1.4}$$

where $u = \mathscr{G}f$ and $\mathscr{G}$ is the Green operator for the problem (1.3). The aim of this paper is to study the problem of minimizing (1.4) on $C(f_0)$, where $n < p < +\infty$. By an easy computation, it is possible to check that Φ is a strictly convex and weakly continuous functional in the weak topology on L^p.

In order to apply the classical variational principles, the minimizer of Φ will be sought in a class of L^p functions larger than $C(f_0)$, closed, weakly compact and convex. This class is $K(f_0)$ consisting of all weak limits in $L^p(\Omega)$ of sequences in $C(f_0)$ (the properties of $K(f_0)$ and of $C(f_0)$ are widely studied in [7], [13], [14], [1]). Then Φ has only one minimizer in $K(f_0)$.

The case $\lambda = 0$ has been treated in [6] and [1] where the authors show that, if $\hat{f}$ is the minimizer of Φ and $\hat{u} = \mathscr{G}\hat{f}$, then $\hat{f}$ is a decreasing function of $\hat{u}$ and, only under the assumption that f_0 does not change sign, $\hat{f} \in C(f_0)$. The goal of this paper is to study the case $\lambda > 0$. It will be shown that the minimizer $\hat{f}$ in $K(f_0)$ is again a decreasing function of $\hat{u}$ and that $\hat{f}$ cannot change the sign, but a counterexample shows that it is not sufficient to assume that f_0 is one-signed in order to have the existence of the minimizer of Φ in $C(f_0)$.

2. Preliminar results.

From now on, let Ω be an open bounded set of $\mathbb{R}^n$, sufficiently smooth: if f is a measurable function in Ω, using the definition of distribution function given in (1.1), the *decreasing* and the *increasing rearrangements* of f are respectively defined by:

$$f^*(s) = \sup\{t \in \mathbb{R} : \mu_f(t) > s\} \qquad \text{for } s \in [0, |\Omega|[$$
$$f_*(s) = f^*(|\Omega| - s) \qquad \text{for } s \in]0, |\Omega|].$$

If $\Omega^\#$ is the ball of $\mathbb{R}^n$ centered in the origin such that $|\Omega| = |\Omega^\#|$ and if C_n is the measure of the unit ball of $\mathbb{R}^n$, the *spherically symmetric decreasing* and *increasing rearrangements* of f are, respectively:

$$f^\#(x) = f^*(C_n|x|^n) \qquad f_\#(x) = f_*(C_n|x|^n) \qquad \text{for } x \in \Omega^\#.$$

For more details on rearrangements see, for example, [7], [2], [11], [10].

Let us consider the following order relation in $L^p(\Omega)$: given $f, g \in L^p(\Omega)$ then

$$f \prec g \quad \text{if} \quad \int_0^t f^*(s)\,ds \le \int_0^t g^*(s)\,ds \quad \text{for} \quad t \in [0, |\Omega|]$$

$$f \preceq g \quad \text{if} \quad f \prec g \quad \text{and} \quad \int_\Omega f(x)\,dx = \int_\Omega g(x)\,dx\,.$$

If $f_0 \in L^p(\Omega)$, let $C(f_0)$ be the set defined in (1.2): if $K(f_0)$ is its weak closure, then ([1], [2], [7], [12], [13], [14]):

Theorem II.1. *Let $f_0 \in L^p(\Omega)$, $1 \le p < +\infty$. Then:*

(i) $K(f_0) = \{f \in L^p(\Omega) : f \preceq f_0\}$

(ii) $K(f_0)$ *is a convex set in* $L^p(\Omega)$.

If $g \in L^{p'}(\Omega)$, where p', here and below, is the conjugate exponent of p, consider the following linear functional:

$$(g, f) = \int_\Omega f(x)g(x)\,dx \qquad \text{for } f \in L^p(\Omega)\,. \tag{2.1}$$

From the well known Hardy inequality ([10], [7]):

$$\int_0^{|\Omega|} f^*(s)g_*(s)\,ds \le (g, f) \le \int_0^{|\Omega|} f^*(s)g^*(s)\,ds \tag{2.2}$$

and from Theorem II.1, it follows ([7], [14]):

Theorem II.2. *If $f_0 \in L^p(\Omega)$, $1 \le p < +\infty$, $g \in L^{p'}(\Omega)$ then the functional defined in (2.1) has maximum and minimum in $K(f_0)$ and:*

(a) $$\max_{f\in K(f_0)} (f, g) = \max_{f\in C(f_0)} (f, g) = \int_0^{|\Omega|} f_0^*(s)g^*(s)\,ds\,;$$

(b) $$\min_{f\in K(f_0)} (g, f) = \min_{f\in C(f_0)} (f, g) = \int_0^{|\Omega|} f_0^*(s)g_*(s)ds.$$

A characterization of the minimum points of the functional (g, f) in $K(f_0)$ is given by the following lemma:

Lemma II.3. *Let $f_0 \in L^p(\Omega)$, $1 \leq p < +\infty$ and $g \in L^{p'}(\Omega)$. Let $\hat{f}$ be a minimizer in $K(f_0)$ of the functional (2.1): then the following propositions are equivalent:*

(i) *$\hat{f}$ is the unique minimizer;*
(ii) *there exists a decreasing function ϕ such that $\hat{f} = \phi \circ g \sim f_0$;*
(iii) $\operatorname{supp} df_0^* \subseteq \operatorname{supp} dg_*$.

Proof. The proof is essentially contained in [4], [8], [1]. More precisely:

(i) $\Longrightarrow$ (ii) is in [4], [8];
(ii) $\Longrightarrow$ (iii) is a trivial consequence of :

$$f_0^*(s) = \phi \circ g_*(s) \qquad \text{for } s \in [0, |\Omega|[\,;$$

(iii) $\Longrightarrow$ (i) is similar to that one in remark (iii) of pag. 195 of [1]. □

This section ends with the varational condition at a minimizer of the functional Φ defined in (1.4) ([1], [5]):

Lemma II.4. *Let $f_0 \in L^p(\Omega)$, $n < p < +\infty$: if $\hat{f}$ is the minimizer of Φ in $K(f_0)$ then $\hat{f}$ is the unique solution of the variational inequality*

$$\int_\Omega (f - \hat{f})\mathcal{G}\hat{f} \geq 0 \qquad \textit{for } f \in K(f_0)\,.$$

Proof. The proof is similar to that one in [1] in the case $\lambda = 0$. We prove it for completennes. Denote by U the convex set in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ consisting of the solutions u of (1.3) when f varies in $K(f_0)$. Obviously, $\hat{u}$ is the unique solution of the variational inequality

$$\int_\Omega [\nabla\hat{u} \cdot \nabla(v - \hat{u}) + \lambda\hat{u}(v - \hat{u})]\,dx \geq 0 \qquad \text{for all } v \in U,\ \hat{u} \in U$$

while $\hat{f}$ is the unique solution of the other variational inequality

$$\int_\Omega \{\mathcal{G}\hat{f}\}(f - \hat{f}) \geq 0 \qquad \text{for all } f \in K(f_0),\ \hat{f} \in K(f_0)\,. \qquad \square$$

3. Main result.

Consider the functional defined in (1.4) that is:

$$\Phi(f) = \frac{1}{2}\int_\Omega f\mathscr{G}f\,. \tag{3.1}$$

and let $n < p < +\infty$. Since the operator $\mathscr{G} : L^p(\Omega) \to W^{2,p}(\Omega)$ is bounded and the embedding $W^{2,p} \hookrightarrow L^{p'}(\Omega)$ is compact, it follows that Φ is a weakly sequentially continuous functional on $L^p(\Omega)$. Moreover Φ is strictly convex and so Φ has a unique minimizer $\hat{f} \in K(f_0)$. Clearly, if

$$\int_\Omega f_0(x)\,dx = 0\,,$$

such minimizer is $\hat{f} \equiv 0$. Then, let us suppose that

$$\int_\Omega f_0(x)\,dx > 0\,. \tag{3.2}$$

If (3.2) is not verified we have to consider $-f_0$.

The following theorem holds:

Theorem III.1. *Let $f_0 \in L^p(\Omega)$, $n < p < +\infty$, and let $\hat{f}$ be the minimizer of the functional (3.1) in $K(f_0)$ with $\lambda > 0$ and $\hat{u} = \mathscr{G}\hat{f}$. Then:*

(a) *there exists at most one $\alpha \in \mathbb{R}$ such that $|\{x \in \Omega : \hat{u}(x) = \alpha\}| > 0$;*
(b) *there exists a decreasing function ϕ such that $\hat{f} = \phi \circ \hat{u}$;*
(c) *$\hat{u}$ has not internal minimum and $\hat{f} > 0$ a.e. in Ω.*

Proof. Since $\hat{f}$ is the minimizer of Φ in $K(f_0)$ clearly it is also the minimizer in $K(\hat{f})$, because $K(\hat{f}) \subseteq K(f_0)$. Hence, from Lemma II.4, $\hat{f}$ minimizes $(\hat{u}, \cdot)$ in $K(\hat{f})$. Assume that $\hat{u}$ has two level set of positive measure, i.e. there exist $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 < \alpha_2$, such that $|A_i| > 0$, $i = 1, 2$, where $A_i = \{x \in \Omega : \hat{u}(x) = \alpha_i\}$. Since $\hat{u}$ solves (1.3) there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that $\hat{f}(x) = \beta_i$ for a.e. $x \in A_i$, $i = 1, 2$. But $\hat{f}$ is a minimizer of $(\hat{u}, \cdot)$ so necessarely, β_1 and β_2 have to be rearranged in the opposite way of α_1 and α_2, then $\beta_1 \geq \beta_2$. Nevertheless (1.3) gives:

$$\beta_i = \lambda\alpha_i \qquad i = 1, 2$$

with $\lambda > 0$, obtaining a contradiction. This proves (a).

If $\hat{u}$ has no level set of positive measure, then (b) follows immediately from Lemma II.3. Otherwise if there exists an $\alpha \in \mathbb{R}$ such that $|\{x \in \Omega : \hat{u} = \alpha\}| > 0$ then, using arguments similar to those in [6], we have

$$\hat{f}(x) \leq \lambda\alpha \qquad \text{a.e. in } A' = \{x \in \Omega : \hat{u}(x) > \alpha\} \tag{3.3}$$

(3.4) $$\hat{f}(x) \geq \lambda\alpha \qquad \text{a.e. in } A'' = \{x \in \Omega : \hat{u}(x) < \alpha\}.$$

In fact, suppose that (3.3), for example, is false. Then there is a subset $S \subseteq A'$ such that $|S| > 0$ and $f(x) > \lambda\alpha$ a.e. in S. Let $T \subseteq \{x \in \Omega : \hat{u}(x) = \alpha\}$ such that $|T| = |S| > 0$. Then, there is a measure preserving bijection $\pi : S \to T$, i.e. π and π^{-1} are measurable and $|\pi(E)| = |E|$, for every measurable subset $E \subseteq S$. Let

$$\tilde{f}(x) = \begin{cases} \hat{f}(\pi(x)) & \text{if } x \in S \\ \hat{f}(\pi^{-1}(x)) & \text{if } x \in T \\ \hat{f}(x) & \text{if } x \in \Omega \setminus (S \cup T). \end{cases}$$

Clearly $\tilde{f} \sim \hat{f}$ and we have

$$\begin{aligned} \int_\Omega \hat{f}\hat{u} - \int_\Omega \tilde{f}\hat{u} &= \int_{S \cup T} (\hat{f}(x) - \tilde{f}(x))\hat{u}(x)\,dx = \\ &= \int_S (\hat{f}(x) - \hat{f}(\pi(x)))\hat{u}(x)\,dx + \\ &+ \int_T (\hat{f}(x) - \hat{f}(\pi^{-1}(x)))\hat{u}(x)\,dx = \\ &= \int_T (\hat{f}(x) - \hat{f}(\pi^{-1}(x)))(\hat{u}(x) - \hat{u}(\pi^{-1}(x)))\,dx > 0 \end{aligned}$$

which is impossible since $\hat{f}$ minimize $(\hat{u}, \cdot)$ relative to $K(\hat{f})$. In a similar way, it is possible to show that (3.4) holds.

Let $\hat{f}_{A'}$, $\hat{f}_{A''}$, $\hat{u}_{A'}$, $\hat{u}_{A''}$ be the restriction of $\hat{f}$ and $\hat{u}$ to A' and A''. Since $\hat{u}_{A'}$ and $\hat{u}_{A''}$ have not level sets of positive measure, from Lemma II.3, there exist two decreasing functions $\phi_1 : (\alpha, +\infty) \to (-\infty, \lambda\alpha)$ and $\phi_2 : (-\infty, \alpha) \to (\lambda\alpha, +\infty)$ such that $\hat{f}_{A'} = \phi_1 \circ \hat{u}_{A'}$ and $\hat{f}_{A''} = \phi_2 \circ \hat{u}_{A''}$.

So, from (3.3) and (3.4), it follows that there exists a decreasing function ϕ such that

(3.5) $$\hat{f} = \phi \circ \hat{u}.$$

This complete the proof of (b).

Suppose that $\hat{u}$ has a positive absolute maximum M at the point $x_M \in \Omega$. From (3.5), $\hat{f}(x) = \phi(\hat{u}(x))$, where ϕ is decreasing: then the function $\phi(t) - \lambda t$ is decreasing in the range of $\hat{u}$. Now, it is easy to prove that

(3.6) $$\phi(\hat{u}(x)) - \lambda\hat{u}(x) \geq 0 \qquad \text{for a.e. } x \in \Omega.$$

In fact, if $|\{\hat{u}(x) = M\}| > 0$ then (3.6) follows by the fact that, from (1.3), $\phi(M) - \lambda M = 0$ and by the monotonicity of the function $\phi(t) - \lambda t$. Otherwise, if $|\{\hat{u}(x) = M\}| = 0$ suppose, ab absurdo, that there exists $t_0 < M$ such that $\phi(t_0) - \lambda t_0 < 0$: then $\phi(t) - \lambda t < 0$ for $t > t_0$. Let $V = \{x \in \Omega : \hat{u}(x) > t_0\}$: clearly V is a neighbourhood of x_M and $-\Delta\hat{u} = \phi(\hat{u}) - \lambda\hat{u} \leq 0$ for a.e. $x \in V$. By the Bony maximum principle, [3] $\hat{u}$ cannot have a maximum in V unless it is constant, obtaining a contradiction. Hence (3.6) holds.

Another way to get the same result is the following. By Bony maximum principle [3]:

$$\lim_{x \to x_M} \inf \text{ess}\, \Delta\hat{u} = l' \leq 0$$

so, there exists a sequence $\{x_n\}_{n \in N} \subseteq \Omega$ such that

$$x_n \to x_M \qquad \text{and} \qquad \lim_n [\phi(\hat{u}(x_n)) - \lambda\hat{u}(x_n)] = -l' .$$

So, $\phi(M^-) - \lambda M \geq -l' \geq 0$ and then (3.6) holds.

Therefore $\hat{u}$ verifies:

$$\begin{cases} \Delta\hat{u} \leq 0 & \text{in } \Omega \\ \hat{u} = 0 & \text{on } \partial\Omega \end{cases}$$

which implies, by the weak maximum principle [9], that $\hat{u}$ cannot have internal minimum that is $\hat{u} > 0$, and then, from (3.6), $\hat{f}(x) = \phi(\hat{u}(x)) > 0$ a.e. in Ω. □

An immediate consequence of this theorem is contained in the following

Corollary III.2. *Under the assumption of the Theorem III.1, if there exists* $\alpha \in \mathbb{R}$ *such that* $|\{x \in \Omega : \hat{u}(x) = \alpha\}| > 0$ *then* $\alpha = \max_{\Omega} \hat{u}$.

4. A counterexample.

From Theorem III.1, it follows that the minimizer $\hat{f}$ of the functional Φ defined in (1.4) do not change the sign, so, if f_0 is not one-signed, clearly Φ has not a minimum in $C(f_0)$.

Suppose that $f_0 > 0$ a.e. in Ω; in [6] it has been shown that, if $\lambda = 0$ and f_0 is one-signed the minimum exists in $C(f_0)$. Our goal, in this section, is to show that the result just exposed is not true for $\lambda > 0$. This is due to the fact that $\hat{u} = \mathcal{G}\hat{f}$ is the solution of (1.3) with $\lambda \neq 0$ and so it can be costant on a set of positive measure. It follows that the linear functional

$$(\hat{u}, \cdot) = \int_{\Omega} f\hat{u} \qquad f \in K(f_0)$$

can have minima that are not in $C(f_0)$. (This case cannot happen for $\lambda = 0$).

Suppose, in fact, that Ω is a ball centered at the origin: from the uniqueness of the minimum $\hat{f}$ of Φ in $K(f_0)$, it follows that $\hat{f}$, and then $\hat{u}$, are spherically symmetric. From Theorem III.1 since $\hat{f} > 0$, it follows that $\hat{u}$ is radially decreasing. Furthermore, summarizing the results of the previous sections, only two cases are possible:

(a) $\hat{u}$ does not have any level set of positive measure and then $\hat{f} = (f_0)_\#$;
(b) there exists only one set S, $|S| > 0$, where $\hat{u}$ is constant and, from Corollary III.2, S must be a ball centered at the origin: however, we still have $\hat{f} = (\hat{f})_\#$ not necessarily in $C(f_0)$.

Now, we exhibit an example when the minimizer of Φ on $K(f_0)$ is not a rearrangement of f_0.

Let $n = 1$ and $f_0(x) = |x|$, and let us consider the following problem:

$$\begin{cases} -u'' + u = f & \text{in }]-2, 2[\\ u(2) = u(-2) = 0. \end{cases} \tag{4.1}$$

with $f \in K(f_0)$. If $\mathcal{G}$ is the Green operator for the problem (4.1) let

$$\Phi(\hat{f}) = \min_{f \in K(f_0)} \frac{1}{2} \int_{-2}^{2} f \mathcal{G} f$$

and let $\hat{u} = \mathcal{G}\hat{f}$. Suppose that $\hat{f} \in C(f_0)$. Taking into account (a), (b) and the fact that $f_0 = (f_0)_\#$ we still have $\hat{f} = f_0$ and the minimum of Φ would be

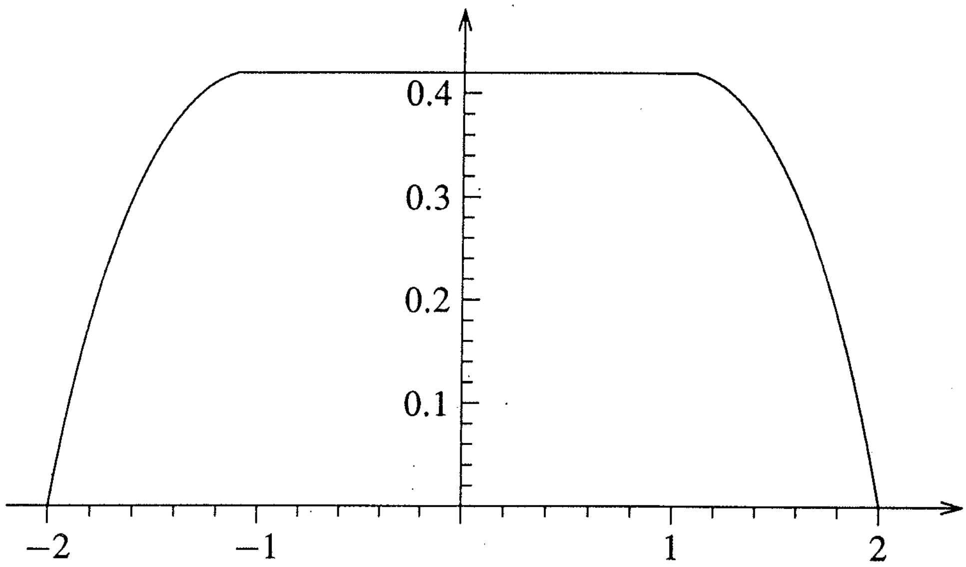
$$\Phi(f_0) = \frac{1}{2} \int_{-2}^{2} f_0(x) u_0(x)\, dx \approx 0.711379. \tag{4.2}$$

In order to prove that $\hat{f} \notin C(f_0)$, we will show that $\hat{f} \neq f_0$. Let us consider the family of functions f_r, with $r \in (0,2)$ defined by

$$f_r r(x) = \begin{cases} \dfrac{1}{2r} \displaystyle\int_{-r}^{r} f_0(\tau) d\tau & \text{if } |x| < r \\ f_0(x) & \text{if } r \leq |x| < 2. \end{cases}$$

Clearly, $f_r \in K(f_0)$, for all $r \in (0, 2)$. Numerically, we found that

$$\min_{r \in (0,2)} \Phi(f_r) \approx 0.705664 \tag{4.3}$$



and this minimum is attained for $\tilde{r} \approx 0.96124$. Comparing (4.2) and (4.3) we have the counterexample. We finally observe that the numerical calculation evidence that $u_{\tilde{r}} = \mathcal{G} f_{\tilde{r}}$ is constant in $(-\tilde{r}, \tilde{r})$ (see figure).

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Dipartimento di Matematica e Appl. "R. Caccioppoli",
Università di Napoli,
Complesso Monte S. Angelo,
80126 Napoli (Italy)