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ON THE K-CONTINUITY OF K-HULL MIDCONVEX SET-VALUED FUNCTIONS

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In the present paper it is proved the K-continuity of a set-valued function which is K-hull midconvex, K-lower semicontinuous at a point and whose values are bounded K-midconvex sets; the concept of K-hull midconvexity is introduced here and it is weaker than the one of K-midconvexity.
Then a generalization of the well known Theorem of Bernstein - Doech is obtained, which gives, in the particular case when the space is locally bounded, a characterization of the K-continuity of set-valued function.
The theorems presented here improve some earlier results obtained by K. Nikodem (Zeszyty Nauk Politech. Lodz 559, Rospravy Mat. 114, 1989).

1. Introduction.

It is well known that convex set-valued functions defined on an infinite dimensional space need not be continuous.
Many papers ([1], [3], [4], [5], [6]) have been devoted to find conditions under which the convex set-valued functions are continuous. In [4] K. Nikodem proved that if X and Y are topological vector spaces and D is a non-empty open convex subset of X, the continuity of the set-valued function $F : D \to n(Y)$ is guaranteed even by midconvexity under the additional conditions that its values are bounded and F itself is bounded on a subset of D with non-empty interior.

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Later, in [5] K. Nikodem studied this problem for the K-midconvex set-valued functions, where K is a cone in Y, and he proved that if F is a set-valued function with bounded values which is K-lower semicontinuous at some point of D, then the K-midconvexity implies the K-continuity.
In this paper we introduce the concept of K-hull midconvex set-valued function: this concept contains the one of hull-midconvex set-valued function introduced in [2] and is weaker than the one of K-midconvex set-valued function.
In the Theorem 1 we prove the K-continuity of a set-valued function F which is K-hull midconvex, K-lower semicontinuous at a point of D and whose values are bounded K-midconvex sets of Y. This Theorem improves the Theorem 3.2 of [5] (cf. Remark 1).
In 4, by using the Theorem 1, we obtain in the Theorem 2 a characterization of the K-continuity of set-valued functions.
We observe that the sufficient part of the Theorem 2 does improve the Theorem 1 (cf. Remark 2). The next Theorem 3 is a generalization of the well known theorem of Bernstein-Doetch and, in the particular case when the space Y is locally bounded, it gives a characterization of the K-continuity of set-valued functions (cf. Theorem 4).
Finally, we wish to observe that the Theorems 2 and 3 improve respectively the Theorems 3.3 and 3.4 of [5] and that the Corollaries 1, 2 and 3 improve similar results obtained by K. Nikodem in [5] (cf. Corollaries 3.1, 3.3, 3.4).

2. - Let X and Y be real vector spaces, α, β two real numbers, S, T two sets, $S, T \subset Y$; we put

$$\alpha S + \beta T = \{y \in Y : y = \alpha s + \beta t,\ s \in S,\ t \in T\}.$$

We denote by

$$n(Y) = \{S \subset Y : S \neq \emptyset\},$$
$$\mathbb{B}(Y) = \{S \subset Y : S \text{ bounded},\ S \neq \emptyset\},$$

and by $\operatorname{co} A$ the convex hull of a set $A \subset Y$.
If D is a non-empty convex subset of X and K is a cone in Y, a set-valued function $F : D \to n(Y)$ is said to be *K-convex* if (cfr. [3], p. 393)

$$(2.1)\quad tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + K, \qquad \forall t \in [0, 1],\ \forall x, y \in D.$$

If the condition (2.1) is verified for $t = \frac{1}{2}$, F is called *K-midconvex*.
We give now an extension of the notion of a K-convex (K-midconvex) set-valued function.

A set-valued function $F : D \to n(Y)$ is said to be *K-hull convex* if

$$(2.2)\quad tF(x)+(1-t)F(y) \subset \operatorname{co} F(tx+(1-t)y)+K, \quad \forall t \in [0,1],\ \forall x, y \in D.$$

If the condition (2.2) is verified for $t = \frac{1}{2}$, F is called *K-hull midconvex.*
If a set-valued function F is K-convex (K-midconvex), then F is K-hull convex (K-hull midconvex), but not vice versa; it is enough to consider the set-valued function $F : [-1,1] \to n(\mathbb{R}^2)$ given by

$$F(x) = \begin{cases} B, & x \in [-1,1] - \{0\} \\ A, & x = 0 \end{cases}$$

where $A = \{(x_1, x_2) : x_1^2 x_2^2 = 1,\ x_2 > 0\}$, $B = \{(x_1, x_2) : x_2 > 0\}$ and $K = [0, +\infty[\times\{0\}$.

A function $f : D \to Y$ is said to be a *selection* of a set-valued function $F : D \to n(Y)$ if $f(x) \in F(x), \forall x \in D$.

Let X and Y be real topological vector spaces T_0, D an open subset of X, $\mathscr{U}(0)$ and $\mathscr{W}(0)$ two basis of balanced neighbourhoods of zero respectively in X and Y.

A set-valued function $F : D \to n(Y)$ is said to be *K-lower semicontinuous* in $x_0 \in D$ if (cf. [3], p. 394)

(K-l.s.c.) $\forall W \in \mathscr{W}(0)\ \exists U \in \mathscr{U}(0),\ x_0 + U \subset D$, such that

$$F(x_0) \subset F(x) + W + K, \quad \forall x \in x_0 + U;$$

moreover F is said to be *K-upper semicontinuous* in $x_0 \in D$ if (cf. [3], p. 394)

(K-u.s.c.) $\forall W \in \mathscr{W}(0)\ \exists U \in \mathscr{U}(0),\ x_0 + U \subset D$, such that

$$F(x) \subset F(x_0) + W + K, \quad \forall x \in x_0 + U.$$

The set-valued function F is said to be *K-continuous* in $x_0 \in D$ if it is K-l.s.c. and K-u.s.c. in x_0.
The set-valued function F is called *weakly K-upper bounded* on a set $A \subset D$ if there exists a bounded set $B \subset Y$ such that

$$F(x) \cap (B - K) \neq \emptyset, \quad \forall x \in A.$$

3. - In this section we present some theorems giving conditions under which K-hull midconvex set-valued functions are K-continuous. To prove Lemma 1 it's enough that X and Y are real vector spaces, D is a convex and non-empty subset of X.

Lemma 1. *The set-valued function F is K-hull convex (K-hull midconvex) iff* $\operatorname{co} F$ *defined by* $(\operatorname{co} F)(x) = \operatorname{co} F(x)$ *is K-convex (K-midconvex).*

If F is K-hull convex, by using Lemma 1.2 of [5], we obtain

$$(3.1) \quad t\operatorname{co} F(x) + (1-t)\operatorname{co} F(y) = \operatorname{co}(tF(x) + (1-t)F(y)) \subset$$
$$\subset \operatorname{co}(F(tx+(1-t)y)+K) = \operatorname{co} F(tx+(1-t)y)+K, \quad \forall t \in [0,1],\ \forall x, y \in D,$$

that is the set-valued function $\operatorname{co} F$ is K-convex.
On the other hand, if $\operatorname{co} F$ is K-convex, it is obvious that F is K-hull convex.

From now on we make the assumption that X and Y are topological vector spaces, D is an open convex and non-empty subset of X, $\mathcal{U}(0)$ and $\mathcal{W}(0)$ two basis of balanced neighbourhoods of zero respectively in X and Y.

Lemma 2. *If $F : D \to \mathbb{B}(Y)$ is K-hull midconvex and K-lower semicontinuous at a point $x_0 \in D$ and $F(x_0)$ is a K-midconvex set, then F is K-upper semicontinuous at this point.*

We may assume that $x_0 = 0$ and we take a neighbourhood $W \in \mathcal{W}(0)$. There exist a $W' \in \mathcal{W}(0)$ such that $W'+W'+W'+W' \subset W$ and a neighbourhood $V \in \mathcal{W}(0)$ such that $V+V+V \subset W'$. Since F is K-lower semicontinuous in 0, it's possible to find a neighbourhood $U \in \mathcal{U}(0)$, $U \subset D$, such that

$$(3.2) \qquad F(0) \subset F(u) + V + K, \qquad \forall u \in U.$$

Because the set $\operatorname{cl} F(0)$ is bounded, there exists some diadic number $q \in (2^{-1}, 1)$ with $(q^{-1} - 1)\operatorname{cl} F(0) \subset V$.
We take an arbitrary point $u \in (1 - q^{-1})U$ and put $u' = (1-q^{-1})^{-1}u \in U$. If $v \in F(0)$, by using (3.2), there exists a point $y' \in F(u')$ such that $v \in y' + V + K$. By Lemma 3.1 of [5] and Lemma 1, we obtain

$$(3.3) \quad q\operatorname{co} F(u) + (1-q)\operatorname{co} F(u') \subset \operatorname{co} F(qu + (1-q)u') + K =$$
$$= \operatorname{co}(F(0) + K) \subset \operatorname{co}(\operatorname{cl}(F(0) + K)).$$

Since $F(0)$ is a K-midconvex set, $\operatorname{cl}(F(0)+K)$ is a convex set, then, from (3.3) we get

$$qF(u) + (1-q)y' + (1-q)v \subset qF(u) + (1-q)F(u') + (1-q)(y'+V+K) \subset$$
$$\subset \operatorname{cl}(F(0)+K) + (1-q)y' + (1-q)(V+K).$$

It follows that

$$F(u) \subset q^{-1}\mathrm{cl}\,(F(0)+K)+V+K+(1-q^{-1})F(0). \tag{3.4}$$

Since

$$q^{-1}\mathrm{cl}(F(0)+K) \subset (q^{-1}-1)\mathrm{cl}(F(0)+K)+\mathrm{cl}(F(0)+K) \tag{3.5}$$

and

$$(1-q^{-1})F(0) \subset V, \tag{3.6}$$

by (3.4) we obtain

$$F(u) \subset F(0)+W+K, \quad \forall u \in (1-q^{-1})U,$$

that is F is K-upper semicontinuous at $x_0 = 0$.

Lemma 3. *Let $F : D \to \mathbb{B}(Y)$ be a K-hull midconvex set-valued function. We assume that there exists a point $x_0 \in D$ in which F is K-lower semicontinuous and that the set $F(x)$ is K-midconvex, $\forall x \in D$. In these conditions F is K-lower semicontinuous on D.*

Let $x \in D$ be an arbitrary point, $x \neq x_0$. There exist a point $x_1 \in D$ and a diadic number $p \in (0, 1)$ such that $x = px_1 + (1-p)x_0$. Let $W \in \mathscr{W}(0)$ and we choose a $W' \in \mathscr{W}(0)$ such that $W' + W' \subset W$ and a $V \in \mathscr{W}(0)$ such that $V + V \subset W'$.
Since F is K-lower semicontinuous at x_0, there exists a neighbourhood $U \in \mathscr{U}(0)$ with the property

$$F(x_0) \subset F(x_0+u)+V+K, \quad \forall u \in U. \tag{3.7}$$

If $y_0 \in F(x_0)$ and $y_1 \in F(x_1)$ are fixed points and $y = py_1 + (1-p)y_0$, by Lemma 3.1 di [5], Lemma 1 and (3.7), we obtain

$$y \in pF(x_1)+(1-p)F(x_0) \subset \mathrm{co}\,F(x+(1-p)u)+V+K, \quad \forall u \in U. \tag{3.8}$$

Since the set $F(x)$ is bounded, there exists a diadic number $q \in (0, 1)$ such that $q(F(x)-y) \subset V$.
By (3.8) and Lemma 3.1 di [5] we have

$$\begin{aligned} F(x) &\subset (1-q)F(x)+V+q\,\mathrm{co}\,F(x+(1-p)u)+V+K \subset \\ &\subset \mathrm{co}(\mathrm{cl}(F(x+q(1-p)u)+K))+V+V+K. \end{aligned} \tag{3.9}$$

Since $\text{co}(\text{cl}\, F(x+q(1-p)u)+K)) = \text{cl}(F(x+q(1-p)u)+K)$, from (3.8) we get

$$\begin{aligned} F(x) &\subset F(x+q(1-p)u)+K+W'+W' \subset \\ &\subset F(x+v)+W+K, \quad \forall v \in q(1-q)U, \end{aligned}$$

namely F is K-lower semicontinuous at x.

As an immediate consequence of Lemma 2 and Lemma 3, we obtain the following.

Theorem 1. *Let $F : D \to \mathbb{B}(Y)$ be a K-hull midconvex set-valued function which is K-l.s.c. at a point $x_0 \in D$. If $F(x)$ is a K-midconvex subset of Y, $\forall x \in D$, then F is K-continuous on D.*

Remark 1. We observe that the Theorem 1 improves the Theorem 3.2 of [5]. Actually, a K-midconvex set-valued function F is K-hull midconvex and its values are K-midconvex subsets of Y; on the other hand there exist set-valued functions F which verify the conditions of our Theorem 1 but which are not K-midconvex. An example is given by the set-valued function $F: \mathbb{R} \to n(\mathbb{R}^2)$ defined by

$$F(x) = \begin{cases} [-1, 1] \times \{0\}, & x \neq 0 \\ A \times \{0\}, & x = 0 \end{cases}$$

with $A = \{k/2^n : n \in \mathbb{N},\ k \in \mathbb{Z},\ |k| \leq 2^n\}$ and $K = \{0\} \times [0, +\infty[$.

4. - In the following Theorem 2, by using the Theorem 1, we obtain a characterization of the K-continuity of set-valued functions.

Theorem 2. *Let $F : D \to \mathbb{B}(Y)$ be a K-hull midconvex set-valued function such that the set $F(x)$ is K-midconvex, $\forall x \in D$. Under these conditions F is K-continuous on D iff there exists a set-valued function $G : D \to \mathbb{B}(Y)$ which is K-l.s.c. at some point of D such that $G(x) \subset F(x) + K$, $\forall x \in D$.*

The necessary part is trivial. Now, we suppose that $0 \in D$, $0 \in G(0)$ and G is K-l.s.c. at 0. Let $W \in \mathscr{W}(0)$ and $V \in \mathscr{W}(0)$ such that $V+V+V \subset W$. Since G is K-l.s.c. at 0, there exists a neighbourhood $U \in \mathscr{U}(0)$ such that

$$G(0) \subset G(u) + V + K, \quad \forall u \in U. \tag{4.1}$$

By assumptions and (4.1), we have

$$0 \in G(u) + V + K \subset F(u) + V + K, \quad \forall u \in U. \tag{4.2}$$

The set $F(0)$ is bounded, then there exists a diadic number $q \in (0, 1)$ such that $qF(0) \subset V$. By using (4.2) and Lemma 3.1 of [5], we have

$$\begin{aligned}(4.3)\quad F(0) &\subset qF(0) + (1-q)F(0) + qF(u) + V + K \subset \\ &\subset qF(0) + \operatorname{co} F(qu) + V \subset \operatorname{co}(\operatorname{cl}(F(qu) + K)) + V + V + K \subset \\ &\subset F(qu) + K + W, \quad \forall u \in U.\end{aligned}$$

Now, since F is K-l.s.c. at 0, by using Theorem 1, it follows that F is K-continuous at 0.

Remark 2. The sufficient part of the Theorem 2 does improve the Theorem 1. Actually, a set-valued function $G : D \to \mathbb{B}(Y)$ with the conditions required in the Theorem 2 there exists if we assume, as in the Theorem 1, that F is K-l.s.c. at some point of D: it's enough to take $G = F$.

As an immediate consequence of the Theorem 2, we have the following

Corollary 1. *Let $F : D \to \mathbb{B}(Y)$ be a K-hull midconvex set-valued function such that the set $F(x)$ is K-midconvex, $\forall x \in D$. If F has a selection $f : D \to Y$ K-l.s.c. at some point of D, then F is K-continuous on D.*

We will prove the following

Theorem 3. *Let $F : D \to \mathbb{B}(Y)$ be a K-hull midconvex set-valued function such that the set $F(x)$ is K-midconvex, $\forall x \in D$. If there exists a set-valued function $G : D \to n(Y)$ weakly K-upper bounded on some subset of D with non-empty interior such that $G(x) \subset F(x) + K$, $\forall x \in D$, then F is K-continuous on D.*

Since G is weakly K-upper bounded on some subset of D with non-empty interior, there exist a point $x_0 \in D$, a neighbourhood $U \in \mathscr{U}(0)$ and a bounded set $B \subset Y$ such that

$$(4.4)\qquad G(x) \cap (B - K) \neq \emptyset, \quad \forall x \in x_0 + U.$$

Let $W \in \mathscr{W}(0)$ and $V \in \mathscr{W}(0)$ such that $V + V + V \subset W$. Since the sets B and $F(x_0)$ are bounded, there exists a diadic number $q \in (0, 1)$ such that

$$(4.5)\qquad qB \subset V \quad \text{and} \quad qF(x_0) \subset V.$$

Because of (4.4) it follows that

$$(4.6)\qquad 0 \in qG(x) - V + K \subset qF(x) + V + K, \quad \forall x \in x_0 + U.$$

Now, by using (4.5), (4.6) and Lemma 3.1 of [5], we have

$$F(x_0) \subset V + (1-q)\operatorname{co} F(x_0) + q \operatorname{co} F(x_0 + u) + V + K \subset$$
$$\subset V + \operatorname{cl}(F(x_0 + qu) + K) + V + K \subset F(x_0 + qu) + W + K, \quad \forall u \in U,$$

i.e. F is K-l.s.c. at x_0.
Then, by Theorem 1, we conclude that F is K-continuous on D.

As an immediate consequence of Theorem 3 we obtain the Corollary 2 and the Corollary 3, which improve respectively Corollary 3.3 and Corollary 3.4 of [5].

Corollary 2. *Let $F : D \to \mathbb{B}(Y)$ be a K-hull midconvex set-valued function weakly K-upper bounded on some subset of D with non-empty interior. Besides, if the set $F(x)$ is K-midconvex, $\forall x \in D$, then F is K-continuous on D.*

Corollary 3. *Let $F : D \to \mathbb{B}(Y)$ be a K-hull midconvex set-valued function such that the set $F(x)$ is K-midconvex, $\forall x \in D$. Besides, if F has a selection $f : D \to Y$ wich is K-upper bounded on a subset of D with non-empty interior, then F is K-continuous on D.*

Remark 3. The Theorem 2 and the Theorem 3 improve respectively the Theorem 3.3 and the Theorem 3.4 of [5]. Actually, a K-midconvex set-valued function F is K-hull midconvex and its values are K-midconvex subsets of Y; on the other hand, there exist set-valued functions F which verify the conditions of our theorems but which are not K-midconvex. An example is given by the same set-valued function $F : \mathbb{R} \to n(\mathbb{R}^2)$ studied in Remark 1.

Finally, it is easy to see that, in the particular case when Y is a locally bounded space, the Theorem 3 offers a new condition so that a set-valued function is K-continuous: we consequently have the following

Theorem 4. *Let Y be a locally bounded space, $F : D \to \mathbb{B}(Y)$ be a K-hull midconvex set-valued function such that the set $F(x)$ is K-midconvex, $\forall x \in D$. Then F is K-continuous on D iff there exists a set-valued function $G : D \to n(Y)$ weakly K-upper bounded on some subset of D with non-empty interior such that $G(x) \subset F(x) + K$, $\forall x \in D$.*

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