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A NEW CONTRIBUTION TO THE $W^{2,p}$ REGULARITY FOR A CLASS OF ELLIPTIC SECOND ORDER EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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In this paper we continue the study of a class of second order elliptic equations we began in [7] improving the assumptions on the lower order terms.

Introduction.

In this paper we continue the study we began in our previous work [7] of the Dirichlet problem for a class of elliptic second order operators with discontinuous coefficients. Precisely we establish an existence and uniqueness result in the class $W^{2,p} \cap W_0^{1,p}$ for equation

$$(*) \qquad \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f$$

assuming that the leading terms' coefficients a_{ij} are in the Sarason's space VMO and the lower order terms' coefficients are taken in suitable L^p spaces (for precise assumptions and definitions see Section 1).

In their paper [2] Chiarenza, Frasca and Longo recently obtained the same result in the case $b_i = c = 0$.

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In our work [7] we considered the complete equation $(*)$ but we had at least to assume $c \in L^n$ (see Theorem 2.3 in [7] for a more precise statement) in order to achieve the uniqueness and then the existence for the Dirichlet problem for equation $(*)$. This in turn because of the use we did of the Alexandrov-Pucci maximum principle (in this following the approach given in [2]). In the present paper we were able to make some more natural assumptions on c (see assumptions (2.4)). The result follows through a standard argument from a uniqueness result (Theorem 3.1) which in turn depends on a maximum principle that is also proved in Section 3 of this paper.

Finally we wish to express our thanks to Prof. E. Fabes for his encouragement and help in the proof of Theorem 3.1.

1. Some functional spaces.

We start this section by recalling the definitions of the spaces BMO and VMO.

We say that a locally integrable function f in $\mathbb{R}^n$ is in the space BMO if

$$\sup_B \fint_B |f(x) - f_B|\, dx = \|f\|_* < +\infty$$

where B ranges in the class of the balls in $\mathbb{R}^n$. Here f_B is the average $\fint_B f(x)dx$.

For $f \in \mathrm{BMO}$ and $r > 0$ we set

$$\sup_{\rho \le r} \fint_B |f(x) - f_B|\, dx = \eta(r) \tag{1.1}$$

where B ranges in the class of the balls with radius ρ.
We say that a function $f \in \mathrm{BMO}$ is in the space VMO (see [6]) if $\lim_{r \to 0^+} \eta(r) = 0$.
We will refer to $\eta(r)$ as the VMO modulus of f.

We will need for further developments the following known property of the space VMO (see e.g. [6], [3]).

Theorem 1.1. *For $f \in \mathrm{BMO}$ the following conditions are equivalent:*

(1) *f is in* VMO;
(2) *f is the* BMO *closure of the set of the uniformly continuous functions which belong to* BMO;
(3) $\lim_{y \to 0} \|f(x - y) - f(x)\|_* = 0$.

By this Theorem and a known result (see [3]) we have that if $f \in \mathrm{VMO}$, the usual mollifiers converge to f in the BMO norm. In other words, given any $f \in \mathrm{VMO}$ with VMO modulus $\eta(r)$, it is possible to find a sequence of C^∞ functions $\{f_h\}_{k\in \mathrm{N}}$ converging to f in BMO as $h \to 0$ and with their VMO moduli $\eta_h(r) \leq \eta(r)$.

Moreover, for $f \in L^p(\Omega)$, we set

$$\sup_{\substack{|E|\leq\sigma \\ E\subseteq\Omega}} \int_E |f(x)|^p\, dx = \omega^p(\sigma) \text{ (}^1\text{)}$$

Clearly $\omega(\sigma)$ is a decreasing function in $]0, |\Omega|]$ such that $\lim_{\sigma\downarrow 0} \omega(\sigma) = 0$. We will refer to $\omega(\sigma)$ as the AC modulus of $|f|^p$.

2. Notations, assumptions and main result.

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 3$, and $p \in]1, +\infty[$. We suppose that the boundary of Ω (denoted by $\partial\Omega$) belongs to $C^{1,1}$.
We consider the elliptic operator

$$L = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + c \tag{2.1}$$

and on the coefficients of L we make the following assumptions

$$\begin{cases} a_{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^n) & i, j = 1, \dots, n \\ a_{ij}(x) = a_{ji}(x) \quad i, j = 1, \dots, n & \text{a.e. in } \Omega \\ \exists \lambda > 0 : \lambda|\xi|^2 \geq \sum\limits_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda^{-1}|\xi|^2 & \text{a.e. in } \Omega \quad \forall \xi \in \mathbb{R}^N. \end{cases} \tag{2.2}$$

$$\begin{cases} b_i \in L^r(\Omega) \quad i = 1, \dots, n \quad \text{where} \quad r > n \quad \text{for} \quad 1 < p \leq n, \\ r = p \quad \text{for} \quad p > n. \end{cases} \tag{2.3}$$

$$c \in L^s(\Omega) \quad \text{with} \quad s > n/2 \quad \text{for} \quad p \in]1, n/2], \qquad s = p \quad \text{for} \quad p > n/2, \quad c \leq 0 \quad \text{a.e. in } \Omega. \tag{2.4}$$

In this paper our main result is the following theorem

(^{1}) If $E \subseteq \Omega$ is Lebesgue measurable we set $|E|$ for its Lebesgue measure.

Theorem 2.1. *Assume* (2.2), (2.3) *and* (2.4), $f \in L^p(\Omega)$ *with* $p \in]1, +\infty[$. *Then, for the Dirichlet problem*

$$\begin{cases} Lu = f & \text{a.e. in } \Omega \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \end{cases} \tag{2.5}$$

exists an unique solution u. *Furthermore there exists a positive constant* K *such that*

$$\|u\|_{W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)} \leq K\|f\|_{L^p(\Omega)}. \tag{2.6}$$

Here the constant K *depends an* n, p, $\partial\Omega$, λ, *on the* VMO *moduli of* a_{ij} $(i, j = 1, \ldots, n)$ *and on the* L^r *and* L^s *norms respectively of the coefficients* b_i $(i = 1, \ldots, n)$ *and* c *and their* AC *moduli.*

For the proof of Theorem 2.1 we will need the following results which have been proved in [7].

Theorem 2.2. *Assume* (2.2), (2.3) *and* (2.4). *Let* $q, p \in]1 + \infty[$, $q \leq p$, $f \in L^p(\Omega)$. *Then for any solution* u *of problem*

$$\begin{cases} Lu = f & \text{a.e. in } \Omega \\ u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \end{cases}$$

we have

$$u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Furthermore there exists a positive constant c_3 *such that*

$$\|u\|_{W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)} \leq c_3\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right). \tag{2.7}$$

Here c_3 *depends on* n, p, $\partial\Omega$, λ, *on the* VMO *moduli of* a_{ij}, $i, j = 1, \ldots, n$, *on the norms and* AC *moduli of* b_i, $i = 1, \ldots, n$, *and* c.

Theorem 2.3. *Assume* (2.2), (2.3) *and* $c = 0$; $f \in L^p(\Omega)$ *with* $p \in]1, +\infty[$. *Then the Dirichlet problem* (2.5) *has a unique solution* u. *Furthermore there exists a positive constant* c_4 *such that*

$$\|u\|_{W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)} \leq c_4\|f\|_{L^p(\Omega)}. \tag{2.8}$$

Here c_4 *depends on* n, p, $\partial\Omega$, λ, *on the* VMO *moduli of* a_{ij}, $i, j = 1, \ldots, n$, *on the norms and* AC *moduli of* b_i, $i = 1, \ldots, n$.

3. Preliminary results. A maximum principle.

In this section we develop some tools which we will need in the proof of Theorem 2.1. We start introducing the Green function for the operator L with $c = 0$. More precisely for $f \in L^p$, $p > n/2$, we consider the Dirichlet problem

$$(3.1)\qquad \begin{cases} L'z = \sum_{i,j=1}^{n} a_{ij}(x) \dfrac{\partial^2 z}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \dfrac{\partial z}{\partial x_i} = f \quad \text{a.e. in } \Omega \\ z \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \end{cases}$$

By Theorem 2.3 problem (3.1) has a unique solution z and the following estimate holds

$$\max_{\overline{\Omega}} |z| \leq C \|f\|_{L^p(\Omega)}$$

where the constant C is of the same kind of constant c_4 in Theorem 2.3.
Then, for all x in Ω, the map $f \to z(x)$ is a bounded linear functional in $L^p(\Omega)$. Therefore, by the Riesz representation theorem there exists $g(x, \cdot) \in L^{p'}(\Omega)$, $p' = p/(p-1)$, such that

$$(3.2)\qquad z(x) = -\int_{\Omega} g(x, y) f(y)\, dy\,.$$

$g(x, y)$ is the Green's function for operator L' in Ω.

By an approximation argument of L' with smooth operators it can be shown that $g(x, y)$ is positive (see e.g. [1]).

By approximation one can also prove the following two theorems (Maximum principle and Harnack inequality). We will give the details of the proof only for the first of these theorems.

Maximum Principle. *Assume* (2.2) *and* (2.3). *Let B a ball, $B \subseteq \Omega$; h and v two functions in $W^{2,p}(B)$, $p \in]n/2, +\infty[$, $h \geq 0$, where v solves the problem*

$$(3.3)\qquad \begin{cases} L'v = \sum_{i,j=1}^{n} a_{ij} \dfrac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \dfrac{\partial v}{\partial x_i} = 0 \quad \text{in } B \\ v_{/\partial B} = h_{/\partial B} \end{cases}$$

Then we have $v \geq 0$ in B.

Proof. We start by transforming the problem (3.3) into problem

$$(3.3') \qquad \begin{cases} L'(v-h) = -L'h \\ (v-h) \in W^{2,p}(B) \cap W_0^{1,p}(B) \end{cases}$$

which has a unique solution by Theorem 2.3.

Moreover, for the solution of the problem (3.3′) we have the following estimate

$$(3.4) \qquad \|v-h\|_{W^{2,p}(B)\cap W_0^{1,p}(B)} \le K\|L'h\|_{L^p(B)}$$

where the positive constant K depends only on n, p, $\partial\Omega$, λ, on the VMO moduli η_{ij} of a_{ij} $(i, j = 1, \dots, n)$ and on the L^r norm and AC moduli of b_i $(i = 1, \dots, n)$.
By (3.4) we obtain:

$$(3.5) \qquad \|v-h\|_{W^{2,p}(B)\cap W_0^{1,p}(B)} \le \\ \le K\left(S\|\,|b|\,\|_{L^r(B)}\|h\|_{W^{2,p}(B)} + \lambda\|h\|_{W^{2,p}(B)}\right)$$

where S is Sobolev's constant and $|b| = \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$.
Finally by (3.5) we have

$$(3.6) \qquad \|v\|_{W^{2,p}(B)} \le K_1\|h\|_{W^{2,p}(B)}$$

where the positive constant K_1 is the same kind of K.
Recalling the remarks following Theorem 1.1 we can find a sequence of smooth functions $\left\{a_{ij}^{(k)}\right\}_{k\in\mathbb{N}}$ converging to a_{ij} in L^p for all p in $]1, +\infty[$ satisfying (2.2) and with VMO moduli uniformly bounded by η_{ij}. Moreover we can consider sequences $\left\{b_i^{(k)}\right\}_{k\in\mathbb{N}}$, $\left\{\zeta^{(k)}\right\}_{k\in\mathbb{N}}$ of smooth functions converging to b_i and h respectively in the relevant spaces with the AC moduli of $\left\{b_i^{(k)}\right\}_{k\in\mathbb{N}}$ uniformly bounded by those of b_i and satisfying

$$\left\|b_i^{(k)}\right\|_{L^r} \le \left\|b_i\right\|_{L^r} \quad \forall i = 1, \dots, n;\ \forall k \in \mathbb{N},$$

$$\left\|\zeta^{(k)}\right\|_{W^{2,p}} \le \|h\|_{W^{2,p}} \quad \forall k \in \mathbb{N}.$$

Also let

$$L'^{(k)} = \sum_{i,j=1}^{n} a_{ij}^{(k)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^{(k)} \frac{\partial}{\partial x_i}.$$

Because of the smoothness of the coefficients, the problem

$$\begin{cases} L'^{(k)} v^{(k)} = 0 & \text{in } B \\ v^{(k)}_{/\partial B} = \zeta^{(k)}_{/\partial B} \end{cases} \tag{3.7}$$

admit a unique solution $v^{(k)}$ and also we have, for the classical weak maximum principle, $v^{(k)} \geq 0$ (see e.g. [4]).
Furthermore by (3.6), for $v^{(k)}$ we have

$$\left\| v^{(k)} \right\|_{W^{2,p}(B)} \leq K_1 \left\| \zeta^{(k)} \right\|_{W^{2,p}(B)} \tag{3.8}$$

and then, from (3.8) we obtain

$$\left\| v^{(k)} \right\|_{W^{2,p}(B)} \leq K \|h\|_{W^{2,p}(B)} \quad \forall k \in \mathbb{N}. \tag{3.9}$$

Recalling that $p > n/2$, by Sobolev lemma, $\left\{v^{(k)}\right\}_{k\in\mathbb{N}}$ is also bounded in $C^{0,\alpha}$. Then from (3.9) we have that there exists a subsequence of $\left\{v^{(k)}\right\}_{k\in\mathbb{N}}$, which we still call $\left\{v^{(k)}\right\}_{k\in\mathbb{N}}$, weakly convergent in $W^{2,p}(B)$ and strongly convergent to v' in $W^{1,q}(B)$, $1 < q < p^* = \frac{np}{n-p}$, and, by the Ascoli-Arzelà theorem the sequence $\left\{v^{(k)}\right\}_{k\in\mathbb{N}}$ converges to $v' \geq 0$ in $C^0(\overline{B})$. Also we observe that $\left\{v^{(k)} - \zeta^{(k)}\right\}_{k\in\mathbb{N}}$ belongs to $W_0^{1,p}(B)$ and converges to $v' - h$ in $W_0^{1,p}(B)$.

We wish now to show that $Lv' = 0$. By the uniqueness for problem (3.3′) (and then (3.7)) the conclusion will follow.

Consider e.g. the case $n/2 < p < n$. Then for $\zeta \in L^{p'}(B)$, $p' = p/(p-1)$, we have

$$\begin{aligned} \int_B \left|(L^{(k)} v^{(k)} - L'v')\zeta\right| dx \leq \int_B \left| \sum_{i,j=1}^{n} (v_{x_i x_j}^{(k)} - v'_{x_i x_j}) a_{ij}\, \zeta \right| dx + \\ + \sum_{i,j=1}^{n} \left\| v_{x_i x_j}^{(k)} \right\|_{L^p(B)} \cdot \left\| (a_{ij}^{(k)} - a_{ij})\zeta \right\|_{L^{p'}(B)} + \\ + \sum_{i=1}^{n} \left\| v_{x_i}^{(k)} - v'_{x_i} \right\|_{L^q(B)} \cdot \left\| b_i^{(k)} \right\|_{L^r(B)} \cdot \left\| \zeta \right\|_{L^{p'}(B)} + \\ + \sum_{i=1}^{n} \left\| b_i^{(k)} - b_i \right\|_{L^n(B)} \cdot \left\| \zeta \right\|_{L^{p'}(B)} \cdot \left\| v_{x_i}^{(k)} \right\|_{L^{p^*}(B)}. \end{aligned} \tag{3.10}$$

(3.10) implies that $\left\{L^{(k)} v^{(k)}\right\}_{k\in\mathbb{N}}$ weakly converges in L^p to $L'v'$. □

Harnack's inequality. *Assume* (2.2), (2.3). *Let* $v \in W^{2,p}(B)$, $p \in]n/2, +\infty[$, *the solution of the problem* (3.3) *and let* $\overline{B}_{2r} \subseteq B$. *Then*

$$\sup_{x \in B_r} v(x) \leq K_4 \inf_{x \in B_r} v(x) \tag{3.11}$$

where the positive constant K_4 *depends only on* λ *and* n.

Proof. We observe that Harnack's inequality holds true (see [5]) for $v^{(k)}$, solutions to the problems (3.7), with constants independent of k and of the regularity of the coefficients. Then using, as in the previous theorem, a compacteness argument we obtain the conclusion for v. □

We are now in position to give the following theorem

Theorem 3.1. *Assume* (2.2), (2.3) *and* (2.4). *Then the solution of the Dirichlet problem*

$$\begin{cases} Lu = 0 \quad \text{a.e. in } \Omega \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \qquad p \in]1, +\infty[\end{cases}$$

is 0 *in* Ω.

Proof. Assume that $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $1 < p < +\infty$, solves the equation $Lu = 0$ a.e. in Ω. Then by Theorem 2.2 we have $u \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)$, $s > n/2$. In particular we have $u \in C^0(\overline{\Omega})$.
Let $M = \max_{\overline{\Omega}} u \geq 0$. To prove our result we argue by contradiction, precisely we suppose $M > 0$. Then they exist an $x_0 \in \Omega$ and $r_0 > 0$ such that

$$u(x_0) = M \quad \text{and} \quad u(x) > 0 \quad \text{for any } x \in B_{r_0}(x_0).$$

Let $0 \leq g_{B_{r_0}}(x, y) \equiv g$ the Green's function for

$$L' = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} \qquad \text{in } B_{r_0}(x_0).$$

Then, by (3.2), we have in $B_{r_0}(x_0)$

$$M - u(x) = -\int_{B_{r_0}(x_0)} g(x, y)\, (L'(M - u)(y))\, dy + v_{r_0}(x)\,, \tag{3.12}$$

where $v_{r_0}(x)$ solves the problem

$$\begin{cases} L' v_{r_0} = 0 \quad \text{in } B_{r_0}(x_0) \\ v_{r_0/\partial B} = (M - u)_{/\partial B} \end{cases}.$$

Moreover, by the maximum principle we have

$$v_{r_0}(x) \geq 0 \quad \text{in } B_{r_0}(x_0).$$

Using now the equation

$$L'(M-u) = -L'u = cu \quad \text{in } B_{r_0}(x_0),$$

(3.12) can be written as

$$M - u(x) = -\int_{B_{r_0}(x_0)} g(x,y)c(y)u(y)\,dy + v_{r_0}(x)\,. \tag{3.13}$$

Also for $x = x_0$

$$0 \leq v_{r_0}(x_0) = \int_{B_{r_0}(x_0)} g(x_0,y)c(y)u(y)\,dy\,.$$

Because $c(y)u(y) \leq 0$ in $B_{r_0}(x_0)$ and $g(x_0,y) > 0$ a.e. in $B_{r_0}(x_0)$ we have $c(y)u(y) = 0$ a.e. in $B_{r_0}(x_0)$ and then $v_{r_0}(x_0) = 0$. This in turn implies, by Harnack's inequality, $v_{r_0}(x) \equiv 0$ in $B_{r_0}(x_0)$ and from (3.13) we obtain $u \equiv M$ in $B_{r_0}(x_0)$. From this easily a contradiction follows because $u = 0$ on $\partial\Omega$.

Moreover, because $-u$ solves the equation $Lu = 0$ a.e. in Ω we obtain $\max_{\overline{\Omega}}(-u) = -\min_{\overline{\Omega}} u = 0$, then u is 0 in Ω. □

Proof of Theorem 2.1. By Theorem 3.1 uniqueness immediately follows. Then it is quite standard to prove the existence by Theorem 2.2 (getting rid first of the $\|u\|$ term on the right hand side of (2.7)) and then using an approximation argument as in Theorem 2.4 of [7]. □

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