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A NEW APPROACH TO SOME TRACE THEOREMS

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We prove a Sobolev-Adams weighted imbedding using an idea of E.M. Stein.

1. Introduction.

In the work [1] D.R. Adams proved that the necessary and sufficient condition for the continuous imbedding of $L^p(\mathbb{R}^n; dx)$ into $L^r(\mathbb{R}^n; d\mu)$, $1 < p < r$ for the Riesz potential operator $\big(I_\alpha f\big)(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, dy$, $0 < \alpha < n$ is the boundedness of

$$\mathscr{M}(x) = \sup_{\varrho>0} \varrho^{-s} \mu\big(B_\varrho(x)\big),$$

where $s = r\big(\frac{n}{p} - \alpha\big)$ and $B_\varrho(x) = \{y \in \mathbb{R}^n : |x-y| < \varrho\}$ (or in the other words that μ belongs to a classical Morrey space $L^{1,\delta}$). The problem of finding a complete caracterization of those measures μ such that I_α is bounded from $L^p(\mathbb{R}^n; dx)$ to $L^r(\mathbb{R}^n; d\mu)$ (including the *difficult* case $p = r$) was settled many years later by Kerman and Sawyer in [2].

One of their conditions is

$$\bigg(\int_{\mathbb{R}^n} \Big(\int_B \frac{d\mu}{|x-y|^{n-\alpha}}\Big)^{p'} dy\bigg)^{\frac{1}{p'}} \le C_0\big(\mu(B)\big)^{\frac{1}{r'}}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \ \frac{1}{r} + \frac{1}{r'} = 1.$$

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In this paper we prove directly (Theorem 1) that the Kerman–Sawyer assumption above implies the Adams hypotesis; the complete equivalence, when $r > p$, is proved in Remark 2. Also in Theorem 1 we give a simple proof that the Kerman–Sawyer assumption implies the weak type (p, r) for the operator I_α. The idea of the proof has been suggested to us by Stein's proof of Sobolev imbedding. As a tool we use (Lemma 1) a generalization of Schur's lemma [5] which we feel is of some interest in itself.

2. Preliminaries.

Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ be σ-finite measures spaces and $p, q \in [1, \infty]$, we recall that an operator T defined on $L^p(\mu)$ to the space of measurable functions on Y is said to be of strong type (p, q) if there exists a constant C such that

$$\|T(f)\|_q \leq C\|f\|_p \ (^1).$$

Similarly an operator T is said to be of weak type (p, r), $p \in [1, \infty]$ and $r \in [1, \infty[$, if there exists a constant C such that for any $\tau > 0$

$$\nu\big(\{x : (Tf)(x) > \tau\}\big) \leq C\left(\frac{\|f\|_p}{\tau}\right)^r$$

where $\nu\big(\{x : (Tf)(x) > \tau\}\big)$ is the "distribution function" of Tf. We also set $\lambda_f(\tau) \equiv \mu\big(\{x : f(x) > \tau\}\big)$ and recall that $L^r_w(\mu)$, $1 \leq r < \infty$, is the space of measurable functions for which "the weak norm"

$$[f]_r \equiv \Big(\sup_{\tau>0} \tau^r \mu\big(\{x : f(x) > \tau\}\big)\Big)^{\frac{1}{r}} < \infty .$$

If $0 < \lambda < n$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we set

$$\|f\|_{1,\lambda} = \sup_{\varrho>0,\ x\in\mathbb{R}^n} \frac{1}{\varrho^\lambda} \int_{B_\varrho(x)} |f(y)|\, dy .$$

The Morrey space $L^{1,\lambda} = L^{1,\lambda}(\mathbb{R}^n)$ is the subset of $L^1_{\text{loc}}(\mathbb{R}^n)$ for which $\|f\|_{1,\lambda}$ is finite.

(^{1}) We denote with $\|\cdot\|_r$ the usual norm in $L^r(\mu)$ space.

3. Results.

Lemma 1. *Suppose* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *are* σ*-finite measure spaces and* $t > 1$. *Let* K *be a real measurable non negative function on* $X \times Y$ *such that, for some* M_0 *and* M_1, *we have*

$$\left(\int_Y \left(\int_X K(x, y)\, d\mu(x)\right)^t d\nu(y)\right)^{\frac{1}{t}} \leq M_0 \quad \text{if} \quad 1 < t < \infty,$$

$$\int_X K(x, y)\, d\mu(x) \leq M_0 \quad \text{for a.e. } y \in Y \quad \text{if} \quad t = \infty,$$

$$\int_Y K(x, y)\, d\nu(y) \leq M_1 \quad \text{for a.e. } x \in X.$$

If $1 < p \leq \infty$ *and* $f \in L^p(\nu)$ *the integral*

$$Tf(x) = \int_Y K(x, y) f(y)\, d\nu(y)$$

converges absolutely for a.e. $x \in X$. *Moreover exists a constant* C_p *independent of* K *such that*

$$\|Tf\|_s \leq C_p \|f\|_p \quad \text{for } s = p\left(1 - \frac{1}{t}\right)$$

and $C_p = M_0^{\frac{1}{s}} M_1^{\frac{1}{s'}}$, $(s' = \frac{s}{s-1})$. *Hence the operator* T *thus defined is of the strong type* (p, s).

Proof. If $1 < t < \infty$ we have

$$\int_X \left(\left|\int_Y K(x, y) f(y)\, d\nu(y)\right|\right)^s d\mu(x) \leq$$

$$\leq \int_X \left(\int_Y (K(x, y))^{\frac{1}{s'}} (K(x, y))^{\frac{1}{s}} |f(y)|\, d\nu(y)\right)^s d\mu(x) \leq$$

$$\leq M_1^{\frac{s}{s'}} \int_X \left(\int_Y K(x, y) |f(y)|^s\, d\nu(y)\right) d\mu(x) =$$

$$= M_1^{\frac{s}{s'}} \int_Y \left(\int_X K(x, y) |f(y)|^s\, d\mu(x)\right) d\nu(y) \leq M_1^{\frac{s}{s'}} M_0 \|f\|_p^s.$$

We observe that in the particular case $t = \infty$ this is Schur's lemma (see [5]).□

Theorem 1. *Let μ be a σ-finite measure in $\mathbb{R}^n$, $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $p \leq r < \infty$. Suppose that there exists a constant C_0 such that for any ball $B \subset \mathbb{R}^n$*

$$\left(\int_{\mathbb{R}^n}\left(\int_B \frac{d\mu(x)}{|x-y|^{n-\alpha}}\right)^{p'} dy\right)^{\frac{1}{p'}} \leq C_0\big(\mu(B)\big)^{\frac{1}{r'}} \tag{1}$$

where $p' = \frac{p}{p-1}$ and $r' = \frac{r}{r-1}$. Then there exists a constant $\widetilde{C}_0$ such that

$$\big(\mu(\{x \in \mathbb{R}^n : |x-y| < A^{\frac{1}{\alpha-n}}\})\big)^{\frac{1}{r}} \leq \widetilde{C}_0 A^{-1+\frac{n}{p'(n-\alpha)}}, \ \forall A > 0. \tag{2}$$

Moreover the Riesz potential

$$(Tf)(x) = (I_\alpha f)(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

maps L^p into $L^r_w(\mu)$ and

$$[I_\alpha f]_r \leq C\|f\|_p\,,$$

where C is a constant independent on f.

Proof. From (1) we obtain

$$\left(\int_B\left(\int_B \frac{d\mu(x)}{|x-y|^{n-\alpha}}\right)^{p'} dy\right)^{\frac{1}{p'}} \leq C_0\big(\mu(B)\big)^{\frac{1}{r'}}\,,$$

for all balls B. If ϱ is the radius of B, we have $|x-y| < 2\varrho$. Then we have

$$(2\varrho)^{\alpha-n}\mu(B)|B|^{\frac{1}{p'}} \leq C_0\big(\mu(B)\big)^{\frac{1}{r'}}$$

and finally

$$\omega_n^{\frac{1}{p'}} 2^{-\frac{n}{p'}}(2\varrho)^{\alpha-n+\frac{n}{p'}}\big(\mu(B)\big)^{1-\frac{1}{r'}} \leq C_0$$

where ω_n is the volume of the unit ball. From the previous estimates we have

$$\big(\mu(B)\big)^{\frac{1}{r}}(2\varrho)^{\alpha-n+\frac{n}{p'}} \leq 2^{\frac{n}{p'}} C_0 \omega_n^{-\frac{1}{p'}}.$$

Let $\varrho = \frac{A^{\frac{1}{\alpha-n}}}{2}$, $A > 0$ we obtain

$$\big(\mu(B)\big)^{\frac{1}{r}} \leq \widetilde{C}_0 A^{-1+\frac{n}{p'(n-\alpha)}},$$

where $\widetilde{C}_0 = 2^{\frac{n}{p'}} C_0 \, \omega_n^{-\frac{n}{p'}}$.

Following the idea of Stein's proof of Sobolev imbedding theorem (see [6], pg.120) let T_1 and T_2 be the integral operators with kernels K_1 and K_2

$$K_1(x, y) \equiv \left(\frac{1}{|x-y|^{n-\alpha}} - A \right) \chi_E ,$$

where χ_E is the characteristic function of the set $E = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| < A^{\frac{1}{\alpha-n}}\}$, and

$$K_2(x, y) \equiv \frac{1}{|x-y|^{n-\alpha}} - K_1(x, y) .$$

By Hölder's inequality

$$\|T_2 f\|_\infty \leq C_1 A^{\frac{\left(p' - \frac{n}{n-\alpha}\right)}{p'}} \|f\|_p ,$$

where $C_1 = C_1(n, \alpha, p)$.

Also, given $0 < \tau < \infty$, we choose $A = \left(C_1^{-1} \|f\|_p^{-1} \frac{\tau}{2}\right)^{\frac{p'}{\left(p' - \frac{n}{n-\alpha}\right)}}$, then from Lemma 1 with $t = p'$ and $M_0 = C_0 \big(\mu(B)\big)^{\frac{1}{r'}}$, $B = \{x \in \mathbb{R}^n : |x-y| < A^{\frac{1}{\alpha-n}}\}$

$$\begin{aligned}
\lambda_{T_1 f}\left(\frac{\tau}{2}\right) &\leq \frac{2}{\tau} \|T_1 f\|_1 \leq \frac{2}{\tau} M_0 \|f\|_p = \frac{2}{\tau} C_0 \big(\mu(B)\big)^{\frac{1}{r'}} \|f\|_p \leq \\
&\leq 2C_0 \|f\|_p \tau^{-1} \left(\widetilde{C}_0 \, A^{-1+\frac{n}{p'(n-\alpha)}} \right)^{\frac{r}{r'}} = \\
&= 2^{1+\frac{pn}{p'r'}} \omega_n^{-\frac{nr}{p'r'}} C_0^r \|f\|_p \tau^{-1} \left[\left(C_1^{-1} \|f\|_p^{-1} \frac{\tau}{2} \right)^{\frac{p'}{p' - \frac{n}{n-\alpha}}} \right]^{-\frac{r}{r'} + \frac{rn}{r'p'(n-\alpha)}} = \\
&= \left[2^{1+\frac{pn}{p'r'}} \omega_n^{-\frac{nr}{p'r'}} C_0^r \left(\frac{1}{2C_1} \right)^{-\frac{r}{r'}} \right] \tau^{-1-\frac{r}{r'}} \|f\|_p^r = C \tau^{-r} \|f\|_p^r
\end{aligned}$$

where we used (2) to estimate $\mu(B)$.

It follows

$$\lambda_{Tf}(\tau) \leq \lambda_{T_1 f}\left(\frac{\tau}{2}\right) + \lambda_{T_2 f}\left(\frac{\tau}{2}\right) \leq C \tau^{-r} \|f\|_p^r$$

i.e.

$$[I_\alpha f]_r \leq C \|f\|_p . \qquad \square$$

Remark 1. *Let $\mu \in L^{1,\lambda}(\mathbb{R}^n)$ with $0 < \alpha < n$, $0 \le \lambda < n - \alpha$, $\frac{n-\lambda}{\alpha} < p < \frac{n}{\alpha}$ and $r = \frac{p\lambda}{n-\alpha p}$. Then it exists a constant C_0' such that for the ball $B = \{x \in \mathbb{R}^n : |x-y| < \varrho\}$, where $y \in \mathbb{R}^n$ and $\varrho > 0$, we have*

$$\left(\int_{\mathbb{R}^n}\left(\int_B \frac{d\mu(x)}{|x-y|^{n-\alpha}}\right)^{p'} dy\right)^{\frac{1}{p'}} \le C_0'\big(\mu(B)\big)^{\frac{1}{r'}}$$

where $C_0' = C_0'(n, p, \alpha, \lambda, \|\mu\|_{1,\lambda})$.

Proof. Set, for any $y \in \mathbb{R}^n$ and $\varrho > 0$, $I_\alpha'\mu = I_\alpha(\mu\chi_{B(y,\varrho)})$ from [1] $I_\alpha'\mu \in L_w^{\frac{n-\lambda}{n-\lambda-\alpha}}$ and

$$[I_\alpha'\mu]_{\frac{n-\lambda}{n-\lambda-\alpha}} \le C_1'(n,\alpha,\lambda)\|\mu\|_{1,\lambda}^{\frac{\alpha}{n-\lambda}}\mu(B)^{\frac{n-\lambda-\alpha}{n-\lambda}}.$$

Because $\mu(B) < +\infty$ we also have $I_\alpha'\mu \in L_w^{\frac{n}{n-\alpha}}$ and

$$[I_\alpha'\mu]_{\frac{n}{n-\alpha}} \le C_2'(n,\alpha)\,\mu(B)$$

(see [6], pg. 120). Then for all $p \in \left]\frac{n-\lambda}{\alpha}, \frac{n}{\alpha}\right[$ we have

$$\|I_\alpha'\mu\|_{p'} \le \left[I_\alpha'\mu\right]^{\vartheta}_{\frac{n-\lambda}{n-\lambda-\alpha}} \left[I_\alpha'\mu\right]^{1-\vartheta}_{\frac{n}{n-\alpha}}$$

where ϑ is such that

$$\frac{1}{p'} = \frac{1-\vartheta}{\frac{n}{n-\alpha}} + \frac{\vartheta}{\frac{n-\lambda}{n-\lambda-\alpha}}$$

(see e.g. [4], pg.236) i.e.

$$\vartheta = \left(\frac{n-\lambda}{\lambda}\right)\left(\frac{n-\alpha p}{\alpha p}\right).$$

Then

$$\|I_\alpha'\mu\|_{p'} \le \left(C_1'(n,\alpha,\lambda)\|\mu\|_{1,\lambda}^{\frac{\alpha}{n-\lambda}}\big(\mu(B)\big)^{1-\frac{\alpha}{n-\lambda}}\right)^{\vartheta}\left(C_2'(n,\alpha)\mu(B)\right)^{1-\vartheta} =$$
$$= C_0'\big(\mu(B)\big)^{\frac{1}{r'}}. \qquad \square$$

Corollary. *Let α, μ, p, r be as in the previous remark, then $I_\alpha f$ is strong type (p, r).*

Proof. Observing that if p_1, p_2 are two real numbers such that $1 < p_1 < p < p_2$ we obtain from Remark 1 and Theorem 1 that I_α is weak type (p_1, r_1) and (p_2, r_2), with $r_1 = \frac{\lambda}{\frac{n}{p_1}-\alpha} < r < r_2 = \frac{\lambda}{\frac{n}{p_2}-\alpha}$ so it follows from Marcinkiewicz theorem that $I_\alpha f$ is strong type (p, r). More precisely if $[I_\alpha f]_{r_i} \le C_i \|f\|_{p_i}$ $i = 1, 2$, then $\|I_\alpha f\|_r \le C\|f\|_p$ where C depends only on p_i, r_i, C_i in addition to p. □

Remark 2. *Let α, λ, p, r as in Remark 1. The condition* (1) *of Theorem 1 is equivalent to the strong type (p, r) for $I_\alpha(f)$.*

Proof. We observe that from the hypoteses (1) of Theorem 1 it follows $\mu \in L^{1,\lambda}$, $\lambda = r(\frac{n}{p} - \alpha)$. In fact (1) implies (2) and then

$$\mu\{x \in \mathbb{R}^n : |x - y| < A^{\frac{1}{n-\alpha}}\} \le \widetilde{C}_0^r \left(A^{\frac{1}{\alpha-n}}\right)^{(\alpha-n)(-r+\frac{nr}{p'(n-\alpha)})}$$

where

$$\begin{aligned}(\alpha - n)\left(-r + \frac{nr}{p'(n-\alpha)}\right) &= r(n-\alpha) - \frac{nr}{p'} = \\ &= r\left(n - \alpha - n\left(1 - \frac{1}{p}\right)\right) = r\left(\frac{n}{p} - \alpha\right).\end{aligned}$$

From this fact and the Remark 1 we have that (1) is equivalent to $\mu \in L^{1,\lambda}$, $\lambda = r(\frac{n}{p} - \alpha)$. But from Adams theorem (see e.g. [3] pg. 52), $\mu \in L^{1,\lambda}$, $\lambda = r(\frac{n}{p} - \alpha)$ is equivalent to the strong type (p, r) for $I_\alpha(f)$, then (1) is equivalent to be $I_\alpha(f)$ strong type (p, r). □

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