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DIOPHANTINE APPROXIMATIONS AND CONVERGENCE OF SERIES IN BANACH SPACES

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In this paper we give a new proof of a known diophantine approximation result, then we apply this to prove convergence of a class of series in a Banach space, whose terms are defined recursively.

Introduction.

Let $\mathscr{F}$ be the class of functions $f : [0, +\infty[\longrightarrow [0, 1]$ and $\mathscr{F}_T$ the subset of $\mathscr{F}$ of the periodical functions of period T. Let $\mathscr{B}$ be a real Banach space. $\forall \lambda \in \mathscr{B}$ and $\forall \varphi \in \mathscr{F}$ we denote by $\sum_\lambda^\varphi$ the series (in $\mathscr{B}$) whose terms are defined recursively by

$$\begin{cases} a_1 = \lambda \\ a_{n+1} = \varphi(n)\, a_n \qquad \forall\, n \in \mathbb{N}. \end{cases}$$

As it is easy to prove the Kronecker's theorem (see, for example, [4], p. 373) implies that, given $a, b \in \mathbb{R}^+$ ($\frac{a}{b} \notin \mathbb{Q}$), $c \in \mathbb{R}$, then $\forall \varepsilon > 0$ there exist two sequences $\{h_n\}$ and $\{k_n\}$ of integers such that

$$|h_n a - k_n b + c| < \varepsilon\,.$$

An interesting property of the sequences $\{h_n\}$ and $\{k_n\}$ is that they have bounded gap (i.e. there exists $p \in \mathbb{N}$ such that $h_{n+1} - h_n \leq p$, $k_{n+1} - k_n \leq p$,

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$\forall\ n \in \mathbb{N}$). This is equivalent to say that the set $\mathscr{A} = \{(h_n, k_n)\}$ is syndetic (see [3] Theorem 1.15 and Lemma 1.25). In the first section we give a new simple proof of this property. In the second section we apply this result to prove the convergence of the series $\sum_\lambda^\varphi$. In doing this we also utilize a general convergence theorem that we have proved to hold in $\mathscr{B}$ (Theorem 2.1). Other results complete the section. At the end some examples are given to explain the theory.

1. Diophantine approximation.

We begin proving the following preliminary result

Lemma 1.1. *Let $a, b \in \mathbb{R}^+$, $a > b$, $c \in \mathbb{R}$. Furthermore let $h, k \in \mathbb{N}$ such that $|ha - kb + c| < a - b$. Then there exists $\bar{h} \in \mathbb{N}$, depending only on a and b, such that:*

1) *if $ha - kb + c > 0$ then $|(h + \bar{h} - 1)a - (k + \bar{h})b + c| < a - b$;*
2) *if $ha - kb + c \leq 0$ then $|(h + \bar{h})a - (k + \bar{h} + 1)b + c| < a - b$.*

Proof. Let $\bar{h}$ be the lowest natural number such that $\bar{h}(a - b) \geq b$. From this it follows

$$\bar{h}(a - b) = b + \gamma' \qquad \text{with} \quad 0 \leq \gamma' < a - b\,. \tag{1}$$

Now we put

$$ha - kb + c = \gamma \tag{2}$$

and distinguish two cases.

1° case: $\gamma > 0$. From (1) and (2) we have

$$(h + \bar{h})a - (k + \bar{h})b + c = b + \gamma + \gamma'$$

from which

$$b < (h + \bar{h})a - (k + \bar{h})b + c < a + a - b$$

hence

$$-(a - b) < (h + \bar{h} - 1)a - (k + \bar{h})b + c < a - b\,,$$

and therefore the thesis.

2° case: $\gamma \leq 0$. From (1) and (2) we have again

$$(h + \bar{h})a - (k + \bar{h})b + c = b + \gamma + \gamma'$$

from which

$$b+\gamma \le (h+\bar{h})a-(k+\bar{h})b+c \le b+\gamma'$$

hence

$$-(a-b) < \gamma \le (h+\bar{h})a-(k+\bar{h}+1)b+c \le \gamma' < a-b\,,$$

and this completes the proof. □

Theorem 1.1. *Let $a, b \in \mathbb{R}^+$ ($\frac{a}{b} \notin \mathbb{Q}$), $c \in \mathbb{R}$. Then $\forall \varepsilon > 0$ there exist a natural number p, depending only on a, b, ε, and two sequences of natural numbers $\{h_n\}$ and $\{k_n\}$, depending only on a, b, c, ε, one not decreasing and the other increasing such that $\forall n \in \mathbb{N}$ it results*

$$h_{n+1}-h_n \le p, \qquad k_{n+1}-k_n \le p$$

and

$$|h_n a - k_n b + c| < \varepsilon\,.$$

Proof. For the Kronecker's theorem there exist $h, k \in \mathbb{N}$, depending only on a, b, ε, such that $0 < |ha-kb| < \varepsilon$. Let us suppose at first $ha-kb > 0$. Again for the Kronecker's theorem there exist $h^*, k^* \in \mathbb{N}$, depending only on a, b, c, ε, such that

$$|h^*(ha)-k^*(kb)+c| < ha-kb\,.$$

By virtue of the Lemma 1.1 there exists $\bar{h} \in \mathbb{N}$ depending only on ha, kb (and hence only on a, b, ε) such that, setting

$$h'_1 = h^*, \qquad k'_1 = k^*$$

and $\forall n \in \mathbb{N}$

$$h'_{n+1} = \begin{cases} h'_n+\bar{h}-1 & \text{if} \quad h'_n(ha)-k'_n(kb)+c > 0 \\ h'_n+\bar{h} & \text{if} \quad h'_n(ha)-k'_n(kb)+c \le 0 \end{cases}$$

$$k'_{n+1} = \begin{cases} k'_n+\bar{h} & \text{if} \quad h'_n(ha)-k'_n(kb)+c > 0 \\ k'_n+\bar{h}+1 & \text{if} \quad h'_n(ha)-k'_n(kb)+c \le 0 \end{cases}$$

it results (proceeding inductively)

$$|h'_n(ha)-k'_n(kb)+c| < ha-kb < \varepsilon \qquad \forall n \in \mathbb{N}\,.$$

At this point, setting

$$p = \max(h\bar{h}, k(\bar{h}+1))$$

and

$$h_n = h'_n h, \qquad k_n = k'_n k \qquad \forall n \in \mathbb{N},$$

we obtain two sequences $\{h_n\}$ and $\{k_n\}$, the first not decreasing and the second increasing, that verify all the conditions of the thesis.

If, otherwise, it is $ha - kb < 0$, for the Kronecker's theorem again, there exist $h^*, k^* \in \mathbb{N}$, depending only on a, b, c, ε, such that

$$|k^*(kb) - h^*(ha) - c| < kb - ha .$$

Proceeding, then, as in the previous case we found the sequences of natural numbers $\{k_n\}$ and $\{h_n\}$, the first not decreasing and the second increasing, and a natural number p, such that $\forall n \in \mathbb{N}$

$$k_{n+1} - k_n \le p, \qquad h_{n+1} - h_n \le p$$

and

$$|k_n b - h_n a - c| < \varepsilon .$$

And from this the thesis follows easily. □

Remark 1.1. The sequences $\{h_n\}$ and $\{k_n\}$ of the previous theorem are both divergent to $+\infty$.

2. Convergence of series in Banach space .

Lemma 2.1. *Let $\sum_{n=1}^{\infty} a_n$ be a series of non-negative real numbers such that the following properties hold:*

1) *the sequence $\{a_n\}$ is not increasing;*
2) *there exist a natural number p and a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that*

$$a_{n_k} \in \{a_{(k-1)p+1}, a_{(k-1)p+2}, \dots, a_{kp}\} \qquad \forall k \in \mathbb{N},$$

and the series $\sum_{k=1}^{\infty} a_{n_k}$ is convergent.

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof. Let us denote by $\{S_n\}$ the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ and let us consider the subsequence $\{S_{kp}\}$ of $\{S_n\}$. Then we have:

$$S_p \leq pa_1$$

and

$$\begin{aligned} S_{kp} &\leq pa_1 + pa_{n_1} + pa_{n_2} + \cdots + pa_{n_{k-1}} = \\ &= pa_1 + p(a_{n_1} + a_{n_2} + \cdots + a_{n_{k-1}}), \qquad \forall k \geq 2 \end{aligned}$$

from which it follows, by virtue of the convergence of the series $\sum_{n=1}^{\infty} a_{n_k}$, that the sequence $\{S_{kp}\}$ is convergent. And this implies that the sequence $\{S_n\}$ is convergent, and therefore the thesis. □

Theorem 2.1. *Let $\sum_{n=1}^{\infty} a_n$ be a series in $\mathscr{B}$ such that the following properties hold:*

1) the sequence $\{\|a_n\|\}$ is not increasing;
2) there exist a natural number p and a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$a_{n_k} \in \{a_{(k-1)p+1}, a_{(k-1)p+2}, \ldots, a_{kp}\} \qquad \forall k \in \mathbb{N},$$

and the series $\sum_{k=1}^{\infty} \|a_{n_k}\|$ is convergent.

Then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Proof. It follows immediately from the previous lemma. □

Theorem 2.2. *Let $\varphi(x) \in \mathscr{F}$; furthermore let $p \in \mathbb{N}$, $q \in]0,1[$ and $\{h_n\}$ a sequence of natural numbers not decreasing and divergent to $+\infty$ such that*

$$h_{n+1} - h_n \leq p, \qquad \varphi(h_n) \leq q \quad \forall n \in \mathbb{N}.$$

Then the series $\sum_{\lambda}^{\varphi}$ is absolutely convergent.

Proof. Being

$$a_{n+1} = \lambda\varphi(1)\varphi(2)\ldots\varphi(n) \qquad \forall n \in \mathbb{N},$$

it results

$$\|a_{h_1+1}\| \leq \|\lambda\| q$$

and the series

$$a_{h_1+2} + a_{h_1+3} + \cdots + a_{h_1+n+1} + \cdots$$

verifies the hypotheses of the Theorem 2.1. In fact the sequence $\{\|a_{h_1+n+1}\|\}$ is not increasing; moreover, by virtue of the hypothesis on the sequence $\{h_n\}$, among the first p terms of it there is at least one, let us say a_{n_1}, such that $\|a_{n_1}\| \leq \|\lambda\|q^2$, among the second p terms of it there is at least one, let us say a_{n_2}, such that $\|a_{n_2}\| \leq \|\lambda\|q^3$, and so on; therefore the series $\sum_{k=1}^{\infty} \|a_{n_k}\|$ is convergent, and this implies, obviously, that the series $\sum_{\lambda}^{\varphi}$ is absolutely convergent. □

Theorem 2.3. *Let $\varphi(x) \in \mathscr{F}_T$, $T \notin \mathbb{Q}$; furthermore let $q \in]0, 1[$ and $[\alpha, \beta]$ an interval included in $[0, T]$ such that $\varphi(x) \leq q \ \forall x \in [\alpha, \beta]$. Then the series $\sum_{\lambda}^{\varphi}$ is absolutely convergent.*

Proof. We put $x_0 = \frac{\alpha+\beta}{2}$, $\delta = \frac{\beta-\alpha}{2}$ and apply the Theorem 1.1 choosing $a = 1$, $b = T$, $c = -x_0$ and $\varepsilon = \delta$.

Let p, $\{h_n\}$, $\{k_n\}$ be the natural number and the sequences whose existence is insured from Theorem 1.1. Then we have:

$$h_{n+1} - h_n \leq p \qquad k_{n+1} - k_n \leq p$$

and

$$|h_n - k_n T - x_0| < \delta\,.$$

From which

$$\alpha + k_n T = x_0 - \delta + k_n T < h_n < x_0 + \delta + k_n T = \beta + k_n T\,.$$

This implies that $\varphi(h_n) \leq q \ \ \forall n \in \mathbb{N}$. Moreover for the Remark 1.1 the sequence $\{h_n\}$ is divergent to $+\infty$ and therefore, for the Theorem 2.2, we have the thesis. □

Remark 2.1. Theorem 2.3 may be proved also using Weyl's Theorem (on uniform distribution) and reasoning in a similar manner as in Theorem 2.1 of [2].

Corollary 2.1. *Let $\varphi(x) \in \mathscr{F}_T$, $T \notin \mathbb{Q}$; furthermore let $\varphi(x)$ be continue in a point $x_0 \in [0, T]$ and it results $\varphi(x_0) < 1$. Then the series $\sum_{\lambda}^{\varphi}$ is absolutely convergent.*

Corollary 2.2. *Let $\varphi(x) \in \mathscr{F}_T$, $T \notin \mathbb{Q}$; furthermore let $\varphi(x)$ be continue in $[0, T]$ and $\lambda \neq 0_{\mathscr{B}}$. Then the series $\sum_{\lambda}^{\varphi}$ is absolutely convergent if and only if there exists a point $x_0 \in [0, T]$ such that $\varphi(x_0) < 1$.*

Theorem 2.4. *Let $\varphi(x) \in \mathscr{F}_T$, $T \in \mathbb{Q}^+$; furthermore let $u \in \mathbb{N} \cap [0, T]$ and let us suppose that $\varphi(u) < 1$. Then the series $\sum_\lambda^\varphi$ is absolutely convergent.*

Proof. We set $T = \frac{r}{s}$, where $r, s \in \mathbb{N}$.Then we have

$$0 < u \leq sT\,,$$

from which

$$(k-1)sT < u + (k-1)sT \leq ksT \qquad \forall k \in \mathbb{N}\,.$$

Therefore setting

$$h_k = u + (k-1)sT \qquad \forall k \in \mathbb{N}$$

we obtain an increasing sequence $\{h_k\}$ of natural numbers such that $\forall k \in \mathbb{N}$ it results

$$h_{k+1} - h_k = sT, \quad \varphi(h_k) = \varphi(u) < 1$$

and hence, for the Theorem 2.2, we have the thesis. □

Theorem 2.5. *Let $\varphi(x) \in \mathscr{F}$ and $S \subseteq \mathscr{B}$ such that $\forall a \in S$ the closed ball in $\mathscr{B}$ of radius $\|a\|$ and center $0_{\mathscr{B}}$ is included in S; furthermore let $f : S \longrightarrow \mathscr{B}$ be a function verifying the condition $\|f(x)\| \leq \|x\|$ $\forall x \in S$; finally we suppose that $\forall \lambda \in S$ let the series $\sum_\lambda^\varphi$ be absolutely convergent. Then, defined*

$$\begin{cases} a_1 = \lambda \in S \\ a_{n+1} = \varphi(n) f(a_n) \qquad \forall\, n \in \mathbb{N}\,, \end{cases}$$

the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.

Proof. Being $\|f(a_n)\| \leq \|a_n\|$ $\forall n \in \mathbb{N}$, denoted by b_n the general term of the series $\sum_\lambda^\varphi$ and proceeding inductively we have

$$\|a_n\| \leq \|b_n\| \qquad \forall n \in \mathbb{N}\,,$$

and from this the thesis follows. □

Corollary 2.3. *Let $\mathscr{B} = \mathbb{R}$ and $\varphi(x) \in \mathscr{F}$; furthermore let $f : [0, a] \longrightarrow \mathbb{R}$ be a function verifying the condition $0 \leq f(x) \leq x$ $\forall x \in [0, a]$; finally we suppose that $\forall \lambda \in [0, a]$ let the series $\sum_\lambda^\varphi$ be convergent. Then, defined*

$$\begin{cases} a_1 = \lambda \in [0, a] \\ a_{n+1} = \varphi(n) f(a_n) \qquad \forall\, n \in \mathbb{N}\,, \end{cases}$$

the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

Proof. It is sufficient to define the function

$$f^*(x) = \begin{cases} f(x) & \forall x \in [0, a] \\ f(-x) & \forall x \in [-a, 0[\end{cases}$$

and apply the Theorem 2.5 choosing as function $f : S \longrightarrow \mathscr{B}$ the function $f^*(x)$. □

Remark 2.2. The Corollary 2.3 realize an interesting connection between the series $\sum_\lambda^\varphi$ and the series $\sum_{\lambda,f}$ which we have studied in [1]. We observe that with the alone condition $0 \le f(x) \le x$ the series $\sum_{\lambda,f}$ may be convergent or divergent (see Theorems 1.3 and 1.5 of [1]).

Example 1. Let $\mathscr{B} = \mathbb{R}$. Let us consider the real functions (defined in $[0, +\infty[$)

$$\varphi_1(x) = |\sin x|, \quad \varphi_2(x) = \sqrt{|\sin x|},$$

$$\varphi_3(x) = \begin{cases} 1 & \text{if} \quad x = k\pi \quad (k \in \mathbb{N}_0) \\ |\sin x|^{|\sin x|} & \text{if} \quad x > 0, \sin x \neq 0 \end{cases}, \qquad \varphi_4(x) = \frac{\sin|\sin x|}{\sin 1},$$

$$\varphi_5(x) = \sin^2\left(\frac{\pi}{2}\frac{\int_0^{\sin x} e^{-t^2}\,dt}{\int_0^1 e^{-t^2}\,dt}\right).$$

It is easy to prove that they verify the hypotheses of the Corollary 2.1. Therefore the series $\sum_\lambda^{\varphi_i}$ $(i = 1, 2, \dots, 5)$ are convergent.
We observe that, for Kronecker's theorem, we have easily, for $i = 1, 2, \dots, 5$,

$$\limsup \varphi_i(n) = 1,$$

and therefore the convergence of the series $\sum_\lambda^{\varphi_i}$ cannot be obtained with the elementary ratio test.

Example 2. Let $\mathscr{B} = \mathbb{R}$. Let us consider the real functions

$$\varphi_1(x) = |\operatorname{sn} x|, \quad \varphi_2(x) = |\operatorname{cn} x|, \quad \varphi_3(x) = \operatorname{dn} x,$$

where $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ are the elliptic fuctions of Jacobi (see, for example, [5]). We have:

-if the period of $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ is irrational, then the series $\sum_\lambda^{\varphi_i}$ $(i = 1, 2, 3)$ are convergent for the Corollary 2.1;

-if the period of $|\operatorname{sn} x|$, $|\operatorname{cn} x|$, $\operatorname{dn} x$ is rational and greater than 1 then the series $\sum_\lambda^{\varphi_i}$ $(i = 1, 2, 3)$ are convergent for the Theorem 2.4.

Example 3. Let $\mathscr{B} = \mathbb{R}$. Let us consider he real functions

$$\varphi(x) = \sin^2 x \quad x \in [0, +\infty[, \qquad f(x) = \arctan x \quad x \in [0, a].$$

We see easily that they verify the hypoteses of the Corollary 2.3, therefore, setting

$$\begin{cases} a_1 = \lambda \in [0, a] \\ a_{n+1} = (\sin^2 n) \arctan a_n , \end{cases}$$

the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Example 4. Let $\mathscr{B} = C^0([0, 1])$ and $K(x, y) \in C^0(\mathbb{R} \times [0, 1])$ such that $|K(x, y)| \leq |x| \quad \forall x \in \mathbb{R}$. Let us define a function $f : \mathscr{B} \longrightarrow \mathscr{B}$ setting

$$f(\psi) = \int_0^1 K(\psi(x), y)\, dy \quad \forall \psi(x) \in \mathscr{B}.$$

Being

$$\left| \int_0^1 K(\psi(x), y)\, dy \right| \leq \int_0^1 |K(\psi(x), y)|\, dy \leq \int_0^1 |\psi(x)|\, dy = |\psi(x)|,$$

it results $\|f(\psi)\| \leq \|\psi\|$. Then by virtue of Theorem 2.5, if the series $\sum_\lambda^\varphi$ is absolutely convergent, we deduce that the series whose terms are given recursively by the formula

$$\begin{cases} a_1 = \lambda \in \mathscr{B} \\ a_{n+1} = \varphi(n) f(a_n) \qquad \forall n \in \mathbb{N}, \end{cases}$$

is absolutely convergent.

Example 5. Let $\mathscr{B} = L^2([0, 1])$ and $K(x, y) \in C^0(\mathbb{R} \times [0, 1])$ such that $0 \leq K(x, y) \leq x \quad \forall x \in \mathbb{R}$. Setting, as in the previous example,

$$f(\psi) = \int_0^1 K(\psi(x), y)\, dy \quad \forall \psi(x) \in \mathscr{B},$$

we obtain a function $f : \mathscr{B} \longrightarrow \mathscr{B}$ that verifies the condition $\|f(\psi)\| \leq \|\psi\|$.

Therefore the same conclusion as in the previous example follows.

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