SPECIAL FANO MANIFOLDS AS AMPLE DIVISORS

CRISTIANA SACCHI

Let $A$ be a projective manifold of dimension $n \geq 3$ contained as an ample divisor in a projective manifold $X$ and let $L = \mathcal{O}_X(A)$. In this paper we study the pairs $(X, L)$ in the following two cases:

i) $A$ a Fano manifold of coindex 3 and Picard number 1;
ii) $A$ a Fano manifold of product type.

Introduction.

It is well known that it is a strong condition for a projective manifold $X$ to contain a fixed projective manifold $A$ as an ample divisor. Let $L = \mathcal{O}_X(A) \in \text{Pic}(X)$: the aim of this paper is to classify pairs $(X, L)$ as above in some special instances.

For $A = \mathbb{P}^n$ or a smooth hyperquadric $Q^n \subseteq \mathbb{P}^{n+1}$ the result is well known [9], [1] and in [6] the classification is given when $A$ is a Del Pezzo manifold. In this paper we study the following two cases:

i) $A$ is a Fano manifold of coindex 3 and Picard number 1;
ii) $A$ is a Fano manifolds of product type.

Recall that $A$ is Fano if $-K_A$ is ample. The index of $A$ is the largest integer $r$ such that $-K_A = rH$, with $H \in \text{Pic}(A)$ ample and the coindex $c$ of $A$ is defined as $c = \dim A + 1 - r$.

Entrato in Redazione il 15 luglio 1994.

AMS 1991 Subject Classification: 14C20, 14J45, 14J40.
The list of possible pairs occurring in case i) is given by Theorem 1.1, combined with the classification by Mukai of coindex 3 Fano manifold [7].

As to case ii), we prove in section 2 that necessarily $A = \mathbb{P}^1 \times Z$ with $Z$ Fano and that if Pic ($Z$) $\cong \mathbb{Z}$ then $X$ is a fibration over $\mathbb{P}^1$ whose general fibre is a Fano manifold $F$ with Picard number 1 and index $r+s$, where $r =$ index ($Z$) and $L_F = sH$, $H$ denoting the ample generator of Pic ($F$). In particular for $Z \subseteq \mathbb{P}^n$ a hypersurface of degree $a \geq 2$, we also prove that $\mathbb{P}^1 \times Z$ cannot be contained fiberwise into a $\mathbb{P}^n$-bundle over $\mathbb{P}^1$ (Theorem 2.4). For $Z = \mathbb{Q}^{n-1}$ the above argument gives as a corollary that an $X$ as above has to be a $\mathbb{Q}^n$-fibration over $\mathbb{P}^1$. This, combined with a result by Paranjape-Srinivas [8] allows us to obtain a small progress on a discussion related to a conjecture by Fania-Sommese [4], p. 216.

All symbols and terminology used in the paper are standard in algebraic geometry. We always consider holomorphic line bundles and, following current abuses, we do not distinguish between a line bundle and the corresponding invertible sheaf; moreover we use the additive notation for the tensor product of line bundles. Furthermore, if $L \in \text{Pic } (X)$ is a line bundle on a projective manifold $X$, $L_Y$ will denote its restriction to a submanifold $Y$.

1. Coindex 3 Fano manifolds with Picard number 1 as ample divisors.

Let $A$ be a Fano $n$-fold contained as an ample divisor in a projective $(n+1)$-fold $X$ and let $L = \mathcal{O}_X(A)$. Let $c$ be the coindex of $A$. It is well known ([9], p. 67, [1], Theorem 4), that if $c = 0$ then $(X, L) = (\mathbb{P}^{n+1}, \mathcal{O}(1))$ while if $c = 1$ then $(X, L) = (\mathbb{P}^{n+1}, \mathcal{O}(2))$ or $(\mathbb{Q}^{n+1}, \mathcal{O}(1))$. The case $c = 2$ is studied in [6], Appendix; here we deal with case $c = 3$.

**Theorem 1.1.** Let $A$ be a coindex 3 Fano $n$-fold with $n \geq 3$, contained as an ample divisor in a smooth projective $(n+1)$-fold $X$ and let $L = \mathcal{O}_X(A)$. Assume Pic ($A$) $\cong \mathbb{Z}$; then $(X, L)$ is one of the following pairs:

i) $(X, H) = (X, L)$ is a coindex 3 Fano $(n+1)$-fold with Pic ($X$) $\cong \mathbb{Z}$ generated by $L$;

ii) $(X, H)$ is a Del Pezzo $(n+1)$-fold with Pic ($X$) $\cong \mathbb{Z}$ and $L = 2H$, where $H$ generates Pic ($X$), except ($\mathbb{P}^3$, $\mathcal{O}(2)$);

iii) $(\mathbb{Q}^{n+1}, \mathcal{O}(3))$;

iv) $(\mathbb{P}^{n+1}, \mathcal{O}(4))$.

**Proof.** Let $-K_A = (n-2)h$, where $h$ is an ample element of Pic ($A$). From $n-2 =$ index ($A$) it is immediate to see that $h$ generates Pic ($A$). As $n \geq 3$, by the Lefschetz theorem (see [9], p. 56) we have Pic ($X$) $\cong \text{Pic } (A) \cong \mathbb{Z}$. Let
\( H \in \text{Pic}(X) \) be the element such that \( H_A = h \). Note that \( H \) generates \( \text{Pic}(X) \), so we can write \( L = aH \) and \( K_X = rH \) for some integers \( r \) and \( a > 0 \). By adjunction we have

\[
(n - 2)h = -K_A = -(K_X + L)_A = -(r + a)H_A = -(r + a)h,
\]

hence

\[
-K_X = ((n - 2) + a)H = (\dim X - (3 - a))H.
\]

This implies \( a \leq 4 \) by the Kobayashi-Ochiai theorem. Since \( a \geq 1 \) we get the following possibilities:

i) \( a = 1 \), in which case \( X \) is a coindex 3 Fano manifold and \( L = H \) generates \( \text{Pic}(X) \);

ii) \( a = 2 \), in which case by definition \( (X, H) \) is a Del Pezzo manifold with \( \text{Pic}(X) \cong \mathbb{Z} \) and \( L = 2H \). From Fujita's classification of Del Pezzo manifolds ([5], I.8.11), we obtain 5 cases. Note that even thought \( (X, H) = (\mathbb{P}^3, \mathcal{O}(2)) \) is a Del Pezzo manifold with \( \text{Pic}(X) \cong \mathbb{Z} \), it cannot contain \( A \) as an ample divisor since in this case \( H \) does not generate \( \text{Pic}(X) \);

iii) \( a = 3 \), in which case \( (X, H) = (\mathbb{P}^{n+1}, \mathcal{O}(1)) \) by the Kobayashi-Ochiai theorem and \( L = \mathcal{O}(3) \);

iv) \( a = 4 \), in which case \( (X, H) = (\mathbb{P}^{n+1}, \mathcal{O}(1)) \) by the Kobayashi-Ochiai theorem and \( L = \mathcal{O}(4) \).

For the classification of Fano manifolds of coindex 3 occurring in i) see [7]. Note also that by adjunction it is immediate to see that the list in Theorem 1.1 is effective; of course in case i) we have to suppose that the linear system of the ample generator of \( \text{Pic}(X) \) contains a smooth element. This is true for \( \dim X = 4 \) ([10], p. 173), and conjectured for \( \dim X > 4 \).

2. Fano products as ample divisors.

2.0. - Let \( A \) be a Fano manifold of product type contained as an ample divisor in a smooth projective \((n + 1)\)-fold \( X \).

**Proposition 2.1.** Let \( A \) be as in 2.0. Then \( A = \mathbb{P}^1 \times Z \), where \( Z \) is a Fano manifold.

**Proof.** If \( A \) is a product contained as an ample divisor in a projective manifold \( X \) then \( A \) has precisely two factors and one of them is 1-dimensional ([9], Prop. IV), so \( A = Z \times Y \) with \( \dim Y = 1 \). By adjunction we have

\[
K_Z = (K_A + \det N_{Z|A})_Z = (K_A)_Z,
\]
since the normal bundle $N_{Z|A}$ is trivial as $Z$ is a factor of $A$; hence $-K_Z$ is ample so $Z$ is Fano. Similarly we prove that $Y$ is Fano, therefore $Y = \mathbb{P}^1$. \hfill $\Box$

2.2. - Let $A = \mathbb{P}^1 \times Z$ be as in 2.0. Consider the projection $p : A \rightarrow \mathbb{P}^1$ (whose fibres are obviously isomorphic to $Z$). If $\dim Z \geq 2$ then $p$ extends to a fibration $\tilde{p} : X \rightarrow \mathbb{P}^1$ ([9], Prop. III). Let $F$ be the general fibre of $\tilde{p}$. As $Z = F \cap A$ we have that $[Z] = [A]_F$ is ample in $\text{Pic} (F)$.

2.2.1. - Assume that $\text{Pic} (Z) \cong \mathbb{Z}$ and $\dim Z \geq 3$; then by the Lefschetz theorem $\text{Pic} (F) \cong \text{Pic} (Z) \cong \mathbb{Z}$. Let $h$ be the ample generator of $\text{Pic} (Z)$; then $-K_Z = rh$. Let $H \in \text{Pic} (F)$ be the element such that $H_Z = h$. Note that $H$ generates $\text{Pic} (F)$, so we can write $[A]_F = sH$ for a suitable integer $s > 0$.

**Proposition 2.3.** Let $Z$ and $F$ be as in 2.2.1. Then $F$ is a Fano manifold with $\text{Pic} (F) \cong \mathbb{Z}$ and index $(F) = r + s$.

**Proof.** By adjunction we have

$$K_Z = (K_F + [Z])_Z = (K_F + [A]_F)_Z,$$

therefore

$$-(K_F)_Z = ([A]_F)_Z - K_Z = sH_Z + rh = (r + s)h,$$

hence $-K_F = (r + s)H$. \hfill $\Box$

It is obvious that $\mathbb{P}^1 \times \mathbb{Q}^n$ embedded in $\mathbb{P}^1 \times \mathbb{P}^{n+1}$ componentwise cannot be an ample divisor. A less obvious fact is that $\mathbb{P}^1 \times \mathbb{Q}^n$ cannot be contained as an ample divisor in a $\mathbb{P}^{n+1}$-bundle $p : P \rightarrow \mathbb{P}^1$ in such a way that $p$ extends the projection onto the first factor. This is shown by the following

**Theorem 2.4.** Let $P = \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$ be a $\mathbb{P}^{n}$-bundle over $\mathbb{P}^1$ and let $L = \sum_{F \in \text{Pic} (P)} a_F \xi + b[F] \in \text{Pic} (P)$ be an ample line bundle, where $\xi$ stands for the tautological bundle of $\mathcal{E}$ and $F$ for a fibre. Then $|L|$ cannot contain an element $Y = S_a \times \mathbb{P}^1$, where $S_a \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree $a \geq 2$, such that $p |_Y$ is the projection onto the second factor.

**Proof.** By contradiction assume that $Y = S_a \times \mathbb{P}^1 \in |L|$. By well known properties of vector bundles on $\mathbb{P}^1$ we have $\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{E}(a_i)$, where $a_i \leq a_{i+1}$ for $0 \leq i \leq n - 1$.

**Claim.** $\mathcal{E}$ is globally generated.
As $\xi_F = \mathcal{O}(1)$, $\xi_{S_a}$ embeds $S_a$ in $\mathbb{P}^n$, so $\xi_{S_a}$ is very ample. Being $\mathcal{E} = p_\ast \xi$, it follows that $H^0(\mathcal{E}) \cong H^0(\xi)$. Let $t$ be any point of $\mathbb{P}^1$ and let

$$(S_a)_t = p |_{\mathbb{P}^1}(t) = p^{-1}(t) \cap Y.$$ 

We have $H^0(\xi_{(S_a)_t}) \cong H^0(\mathcal{E}_t) \cong \mathcal{E}_t$, where $\mathcal{E}_t$ is the fibre of $\mathcal{E}$ at the point $t$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^0(\mathcal{E}) & \overset{\sim}{\longrightarrow} & H^0(\xi) \\
\downarrow & & \downarrow \\
H^0(\mathcal{E}_t) & \overset{\sim}{\longrightarrow} & H^0(\xi_{(S_a)_t})
\end{array}
\]

where the vertical arrows are restrictions. Since the morphism $H^0(\xi) \longrightarrow H^0(\xi_{(S_a)_t})$ is surjective, then even the morphism $H^0(\mathcal{E}) \longrightarrow H^0(\mathcal{E}_t)$ is surjective, which proves the claim.

As a consequence of the claim we have $a_i \geq 0$ for $0 \leq i \leq n$; up to tensoring $\mathcal{E}$ by $\mathcal{O}(-a_0)$ we can assume $a_0 = 0$. Then $a > 0$, $b > 0$ since $L$ is ample ([2], 3.2.4), and $c_1(\mathcal{E}) = a_0 + \cdots + a_n \geq 0$. Let $\delta = c_1(\mathcal{E})$; by the canonical bundle formula (see [5], p. 2) we have

$$K_P = -(n + 1)\xi + p^\ast(c_1(\mathcal{E}) + K_{\mathbb{P}^1}) = -(n + 1)\xi + (\delta - 2)[F],$$

hence by adjunction

\[(2.4.1) \quad K_Y = (K_P + L)_Y = (a - n - 1)\xi_Y + (\delta + b - 2)[F]_Y.\]

On the other hand, as $Y = S_a \times \mathbb{P}^1$ we have

\[(2.4.2) \quad K_Y = \mathcal{O}(a - n - 1, -2) = (a - n - 1)\xi_Y - 2[F]_Y.\]

Comparing (2.4.1) with (2.4.2) we get $\delta + b = 0$ but this leads to a contradiction since $b > 0$ and $\delta \geq 0$. $\square$

As a consequence of Theorem 2.4 we have the following

**Corollary 2.5.** *The Fano manifold $A = \mathbb{Q}^{n-1} \times \mathbb{P}^1$ can be contained fiberwise as an ample divisor only in a $\mathbb{Q}^n$-fibration over $\mathbb{P}^1$.***
\textbf{Proof.} Let $p : X \longrightarrow \mathbb{P}^1$ be the fibration extending the projection $A \longrightarrow \mathbb{P}^1$ and assume that $A \subseteq X$ is an ample divisor. By Proposition 2.3 the general fibre $F$ of $p$ has to be either $\mathbb{Q}^n$ or $\mathbb{P}^n$; therefore by applying Theorem 2.4 with $a = 2$ we conclude that $F = \mathbb{Q}^n$. $\square$

In the special case $n = 4$, Corollary 2.5 combined with a theorem by Paranjape-Srinivas [8] gives a result related to the last remark in the paper by Fania-Sommese [4], p. 216. We have in fact

\textbf{Corollary 2.6.} Let $Z$ be a Fano 3-fold and assume that $A = Z \times \mathbb{P}^1$ is contained as a very ample divisor in a smooth projective 5-fold $X$. Assume furthermore that $Z \neq \mathbb{P}^3$. Then $Z = \mathbb{Q}^3$ and $X$ is a $\mathbb{Q}^4$-fibration over $\mathbb{P}^1$.

\textbf{Proof.} By a result of Sommese quoted in [4], p. 216, there is a 2 to 1 morphism $f : \mathbb{Q}^3 \longrightarrow Z$. The existence of such a morphism implies that $Z = \mathbb{Q}^3$ by [8] (see also [3]), then the assertion follows from Corollary 2.5. $\square$

\textbf{REFERENCES}


*Dipartimento di Matematica “F. Enriques”,
Università di Milano,
Via C. Saldini 50,
20133 Milano (Italy)*