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SPECIAL FANO MANIFOLDS AS AMPLE DIVISORS

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Let A be a projective manifold of dimension $n \geq 3$ contained as an ample divisor in a projective manifold X and let $L = \mathcal{O}_X(A)$. In this paper we study the pairs (X, L) in the following two cases:

i) A a Fano manifold of coindex 3 and Picard number 1;
ii) A a Fano manifold of product type.

Introduction.

It is well known that it is a strong condition for a projective manifold X to contain a fixed projective manifold A as an ample divisor. Let $L = \mathcal{O}_X(A) \in \text{Pic}(X)$: the aim of this paper is to classify pairs (X, L) as above in some special instances.

For $A = \mathbb{P}^n$ or a smooth hyperquadric $\mathbb{Q}^n \subseteq \mathbb{P}^{n+1}$ the result is well known [9], [1] and in [6] the classification is given when A is a Del Pezzo manifold. In this paper we study the following two cases:

i) A is a Fano manifold of coindex 3 and Picard number 1;
ii) A is a Fano manifolds of product type.

Recall that A is Fano if $-K_A$ is ample. The index of A is the largest integer r such that $-K_A = rH$, with $H \in \text{Pic}(A)$ ample and the coindex c of A is defined as $c = \dim A + 1 - r$.

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The list of possible pairs occurring in case i) is given by Theorem 1.1, combined with the classification by Mukai of coindex 3 Fano manifold [7].

As to case ii), we prove in section 2 that necessarily $A = \mathbb{P}^1 \times Z$ with Z Fano and that if Pic $(Z) \cong \mathbb{Z}$ then X is a fibration over $\mathbb{P}^1$ whose general fibre is a Fano manifold F with Picard number 1 and index $r+s$, where $r =$ index (Z) and $L_F = sH$, H denoting the ample generator of Pic (F). In particular for $Z \subseteq \mathbb{P}^n$ a hypersurface of degree $a \geq 2$, we also prove that $\mathbb{P}^1 \times Z$ cannot be contained fiberwise into a $\mathbb{P}^n$-bundle over $\mathbb{P}^1$ (Theorem 2.4). For $Z = \mathbb{Q}^{n-1}$ the above argument gives as a corollary that an X as above has to be a $\mathbb{Q}^n$-fibration over $\mathbb{P}^1$. This, combined with a result by Paranjape-Srinivas [8] allows us to obtain a small progress on a discussion related to a conjecture by Fania-Sommese [4], p. 216.

All symbols and terminology used in the paper are standard in algebraic geometry. We always consider holomorphic line bundles and, following current abuses, we do not distinguish between a line bundle and the corresponding invertible sheaf; moreover we use the additive notation for the tensor product of line bundles. Furthermore, if $L \in$ Pic (X) is a line bundle on a projective manifold X, L_Y will denote its restriction to a submanifold Y.

1. Coindex 3 Fano manifolds with Picard number 1 as ample divisors.

Let A be a Fano n-fold contained as an ample divisor in a projective $(n+1)$-fold X and let $L = \mathcal{O}_X(A)$. Let c be the coindex of A. It is well known ([9], p. 67, [1], Theorem 4), that if $c = 0$ then $(X, L) = (\mathbb{P}^{n+1}, \mathcal{O}(1))$ while if $c = 1$ then $(X, L) = (\mathbb{P}^{n+1}, \mathcal{O}(2))$ or $(\mathbb{Q}^{n+1}, \mathcal{O}(1))$. The case $c = 2$ is studied in [6], Appendix; here we deal with case $c = 3$.

Theorem 1.1. *Let A be a coindex 3 Fano n-fold with $n \geq 3$, contained as an ample divisor in a smooth projective $(n+1)$-fold X and let $L = \mathcal{O}_X(A)$. Assume Pic $(A) \cong \mathbb{Z}$; then (X, L) is one of the following pairs:*

i) $(X, H) = (X, L)$ *is a coindex 3 Fano $(n+1)$-fold with Pic $(X) \cong \mathbb{Z}$ generated by L;*
ii) (X, H) *is a Del Pezzo $(n+1)$-fold with Pic $(X) \cong \mathbb{Z}$ and $L = 2H$, where H generates Pic (X), except $(\mathbb{P}^3, \mathcal{O}(2))$;*
iii) $(\mathbb{Q}^{n+1}, \mathcal{O}(3))$;
iv) $(\mathbb{P}^{n+1}, \mathcal{O}(4))$.

Proof. Let $-K_A = (n-2)h$, where h is an ample element of Pic (A). From $n - 2 =$ index (A) it is immediate to see that h generates Pic (A). As $n \geq 3$, by the Lefschetz theorem (see [9], p. 56) we have Pic $(X) \cong$ Pic $(A) \cong \mathbb{Z}$. Let

$H \in \mathrm{Pic}(X)$ be the element such that $H_A = h$. Note that H generates $\mathrm{Pic}(X)$, so we can write $L = aH$ and $K_X = rH$ for some integers r and $a > 0$. By adjunction we have

$$(n-2)h = -K_A = -(K_X + L)_A = -(r+a)H_A = -(r+a)h,$$

hence

$$-K_X = ((n-2)+a)H = (\dim X - (3-a))H.$$

This implies $a \leq 4$ by the Kobayashi-Ochiai theorem. Since $a \geq 1$ we get the following possibilities:

i) $a = 1$, in which case X is a coindex 3 Fano manifold and $L = H$ generates $\mathrm{Pic}(X)$;

ii) $a = 2$, in which case by definition (X, H) is a Del Pezzo manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $L = 2H$. From Fujita's classification of Del Pezzo manifolds ([5], I.8.11), we obtain 5 cases. Note that even thought $(X, H) = (\mathbb{P}^3, \mathcal{O}(2))$ is a Del Pezzo manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}$, it cannot contain A as an ample divisor since in this case H does not generate $\mathrm{Pic}(X)$;

iii) $a = 3$, in which case $(X, H) = (\mathbb{Q}^{n+1}, \mathcal{O}(1))$ by the Kobayashi-Ochiai theorem and $L = \mathcal{O}(3)$;

iv) $a = 4$, in which case $(X, H) = (\mathbb{P}^{n+1}, \mathcal{O}(1))$ by the Kobayashi-Ochiai theorem and $L = \mathcal{O}(4)$. □

For the classification of Fano manifolds of coindex 3 occurring in i) see [7]. Note also that by adjunction it is immediate to see that the list in Theorem 1.1 is effective; of course in case i) we have to suppose that the linear system of the ample generator of $\mathrm{Pic}(X)$ contains a smooth element. This is true for $\dim X = 4$ ([10], p. 173), and conjectured for $\dim X > 4$.

2. Fano products as ample divisors.

2.0. - Let A be a Fano manifold of product type contained as an ample divisor in a smooth projective $(n+1)$-fold X.

Proposition 2.1. *Let A be as in 2.0. Then $A = \mathbb{P}^1 \times Z$, where Z is a Fano manifold.*

Proof. If A is a product contained as an ample divisor in a projective manifold X then A has precisely two factors and one of them is 1-dimensional ([9], Prop. IV), so $A = Z \times Y$ with $\dim Y = 1$. By adjunction we have

$$K_Z = (K_A + \det N_{Z|A})_Z = (K_A)_Z,$$

since the normal bundle $N_{Z|A}$ is trivial as Z is a factor of A; hence $-K_Z$ is ample so Z is Fano.
Similarly we prove that Y is Fano, therefore $Y = \mathbb{P}^1$. □

2.2. - Let $A = \mathbb{P}^1 \times Z$ be as in 2.0. Consider the projection $p : A \longrightarrow \mathbb{P}^1$ (whose fibres are obviously isomorphic to Z). If $\dim Z \geq 2$ then p extends to a fibration $\tilde{p} : X \longrightarrow \mathbb{P}^1$ ([9], Prop. III). Let F be the general fibre of $\tilde{p}$. As $Z = F \cap A$ we have that $[Z] = [A]_F$ is ample in $\mathrm{Pic}\,(F)$.

2.2.1. - Assume that $\mathrm{Pic}\,(Z) \cong \mathbb{Z}$ and $\dim Z \geq 3$; then by the Lefschetz theorem $\mathrm{Pic}\,(F) \cong \mathrm{Pic}\,(Z) \cong \mathbb{Z}$. Let h be the ample generator of $\mathrm{Pic}\,(Z)$; then $-K_Z = rh$. Let $H \in \mathrm{Pic}\,(F)$ be the element such that $H_Z = h$. Note that H generates $\mathrm{Pic}\,(F)$, so we can write $[A]_F = sH$ for a suitable integer $s > 0$.

Proposition 2.3. *Let Z and F be as in 2.2.1. Then F is a Fano manifold with $Pic\,(F) \cong \mathbb{Z}$ and index $(F) = r + s$.*

Proof. By adjunction we have

$$K_Z = (K_F + [Z])_Z = (K_F + [A]_F)_{Z'}$$

therefore

$$-(K_F)_Z = ([A]_F)_Z - K_Z = sH_Z + rh = (r + s)h,$$

hence $-K_F = (r + s)H$. □

It is obvious that $\mathbb{P}^1 \times \mathbb{Q}^n$ embedded in $\mathbb{P}^1 \times \mathbb{P}^{n+1}$ componentwise cannot be an ample divisor. A less obvious fact is that $\mathbb{P}^1 \times \mathbb{Q}^n$ cannot be contained as an ample divisor in a $\mathbb{P}^{n+1}$-bundle $p : P \longrightarrow \mathbb{P}^1$ in such a way that p extends the projection onto the first factor. This is shown by the following

Theorem 2.4. *Let $P = \mathbb{P}(\mathscr{E}) \xrightarrow[p]{} \mathbb{P}^1$ be a $\mathbb{P}^n$-bundle over $\mathbb{P}^1$ and let $L = a\xi + b[F] \in Pic\,(P)$ be an ample line bundle, where ξ stands for the tautological bundle of $\mathscr{E}$ and F for a fibre. Then $|L|$ cannot contain an element $Y = S_a \times \mathbb{P}^1$, where $S_a \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree $a \geq 2$, such that $p\,|_Y$ is the projection onto the second factor.*

Proof. By contradiction assume that $Y = S_a \times \mathbb{P}^1 \in |L|$. By well known properties of vector bundles on $\mathbb{P}^1$ we have $\mathscr{E} = \bigoplus_{i=0}^{n} \mathscr{O}(a_i)$, where $a_i \leq a_{i+1}$ for $0 \leq i \leq n - 1$.

Claim. $\mathscr{E}$ is globally generated.

As $\xi_F = \mathcal{O}(1)$, ξ_{S_a} embeds S_a in $\mathbb{P}^n$, so ξ_{S_a} is very ample.
Being $\mathcal{E} = p_*\xi$, it follows that $H^0(\mathcal{E}) \cong H^0(\xi)$. Let t be any point of $\mathbb{P}^1$ and let

$$(S_a)_t = p\,|_Y^{-1}(t) = p^{-1}(t) \cap Y.$$

We have $H^0(\xi_{(S_a)_t}) \cong H^0(\mathcal{E}_t) \cong \mathcal{E}_t$, where $\mathcal{E}_t$ is the fibre of $\mathcal{E}$ at the point t. Consider the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{E}) & \xrightarrow{\sim} & H^0(\xi) \\ \downarrow & & \downarrow \\ H^0(\mathcal{E}_t) & \xrightarrow{\sim} & H^0(\xi_{(S_a)_t}) \end{array}$$

where the vertical arrows are restrictions. Since the morphism $H^0(\xi) \longrightarrow H^0(\xi_{(S_a)_t})$ is surjective, then even the morphism $H^0(\mathcal{E}) \longrightarrow H^0(\mathcal{E}_t)$ is surjective, which proves the claim.

As a consequence of the claim we have $a_i \geq 0$ for $0 \leq i \leq n$; up to tensoring $\mathcal{E}$ by $\mathcal{O}(-a_0)$ we can assume $a_0 = 0$. Then $a > 0$, $b > 0$ since L is ample ([2], 3.2.4), and $c_1(\mathcal{E}) = a_0 + \cdots + a_n \geq 0$. Let $\delta = c_1(\mathcal{E})$; by the canonical bundle formula (see [5], p. 2) we have

$$K_P = -(n+1)\xi + p^*(c_1(\mathcal{E}) + K_{\mathbb{P}^1}) = -(n+1)\xi + (\delta - 2)[F],$$

hence by adjunction

$$K_Y = (K_P + L)_Y = (a - n - 1)\xi_Y + (\delta + b - 2)[F]_Y. \tag{2.4.1}$$

On the other hand, as $Y = S_a \times \mathbb{P}^1$ we have

$$K_Y = \mathcal{O}(a - n - 1, -2) = (a - n - 1)\xi_Y - 2[F]_Y. \tag{2.4.2}$$

Comparing (2.4.1) with (2.4.2) we get $\delta + b = 0$ but this leads to a contradiction since $b > 0$ and $\delta \geq 0$. $\square$

As a consequence of Theorem 2.4 we have the following

Corollary 2.5. *The Fano manifold $A = \mathbb{Q}^{n-1} \times \mathbb{P}^1$ can be contained fiberwise as an ample divisor only in a $\mathbb{Q}^n$-fibration over $\mathbb{P}^1$.*

Proof. Let $p : X \longrightarrow \mathbb{P}^1$ be the fibration extending the projection $A \longrightarrow \mathbb{P}^1$ and assume that $A \subseteq X$ is an ample divisor. By Proposition 2.3 the general fibre F of p has to be either $\mathbb{Q}^n$ or $\mathbb{P}^n$; therefore by applying Theorem 2.4 with $a = 2$ we conclude that $F = \mathbb{Q}^n$. □

In the special case $n = 4$, Corollary 2.5 combined with a theorem by Paranjape-Srinivas [8] gives a result related to the last remark in the paper by Fania-Sommese [4], p. 216. We have in fact

Corollary 2.6. *Let Z be a Fano 3-fold and assume that $A = Z \times \mathbb{P}^1$ is contained as a very ample divisor in a smooth projective 5-fold X. Assume furthermore that $Z \neq \mathbb{P}^3$. Then $Z = \mathbb{Q}^3$ and X is a $\mathbb{Q}^4$-fibration over $\mathbb{P}^1$.*

Proof. By a result of Sommese quoted in [4], p. 216, there is a 2 to 1 morphism $f : \mathbb{Q}^3 \longrightarrow Z$. The existence of such a morphism implies that $Z = \mathbb{Q}^3$ by [8] (see also [3]), then the assertion follows from Corollary 2.5. □

REFERENCES

[1] L. Bădescu, *On ample divisors,* Nagoya Math. J., 86 (1982), pp. 155–171.

[2] M.C. Beltrametti - A.J. Sommese, *The adjunction theory of complex projective varieties,* Exposition in Math., De Gruyter, to appear.

[3] K. Cho - E. Sato, *Smooth projective varieties dominated by G/P,* preprint, Kyushu Univ., 1989.

[4] M.L. Fania - A.J. Sommese, *Varieties whose hyperplane sections are $\mathbb{P}^k_{\mathbb{C}}$-bundles,* Ann. Sc. Norm. Sup. Pisa, (IV) 15 (1988), pp. 193–218.

[5] T. Fujita, *Classification theories of polarized varieties,* London Math. Soc., Lect. Notes Series, 155, Cambridge Univ. Press, Cambridge, 1990.

[6] A. Lanteri - M. Palleschi - A.J. Sommese, *Del Pezzo surfaces as hyperplane sections,* Preprint, 1994.

[7] S. Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3,* Proc. Natl. Acad. Sci. U.S.A., 86 (1989), pp. 3000–3002.

[8] K.H. Paranjape - V. Srinivas, *Self maps of homogeneous spaces,* Invent. Math., 98 (1985), pp. 425–444.

[9] A.J. Sommese, *On manifolds that cannot be ample divisors,* Math. Ann., 221 (1976), pp. 55–72.

[10] P.M.H. Wilson, *Fano manifolds of index greater than one,* J. Reine Angew. Math., 378 (1987), pp. 172–181.

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