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THE USE OF THE E-METRIC SPACES IN THE SEARCH FOR FIXED POINTS

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We prove some general theorems on the convergence of the successive approximations $x_n = Ax_{n-1}$ to a fixed point of a nonlinear contraction mapping A defined on an E-metric space. We derive, as applications, some fixed point theorems in uniform spaces and an abstract Cauchy-Kowalewski type theorem.

1. Introduction.

Let (X, d) be a complete metric space and $A: X \to X$ a continuous operator. We are interested in the solution of

(1) $$Ax = x\ .$$

If A satisfies a condition of the type

(2) $$\exists k < 1 \quad \text{such that} \quad d(Ax, Ay) \le kd(x, y) \quad \forall x, y \in X,$$

then the Contraction Mapping Principle solves positively (1) for the existence and uniqueness as well. Anyway condition (2) is too much restrictive. Many concrete situations can be reduced to verify (2) only with a big loss of generality and

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in some cases, condition (2) is not completely feasible. The class of "contractive" operators becomes wider if we allow the metric d to assume its values in a linear space E ordered by a cone E_+. In fact the product by the Lipschitz constant k in the condition (2) can be replaced by the action on $d(x, y)$ of a suitable (possibly nonlinear) operator $S: E_+ \to E_+$.

Perhaps the role of the E-metric spaces, introduced about forty years ago ([3], [8]), is not yet sufficiently explored. It seems to us that the fixed point theorems in E-metric spaces are, possibly, a useful tool in dealing with differential equations. For example, we obtain in a simple way an abstract nonlinear Cauchy-Kowalewski theorem in the case of weak singularity (for a general formulation and related problems for the abstract Cauchy-Kowalewski type theorems cfr. [2], [5], [6], [7], [9]).

The general setting of E-metric spaces also favours the linguistic point of view, because it allows us to unify and simplify results that are involved in other contexts (e.g. we obtain from the simple Corollary 1 some fixed point theorems in uniform spaces due to Angelov [1]).

Now it is necessary to introduce some notations and definitions. We follow [10].

Let E be a real linear space partially ordered by $\leq$ and let $E_+ = \{e \in E: e \geq 0\}$ be the positive cone of E.

A notation of convergence of sequences in $E (e_n \to e$ or $e = \lim e_n)$ is a *linear convergence* if the following properties are satisfied:

1. if $e_n = e \ \forall n$, then $\lim e_n = e$;
2. $\lim e_n = e$ implies $\lim e_{n'} = e$ for every subsequence $(e_{n'})$ of (e_n);
3. $\lim e_n = e$ and $\lim f_n = f$ imply $\lim(e_n + f_n) = e + f$;
4. $\lim e_n = e$ implies $\lim(re_n) = re \quad \forall r \in \mathbb{R}$;
5. if $e_n \leq f_n \ \forall n$ and $\lim e_n = e$, $\lim f_n = f$ then $e \leq f$;
6. if $f_n \leq e_n \leq g_n$ and $\lim f_n = \lim g_n = e$ then also $\lim e_n = e$.

Let X be a nonempty set and let E be an ordered linear space with a notion of linear convergence. An *E-metric on X* is a mapping $d: XxX \to E_+$ subject to the usual axioms:

$$d(x, y) = 0 \text{ if and only if } x = y;$$
$$d(x, y) = d(y, x);$$
$$d(x, y) \leq d(x, z) + d(z, y).$$

By *E-metric space* we mean a nonempty set X with an E-metric on X. The ordered space E is briefly called the metrizing space for X.

A sequence (x_n) of elements of an E-metric space X is said *convergent* toward $x \in X$ (and we write $x_n \to x$) if $d(x_n, x) \to 0$ as $n \to \infty$.

A sequence (x_n) in X is said to be a *Cauchy sequence* if $d(x_n, x_m) \to 0$ as $n, m \to \infty$ (to be more precise, in the ordered linear space E, the limit is defined only for sequences and not for double sequences. To avoid any misunderstanding, $d(x_n, x_m) \to 0$ for $n, m \to \infty$, means $d(x_{n_k}, x_{m_k}) \to 0$ as $k \to \infty$ for all choice of the subsequences (n_k), (m_k).

The E-metric space X is said to be *sequentially complete* if each Cauchy sequence in X converges to a point in X.

A subset Y of an E-metric space X is said to be *bounded* if the set $\{d(x, y): x, y \in Y\}$ has an upper bound in E.

For more informations concerning ordered linear spaces, E-metric spaces and related topics, see [3] and [10].

We give some remarks with respect to the above definitions.

(a) In an ordered topological vector space $(E, \tau, \leq)$, the τ-convergence is not, in general, a linear convergence. In fact, Axioms 1 - 5 hold but Axiom 6, essential in our proofs, doesn't always hold. For this, some additional hypotheses on E are needed. For example, if E is an ordered Banach space, the "policeman lemma" (the more popular name for Axiom 6) holds if and only if the norm on E is semimonotone, i.e. if there exists $b > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq b\|y\|$ (Theorem 4.3 of [4]).

(b) In solving (1) we are interested only on sequences convergence in X (more precisely, we are interested on the existence of an $\hat{x} \in X$ such that $A^n\hat{x}$ converges). More general types of convergence (nets, filters, etc.) are not involved in this context.

(c) The space X in which we want to solve (1), is usually equipped with an algebraic-topological structure that can be described independently from an E-metric d. The introduction of a such E-metric d (not uniquely determined) on X is justified by the fact that d is choosen in such a way that the operator A is persuaded to become "contractive". In this situation d is subject to the condition: $d(x_n, x) \to 0$ implies $x_n \to x$ in the topology of X.

2. Results.

The following Theorems 1 and 2 are our main results.

Theorem 1. *Let E be an ordered linear space with a notion of linear convergence in which every nonempty countable subset having an upper bound has the supremum. Let X be a sequentially complete E-metric space. Let $S: E_+ \to E_+$ be an increasing operator and let $A: X \to X$ be an S-contraction i.e. $d(Ax, Ay) \leq Sd(x, y)$ $\forall x, y \in X$. Furthermore S is "small" in the following sense:*

(i) $f_n \in E_+$, $\lim f_n = 0$ *imply* $\lim Sf_n = S0$.
(ii) *there exists* $x_0 \in X$ *such that the orbit* $(A^n x_0)$ *is bounded and* $S^n f_0 \to 0$ *where* f_0 *is the diameter of the orbit,*

$$f_0 = \sup_{n,m \geq 0} d(A^n x_0, A^m x_0).$$

Then there exists $\hat{x} = \lim A^n x_0$ *and* $\hat{x} = A\hat{x}$. *Moreover, if* $S^n d(x, y) \to 0$ $\forall x, y \in X$, *then* A *has a unique fixed point.*

Proof. Let

$$f_n = \sup_{h,k \geq n} d(A^h x_0, A^k x_0), \quad n = 0, 1, 2 \ldots .$$

Then

$$f_n \leq \sup_{h,k \geq n} Sd(A^{h-1}x_0, A^{k-1}x_0) \leq S\left[\sup_{h,k \geq n-1} d(A^h x_0, A^k x_0)\right] =$$

$$= Sf_{n-1} \leq S^2 f_{n-2} \leq S^n f_0 \to 0,$$

and consequently $(A^n x_0)$ is a Cauchy sequence in X. From the sequential completeness of the space X, we have that there exists $\hat{x} \in X$ such that $\hat{x} = \lim A^n x_0$. Now we show that $\hat{x} = A\hat{x}$. First, we note that $S0 = 0$, since $d(\hat{x}, A\hat{x}) \leq d(\hat{x}, A^n x_0) + d(A^n x_0, A\hat{x}) \leq d(\hat{x}, A^n x_0) + Sd(A^{n-1}x_0, \hat{x}) \to 0$. Moreover, if $S^n d(x, y) \to 0$ $\forall x, y \in X$ and $Ax_1 = x_1$, $Ax_2 = x_2$, then $d(x_1, x_2) = d(Ax_1, Ax_2) \leq S^n d(x_1, x_2)$, so $x_1 = x_2$. □

Theorem 2. *Let* E *be an ordered linear space with a notion of linear convergence. Let* X *be a sequentially complete* E*-metric space. Let* (S_n) *be a family of operators* $S_n\colon E_+ \to E_+$ *and let* $A\colon X \to X$ *be a map such that*

$$d(A^n x, A^n y) \leq S_n d(x, y) \quad \forall n \in N \quad \textit{and} \quad x, y \in X.$$

Suppose that:

(i) $f_k \in E_+$, $\lim f_k = 0$ *imply* $\lim S_1 f_k = 0$;
(ii) *there exists* $x_0 \in X$ *such that the series* $\sum S_n d(x_0, Ax_0)$ *is convergent.*

Then A *has at least one fixed point. Moreover, if either* S_1 *is an increasing operator such that* $S_1^k d(x, y) \to 0$ $\forall x, y \in X$ *or* $S_n d(x, y) \to 0$ $\forall x, y \in X$, *then* A *has a unique fixed point.*

Proof. We note that the sequence $(A^n x_0)$ is a Cauchy sequence. Indeed,

$$d(A^{n+m}x_0, A^n x_0) \le \sum_{i=1}^{m} d(A^{n+1-1}x_0, A^{n+1}x_0) \le \sum_{i=1}^{m} S_{n+i-1} d(x_0, Ax_0) =$$
$$= \sum_{k=n}^{n+m-1} S_k d(x_0, Ax_0) \to 0.$$

Let $\hat{x} = \lim A^n x_0$. Then

$$d(\hat{x}, A\hat{x}) \le d(\hat{x}, A^{n+1}x_0) + d(A^{n+1}x_0, A\hat{x}) \le$$
$$\le d(\hat{x}, A^{n+1}x_0) + S_1 d(A^n x_0, \hat{x}) \to 0$$

so $\hat{x}$ is the required fixed point of A. □

The following Corollary 1 is a more or less known formulation of the Contraction Mapping Principle in the setting of E-metric spaces.

Corollary 1. *Let E, X be as in Theorem 2. Let $S: E_+ \to E_+$ be an increasing operator and let $A: X \to X$ be an S-contraction.*
Suppose that:

(i) $f_n \in E_+$, $\lim f_n = 0$ *imply* $\lim Sf_n = S0$;
(ii) there exists $x_0 \in X$ such that the series $\sum S^n d(x_0, Ax_0)$ is convergent.

Then there exists $\hat{x} = \lim A^n x_0$ and $\hat{x} = A\hat{x}$. Moreover, if $S^n d(x; y) \to 0$ $\forall x, y \in X$, then A has a unique fixed point.

Remark 1. Suppose that in the Corollary 1, E is an ordered Banach space with the norm convergence and $S: E_+ \to E_+$ is the restriction to E_+ of a continuous linear operator $S: E \to E$.
If the spectral radius of S is less than 1, then (ii) holds for every $x \in X$, as consequence of the root test for the serie $\sum S^n d(x, Ax)$.

Remark 2. In the Corollary 1, the hypothesis $S^n d(x, y) \to 0$ $\forall x, y \in X$ is essential for the uniqueness. As counterexample one can think to A and S equal to the identity map.

The Corollaries 2 and 3 below exemplify the possibility to absorb in the setting of E-metric spaces, fixed points theory in uniform spaces.

Corollary 2. (Theorem 2 of [1]) *Let X be a Hausdorff sequentially complete uniform space with uniformity defined by a satured family of semimetrics $(d_i)_{i\in I}$, I being an index set. Let $\mathscr{F} = (F_i)_{i\in I}$ be a family of functions $F_i\colon R_+ \to R_+$ with the properties:*

(i) *F_i is monotone increasing and continuous from the right on R_+ $\forall i \in I$.*
(ii) *$F_i(t) < t$ if $t > 0$ and $g\colon I \to I$ is a map of the index set I into itself.*
(iii) *$\forall i \in I$ there exists a function F_i^* in the family $\mathscr{F}$ such that*

$$\sup_{n\geq 0} F_{g^n(i)}(t) \leq F_i^*(t)$$

and $F_i^*(t)/t$ is increasing.
Let $A\colon X \to X$ be a $\mathscr{F}$-contraction on X, i.e.

$$d_i(Ax, Ay) \leq F_i(d_{g(i)}(x, y)) \quad \forall i \in I.$$

Suppose that there exists an element $x_0 \in X$ such that
(iv) *$d_{g^n(i)}(x_0, Ax_0) \leq p(i) < \infty$ $\forall n$.*
Then A has at least one fixed point. If in addition we suppose that
(v) *the sequence $(d_{g^n(i)}(x, y))_{n\geq 0}$ is bounded $\forall i \in I$, $\forall x, y \in X$, i.e.*

$$d_{g^n(i)}(x, y) \leq q(x, y, i) < \infty, \quad n \geq 0,$$

then the fixed point of a is unique.

Proof. We choose as metrizing space for X the space $\mathbb{R}^I$, endowed with the pointwise operations, ordering and convergence. We define $d\colon X \times X \to E_+$ and $S\colon E_+ \to E_+$ respectively by the equalites

$$[d(x, y)](i) = d_i(x, y)$$

$$[Sf](i) = F_i(f(g(i))).$$

Now, it is enough to show that the hypotheses of Corollary 1 are satisfied. We proceed in three steps.

1. *S is an increasing operator.*
   Indeed, let $f_1 \leq f_2$. Then, $\forall i \in I$

$$[Sf_1](i) = F_i(f_1(g(i))) \leq F_i(f_2(g(i))) = [Sf_2](i).$$

2. $f_n \in E_+$, $\lim f_n = 0$ *imply* $\lim Sf_n = S0$.

Indeed,

$$0 \leq [Sf_n](i) = F_i(f_n(g(i))) \leq f_n(g(i)) \to 0 = [S0](i).$$

3. $\sum S^n d(x_0, Ax_0)$ *is a convergent series.*
Indeed,

$$[S^n d(x_0, Ax_0)](i) = F_i(F_{g(i)} \dots F_{g^{n-1}(i)}(d_{g^n(i)}(x_0, Ax_0)) \dots) \leq$$

$$\leq F_i(F_{g(i)} \dots F_{g^{n-1}(i)}(p(i)) \dots) \leq F_i^{*n}(p(i)),$$

so it is enough to show that

$$\sum F_i^{*n}(p(i)) < \infty.$$

If there exists n such that $F_i^{*n}(p(i)) = 0$, the series is really a finite sum. Otherwise, from (iii) and (ii) we have

$$\frac{F_i^{*n+1}(p(i))}{F_i^{*n}(p(i))} = \frac{F_i^*(F_i^{*n}(p(i))}{F_i^{*n}(p(i))} \leq \frac{F_i^*(p(i))}{p(i)} < 1$$

and so, by the ratio test, we are done.
If, in addition, the hypothesis (v) is satisfied, then

$$[S^n d(x, y)](i) = F_i(F_{g(i)}(\dots(d_{g^n(i)}(x, y))\dots)) \leq$$
$$\leq F_i(F_{g(i)}(\dots q(x, y, i))\dots)) \leq F_i^{*n}(q(x, y, i))$$

and the last sequence converges to 0 from hypotheses (i) and (ii). □

Corollary 3. *Let X be as in Corollary 2. Let $\mathscr{F} = (F_i)_{i \in I}$ be a family of functions $F_i \colon R_+ \to R_+$ with the properties:*

(i) F_i is monotone, increasing and continuous from the right on R_+, $\forall i \in I$.

Let $A \colon X \to X$ be a generalized contraction, i.e.

(ii) $\forall i \in I$, $\forall n \in N$, there exists $F_{i,n}$ in the family $\mathscr{F}$ such that

$$d_i(A^n x, A^n y) \leq F_{i,n}(d_{g(i,n)}(x, y)), \quad g \colon IxN \to I.$$

Suppose that there exists an element $x_0 \in X$ such that

$$d_{g(i,n)}(x_0, Ax_0) \leq p(i) < \infty (n = 0, 1, \dots) \quad \text{and} \quad \sum F_{i,n}(p(i)) < \infty.$$

Then A has at least one fixed point.

Proof. Put $[S_n f](i) = F_{i,n}(f(g(i,n)))$.
Following the line of the proof of Corollary 2, it is easy to show that the hypotheses of Theorem 2 are satisfied. □

In the next theorem, we prove an abstract nonlinear Cauchy-Kowalewski Theorem in a scale of Banach spaces with conditions similar to those used by Zabrejko-Makarevich [11].

Theorem 3. (weak singularity) *Let $(B_s, \| \|_s)_{s \in [0,1[}$ be a family of Banach spaces such that $r > s$ implies $B_r \subset B_s$ and $\| \|_s \leq \| \|_r$.*
Let $B = \cap B_s$ be the intersection of the spaces B_s. Assume that a given function $f: [0,T] \times B \to B$ satisfies the following conditions:

(i) For every pair of numbers r, s such that $0 \leq s < r < 1$, f is a continuous mapping from the $[0,T] \times B_r$-topology to the B_s-topology.
(ii) $\|f(t,x_1) - f(t,x_2)\|_s \leq c(r-s)^{-a} \|x_1 - x_2\|_r$ $\forall r > s$ ($c \in \mathbb{R}$, $0 < a < 1$).

Then the Cauchy Problem

$$(\mathrm{CP}) \begin{cases} x' = f(t,x) \\ x(0) = x_0 \end{cases}$$

has a unique continuously differentiable solution $x: [0,T] \to B$.

Proof. We set

(a) $E = \mathbb{R}^{[0,T] \times [0,1[}$ equipped with the pointwise algebraic operations and order. The linear convergence on E useful in the sequel is the following: $z_n \to z$ if and only if $\forall s$, $z_n(t,s) \to z(t,s)$ uniformly in t.

(b) $X = \{x: [0,T] \to B$ continuous as B_s-valued function, $\forall s\}$.

$$d: XxX \to E_+ \quad \text{defined by} \quad [d(x,y](t,s) = \|x(t) - y(t)\|_s.$$

(c) $A: X \to X$ defined by

$$(Ax)(t) = x_0 + \int_0^t f(v, x(v))\, dv.$$

(d) $S: d(XxX) \to E_+$ defined by

$$(Sz)(t,s) = c \inf_{r>s} (r-s)^{-a} \int_0^t z(v,r)\, dv.$$

First we note that:

The E-metric space X is sequentially complete. Indeed, if (x_n) is a Cauchy sequence in X, i.e. $\|x_n(t)-x_m(t)\|_s \to 0$ uniformly in t for every fixed $s \in [0, 1[$, then there exists $\lim x_n(t) =: x(t)$ in B_s for every s. Moreover, the uniform convergence in t and the hypothesis $\| \|_s \leq \| \|_r$ for $s < r$ yield $x \in X$.

From the definition of convergence in E, the nonlinear increasing operator S is sequentially continuous in 0.

From the hypotheses (i) and (ii) it follows that the operator A is an S-contraction.

Now we show that the series $\sum S^n d(x, y)$ is convergent for every $x, y \in X$, i.e. $\sum (S^n d(x, y))(t, s)$ is a convergent series, uniformly in t for every fixed s. Indeed, if $s < r_1$, then

$$(*) \qquad [Sd(x, y)](t, s) \leq c(r_1 - s)^{-a} M(r_1)t$$

where $M(r_1) = \max\limits_{t \in [0,T]} \|x(t) - y(t)\|_{r_1}$.

In particular, from $(*)$ it follows, for $s < r_2 < r_1$,

$$[Sd(x, y)](t, r_2) \leq c(r_1 - r_2)^{-a} M(r_1)t$$

and

$$[S^2 d(x, y)](t, s) \leq c(r_2 - s)^{-a} \int_0^t [Sd(x, y)](v, r_2) dv \leq$$

$$\leq c^2 M(r_1)(r_1 - r_2)^{-a}(r_2 - s)^{-a} t^2/2!$$

And by induction, for every $n \in \mathbb{N}$ and for every choice of $s < r_n < \ldots < r_1$, we have

$$[S^n d(x, y)](t, s) \leq c^n M(r_1)(r_1 - r_2)^{-a}(r_2 - r_3)^{-a} \ldots (r_n - s)^{-a} t^n/n!.$$

From the last inequality, if for every n we choice the scalars r_j equidistributed, (i.e. $r_j = s + (n - j + 1)(r_1 - s)/n,\ j = 1, \ldots, n$) we have

$$[S^n d(x, y)](t, s) \leq c^n M(r_1) n^{an} (r_1 - s)^{-an} t^n/n!.$$

Since

$$\lim_n [[S^n d(x, y)](t, s)]^{1/n} \leq \lim_n ce(1 - s)^{-a} n^{a-1} t = 0$$

for every s uniformly in t, the root test assures that the series $\sum S^n d(x, y)$ is convergent in E.

From Corollary 1 it follows that there exists a unique fixed point $\hat{x}$ of the operator A and $A^n x \to \hat{x}$ for every $x \in X$.

Of course $\hat{x}$ is the unique solutions of (CP). $\square$

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