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INFINITELY MANY SOLUTIONS TO THE DIRICHLET PROBLEM FOR QUASILINEAR ELLIPTIC SYSTEMS INVOLVING THE P(X) AND Q(X)-LAPLACIAN

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In this paper we consider the Dirichlet problem involving the $p(x)$ and $q(x)$ -Laplacian of the type

$$\begin{cases} -\Delta_{p(x)}u = f(u,v) \text{ in } \Omega \\ -\Delta_{q(x)}v = g(u,v) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ v = 0 \text{ on } \partial\Omega \end{cases}$$

and, by applying a critical point variational principle obtained by Ricceri as a consequence of a more general variational principle, we prove the existence of infinitely many solutions.

1. Introduction

The study of nonlinear elliptic differential equations and their variational problems, involving the $p(x)$-Laplacian operator, has received considerable attention in recent years. Such equations arise, for istance, by modelling of electrorheological fluids, which are special viscous fluids characterized by their ability to undergo significant changes in their mechanical properties when an electric

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field is applied (see [7] and references therein) and as well as within the mathematical frame-work of the elasticity theory (see [8]). All these approaches are based on the theory of variable exponent Lebesgue and Sobolev spaces which are the generalization of the corresponding standard theories and also particular cases of the generalized Orlicz and Orlicz-Sobolev spaces, introduced and investigated earlier by Hudzik (see [4]). In this paper we study the quasilinear elliptic system:

$$(P) \qquad \begin{cases} -\Delta_{p(x)}u = f(u,v) \text{ in } \Omega \\ -\Delta_{q(x)}v = g(u,v) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ v = 0 \text{ on } \partial\Omega \end{cases}$$

where the following conditions are satisfied:

$(D1)$ $\Omega \subseteq \mathbb{R}^N$ a bounded open with C^1-boundary $\partial\Omega$, $N \geq 1$, $p,q \in C^0(\overline{\Omega})$,
$N < p^- = \inf_{x\in\Omega} p(x)$, $p^+ = \sup_{x\in\Omega} p(x) < +\infty$
$N < q^- = \inf_{x\in\Omega} q(x)$, $q^+ = \sup_{x\in\Omega} q(x) < +\infty$

$(D2)$ $f,g \in C^0(\mathbb{R}^2)$ such that the differential form $f(u,v)du + g(u,v)dv$ be exact.

The operator $-\Delta_{p(x)}u = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be $p(x)$-Laplacian, which restores the usual p-Laplacian in the $p(x)$ constant case. In [1] the author has considered the existence of infinitely many solutions for the Dirichlet problem (P) under the hypotheses $(D1)$and $(D2)$ in the $p(x)$ and $q(x)$ constant cases. The aim of the present paper is to generalize the main results of [1] to the $p(x), q(x)$-Laplacian non-constant cases. This paper is organized as follows: in section 2, we present some necessary preliminary knowledge on variable-exponent Lebesgue and Sobolev spaces; in section 3, we give the main results of this work by using the following critical point theorem whose proof (see theorem 2.5 of [6]), will be omitted.

Theorem A Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gateaux-differentiable functionals. Let also assume that Ψ is strongly continuous and coercive. For each $\rho > \inf_X \Psi$, we define

$$\varphi(\rho) := \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - \inf_{\overline{(\Psi^{-1}(]-\infty,\rho[))}_w} \Phi}{\rho - \Psi(x)}, \tag{1}$$

where $\overline{(\Psi^{-1}(]-\infty,\rho[))}_w$ stands for the closure of $\Psi^{-1}(]-\infty,\rho[)$ in the weak topology. Fixed $\lambda > 0$, then

(a) if $\{\rho_n\}_{n\in\mathbb{N}}$ is a real sequence with $\lim_{n\to+\infty}\rho_n = +\infty$ such that $\varphi(\rho_n) < \lambda$, $\forall n \in \mathbb{N}$, the following alternative holds: either $\Phi+\lambda\Psi$ has a global minimum, or there exists a sequence $\{x_n\}_n$ of critical points of $\Phi+\lambda\Psi$ such that $\lim_{n\to+\infty}\Psi(x_n) = +\infty$.

(b) if$\{\varepsilon_n\}_{n\in\mathbb{N}}$ is a real sequence with $\varepsilon_n > \inf_X \Psi$ $\forall n \in \mathbb{N}$ and $\lim_{n\to+\infty}\varepsilon_n = \inf_X \Psi$ such that $\varphi(\varepsilon_n) < \lambda$, $\forall n \in \mathbb{N}$, the following alternative holds: either there exists a global minimum of Ψ which is a local minimum of $\Phi+\lambda\Psi$, or there exists a sequence $\{x_n\}_n$ of pairwise distinct critical points of $\Phi+\lambda\Psi$, with $\lim_{n\to+\infty}\Psi(x_n) = \inf_X \Psi$, which weakly converges to a global minimum of Ψ.

In the following, we will denote with H the integral of the differential form $f(u,v)du + g(u,v)dv$ such that $H(0,0) = 0$.

2. Requirements

Let us denote by $M(\Omega)$ the set of all measurable real functions defined on a set $\Omega \subseteq \mathbb{R}^N$ with $|\Omega| > 0$. Two functions in $M(\Omega)$ are considered as the same element of $M(\Omega)$ when they are equal almost everywhere. Let $r(x) \in C^0(\overline{\Omega})$ such that $N < r^-$ and $r^+ < +\infty$, where $r^- = \inf_{x\in\Omega} r(x)$ and $r^+ = \sup_{x\in\Omega} r(x)$. In the following we'll denote as

$$L^{r(x)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u(x)|^{r(x)} dx < +\infty \right\}$$

with the norm

$$\| u \|_{L^{r(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{r(x)} dx \leq 1 \right\}$$

and

$$W^{1,r(x)}(\Omega) = \left\{ u \in L^{r(x)}(\Omega) : |\nabla u| \in L^{r(x)}(\Omega) \right\}$$

with the norm

$$\| u \|_{W^{1,r(x)}(\Omega)} = \| u \|_{L^{r(x)}(\Omega)} + \| \nabla u \|_{L^{r(x)}(\Omega)}$$

We'll use also the compact notation $|u|_{r(x)} = \| u \|_{L^{r(x)}(\Omega)}$ and $\| u \|_{1,r(x)} =$ $= \| u \|_{W^{1,r(x)}(\Omega)}$, while we'll denote as $W_0^{1,r(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,r(x)}(\Omega)$.

Proposition 2.1. (see [3, 5]) The $L^{r(x)}(\Omega)$, $W^{1,r(x)}(\Omega)$ and $W_0^{1,r(x)}(\Omega)$ spaces are separable, reflexive and uniformly convex Banach spaces.

Proposition 2.2. (see [3]) Let $u \in L^{r(x)}(\Omega)$ and $\rho(u) = \int_\Omega |u(x)|^{r(x)}dx$. Then for any $u \in L^{r(x)}(\Omega)$ the following assertions hold:

(1) For $u \neq 0$, $|u|_{r(x)} = \lambda \Longleftrightarrow \rho(\frac{u}{\lambda}) = 1$

(2) $|u|_{r(x)} < 1 (= 1; > 1) \Longleftrightarrow \rho(u) < 1 (= 1; > 1)$

(3) If $|u|_{r(x)} > 1$, then $|u|_{r(x)}^{r^-} \leq \rho(u) \leq |u|_{r(x)}^{r^+}$

(4) If $|u|_{r(x)} < 1$, then $|u|_{r(x)}^{r^+} \leq \rho(u) \leq |u|_{r(x)}^{r^-}$

(5) $\lim_{k\to+\infty} |u_k|_{r(x)} = 0 \Longleftrightarrow \lim_{k\to+\infty} \rho(u_k) = 0$

(6) $\lim_{k\to+\infty} |u_k|_{r(x)} = +\infty \Longleftrightarrow \lim_{k\to+\infty} \rho(u_k) = +\infty$.

Proposition 2.3. (see [2]) In $W_0^{1,r(x)}(\Omega)$ the Poincaré inequality holds, that is, it does exist a positive constant K such that

$$|u|_{r(x)} \leq K|\nabla u|_{r(x)}, \ \forall u \in W_0^{1,r(x)}(\Omega)$$

The expression $|\nabla u|_{r(x)}$ defines a norm on the $W_0^{1,r(x)}(\Omega)$ space, equivalent to the norm $\| u \|_{1,r(x)}$. In the following we'll use such an equivalent norm, that is, we'll consider the spaces $W_0^{1,p(x)}(\Omega)$ and $W_0^{1,q(x)}(\Omega)$ with the norms $\| u \|_{W_0^{1,p(x)}(\Omega)} = |\nabla u|_{p(x)}$ and $\| v \|_{W_0^{1,q(x)}(\Omega)} = |\nabla v|_{q(x)}$ respectively.

3. Main results

Let X be the Cartesian product of $W_0^{1,p(x)}(\Omega)$ and $W_0^{1,q(x)}(\Omega)$ Sobolev spaces with the norm $\| (u,v) \|_X = \sqrt{|\nabla u|_{p(x)}^2 + |\nabla v|_{q(x)}^2}$ or another to its equivalent. We define two functionals $\Psi : X \longrightarrow \mathbb{R}$ and $\Phi : X \longrightarrow \mathbb{R}$ as

$$\Psi(u,v) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla v|^{q(x)} dx$$

$$\Phi(u,v) = -\int_\Omega H(u(x),v(x))dx$$

They are well defined, sequentially weakly lower semicontinuous and Gateaux differentiable in X, the critical points of $\Phi + \Psi$ being precisely the weak solutions to Problem (P). Moreover Ψ is coercive and strongly continuous.

Recall that $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and that, since $N < p^-$, $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus $W^{1,p(x)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Analogously, since $N < q^-$, $W^{1,q(x)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Let

$$C_1 = \sup_{u \in W^{1,p(x)}(\Omega) - \{0\}} \frac{|u|_{+\infty}}{\| u \|_{1,p(x)}}$$

and

$$C_2 = \sup_{v \in W^{1,q(x)}(\Omega) - \{0\}} \frac{|v|_{+\infty}}{\| v \|_{1,q(x)}}$$

be the relative imbedding constants.
We introduce some notation by setting

$$\alpha = \min\left\{ \frac{1}{p^+ {C_1}^{p^-} (K_1+1)^{p^-}}, \frac{1}{p^+ {C_1}^{p^+} (K_1+1)^{p^+}} \right\}$$

and

$$\beta = \min\left\{ \frac{1}{q^+ {C_2}^{q^-} (K_2+1)^{q^-}}, \frac{1}{q^+ {C_2}^{q^+} (K_2+1)^{q^+}} \right\}$$

where K_1 and K_2 are the constants according to the proposition 2.3. in the $p(x)$, $q(x)$ cases respectively and we define for each $t > 0$

$$A(t) = \left\{ (\xi, \eta) \in \mathbb{R}^2 : \alpha D_p(\xi) + \beta D_q(\eta) \le t \right\}$$

$$S(t) = \left\{ (\xi, \eta) \in \mathbb{R}^2 : D_p(\xi) + D_q(\eta) \le t \right\}$$

$$A'(t) = \left\{ (\xi, \eta) \in \mathbb{R}^2 : \alpha F_p(\xi) + \beta F_q(\eta) \le t \right\}$$

$$S'(t) = \left\{ (\xi, \eta) \in \mathbb{R}^2 : F_p(\xi) + F_q(\eta) \le t \right\}$$

where we have put

$$D_r(\mu) = \max\left(|\mu|^{r^+}, |\mu|^{r^-} \right) \quad F_r(\mu) = \min\left(|\mu|^{r^+}, |\mu|^{r^-} \right)$$

with $r \in \{p, q\}$ and $\mu \in \{\xi, \eta\}$. Moreover we indicate with ω the measure $\frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})}$ of the n-dimensional unit ball.
The following inclusions hold:

$$S\left(\frac{t}{\max(\alpha, \beta)} \right) \subseteq A(t) \subseteq A'(t) \subseteq S'\left(\frac{t}{\min(\alpha, \beta)} \right)$$

Teorema 3.1. Let us suppose there exist two real sequences $\{a_n\}_n$ and $\{b_n\}_n$ in $]0,+\infty[$ with $a_n < b_n$, $\forall n \in \mathbb{N}$, $\lim_{n\to+\infty} b_n = +\infty$ such that

$$\lim_{n\to+\infty} \frac{b_n}{a_n} = +\infty$$

$$\max_{S(a_n)} H = \max_{S'(b_n)} H > 0 \ \ \forall n \in \mathbb{N}$$

$$\max\left\{ \frac{2^{p^+}(2^N-1)}{p^- D^{p^*}}, \frac{2^{q^+}(2^N-1)}{q^- D^{q^*}} \right\} < \limsup_{(\xi,\eta)\to+\infty} \frac{H(\xi,\eta)}{D(\xi,\eta)} < +\infty$$

where $D = \sup_{x\in\Omega} d(x,\partial\Omega)$, $D^{p^*} = \min\{D^{p^-}, D^{p^+}\}$, $D^{q^*} = \min\{D^{q^-}, D^{q^+}\}$ and $D(\xi,\eta) = D_p(\xi) + D_q(\eta)$. Let us assume also that $\inf_{\mathbb{R}^2} H \geq 0$. Then the problem (P) admits an unbounded sequence of weak solutions.

Proof. We fix, for each $n \in \mathbb{N}$, $(\xi_n, \eta_n) \in S(a_n)$ such that $\max_{S'(b_n)} H = H(\xi_n, \eta_n)$ and we put $\rho_n = \delta b_n \ \forall n \in \mathbb{N}$ with $\delta = \min(\alpha, \beta)$. Then $\{\rho_n\}_n$ is real sequence of positive terms, divergent and such that

$$\lim_{n\to+\infty} \frac{\rho_n}{D(\xi_n,\eta_n)} = +\infty \tag{2}$$

We wish to prove that $\varphi(\rho_n) < 1$ for $n \in \mathbb{N}$ large enough (here φ is the function (1) of the theorem A and $\lambda = 1$ is assumed), precisely we'll show that there exists a sequence $\{u_n, v_n\}_n \subset X$ with $\Psi(u_n, v_n) < \rho_n$ and such that

$$\frac{\Phi(u_n,v_n) - \inf_{\Psi^{-1}(]-\infty,\rho_n])} \Phi}{\rho_n - \Psi(u_n,v_n)} < 1$$

for $n \in \mathbb{N}$ large enough.
By choosing a constant h satisfying

$$\max\left\{ \frac{2^{p^+}(2^N-1)}{p^- D^{p^*}}, \frac{2^{q^+}(2^N-1)}{q^- D^{q^*}} \right\} < h < \limsup_{(\xi,\eta)\to+\infty} \frac{H(\xi,\eta)}{D(\xi,\eta)}$$

it must exist a point $x^* \in \Omega$ such that

$$\max\{C_{p^*}, C_{q^*}\} < d(x^*, \partial\Omega)$$

where

$$C_{r^*} = \max\{C_{r^-}, C_{r^+}\}$$

$$C_{r^-} = \left(\frac{2^{r^+}(2^N-1)}{r^- h}\right)^{\frac{1}{r^-}} \quad C_{r^+} = \left(\frac{2^{r^+}(2^N-1)}{r^- h}\right)^{\frac{1}{r^+}}$$

with $r \in \{p,q\}$. Now, we fix γ such that

$$\max\left\{C_{p^*}, C_{q^*}\right\} < \gamma < d(x^*, \partial\Omega)$$

hence, by setting

$$\Gamma_r = \frac{2^{r^+}(2^N-1)}{r^-}\max\left(\frac{1}{\gamma^{r^-}}, \frac{1}{\gamma^{r^+}}\right)$$

with $r \in \{p,q\}$, it follows $\max\{\Gamma_p, \Gamma_q\} < h$.
For each $n \in \mathbb{N}$ let us consider the functions $u_n \in W_0^{1,p(x)}(\Omega)$ and $v_n \in W_0^{1,q(x)}(\Omega)$ defined by

$$u_n(x) = \begin{cases} 0 & \text{if } x \in \Omega\backslash B(x^*,\gamma) \\ \xi_n & \text{if } x \in B(x^*, \frac{\gamma}{2}) \\ \frac{2\xi_n}{\gamma}(\gamma - |x-x^*|_N) & \text{if } x \in B(x^*,\gamma)\backslash B(x^*, \frac{\gamma}{2}) \end{cases}$$

$$v_n(x) = \begin{cases} 0 & \text{if } x \in \Omega\backslash B(x^*,\gamma) \\ \eta_n & \text{if } x \in B(x^*, \frac{\gamma}{2}) \\ \frac{2\eta_n}{\gamma}(\gamma - |x-x^*|_N) & \text{if } x \in B(x^*,\gamma)\backslash B(x^*, \frac{\gamma}{2}) \end{cases}$$

Thus one has

$$\Psi(u_n,v_n) = \int_\Omega \frac{1}{p(x)}|\nabla u_n(x)|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v_n(x)|^{q(x)}dx =$$

$$= \int_{B(x^*,\gamma)\backslash B(x^*,\frac{\gamma}{2})} \frac{1}{p(x)}\left|\frac{2\xi_n}{\gamma}\right|^{p(x)}dx + \int_{B(x^*,\gamma)\backslash B(x^*,\frac{\gamma}{2})} \frac{1}{q(x)}\left|\frac{2\eta_n}{\gamma}\right|^{q(x)}dx \leq$$

$$\leq \left[\Gamma_p D_p(\xi_n) + \Gamma_q D_q(\eta_n)\right]\frac{\omega\gamma^N}{2^N} < h\cdot D(\xi_n,\eta_n)\frac{\omega\gamma^N}{2^N} < \rho_n \tag{3}$$

for $n \in \mathbb{N}$ large enough.
Finally, since

$$\limsup_{(\xi,\eta)\to+\infty} \frac{H(\xi,\eta)}{D(\xi,\eta)}$$

is finite, it does exist $L > 0$ such that $\frac{H(\xi_n,\eta_n)}{D(\xi_n,\eta_n)} < L\, \forall n \in \mathbb{N}$, and from (2) follows

$$\frac{\rho_n}{D(\xi_n,\eta_n)} > L\left(|\Omega| - \frac{\omega\gamma^N}{2^N}\right) + \frac{h\omega\gamma^N}{2^N} \tag{4}$$

for all $n \in \mathbb{N}$ large enough.
Consequently from (4) and (3) we have

$$\Phi(u_n,v_n) - \inf_{\Psi^{-1}(]-\infty,\rho_n])}\Phi =$$

$$= \sup_{(u,v)\in\Psi^{-1}(]-\infty,\rho_n])} \int_\Omega H(u(x),v(x))dx - \int_\Omega H(u_n(x),v_n(x))dx \leq$$

$$\leq \int_\Omega \sup_{(\xi,\eta)\in A'(\rho_n)} H(\xi,\eta)dx - \int_{B(x^*,\frac{\gamma}{2})} H(u_n(x),v_n(x))dx \leq$$

$$\leq H(\xi_n,\eta_n)\left(|\Omega| - \frac{\omega\gamma^N}{2^N}\right) < H(\xi_n,\eta_n)\left[\frac{\rho_n}{D(\xi_n,\eta_n)} - \frac{h\omega\gamma^N}{2^N}\right]\cdot\frac{1}{L} =$$

$$= \frac{1}{L}\left[\rho_n - D(\xi_n,\eta_n)\frac{h\omega\gamma^N}{2^N}\right]\cdot\frac{H(\xi_n,\eta_n)}{D(\xi_n,\eta_n)} < \rho_n - \left[\Gamma_p D_p(\xi_n) + \Gamma_q D_q(\eta_n)\right]\frac{\omega\gamma^N}{2^N} <$$

$$< \rho_n - \left[\int_\Omega \frac{1}{p(x)}|\nabla u_n(x)|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v_n(x)|^{q(x)}dx\right] = \rho_n - \Psi(u_n,v_n)$$

for $n\in\mathbb{N}$ large enough.
In the previous inequalities we took into account that, $\forall n\in\mathbb{N}$ and for every $(u,v)\in\Psi^{-1}(]-\infty,\rho_n])$ there holds that $(|u|_{+\infty},|v|_{+\infty})\in A'(\rho_n)$ and consequently $(u(x),v(x))\in A'(\rho_n)$ $\forall x\in\Omega$. Moreover $S(a_n)\subseteq A(\rho_n)\subseteq A'(\rho_n)\subseteq S'(b_n)$ for all $n\in\mathbb{N}$ large enough holds. Now, we verify that the functional $\Phi+\Psi$ has no global minimum in X.
From $h<\limsup_{(\sigma,\tau)\to+\infty}\frac{H(\sigma,\tau)}{D(\sigma,\tau)}$, it follows that

$$h < \inf_{n\in\mathbb{N}}\left(\sup_{\sqrt{\sigma^2+\tau^2}\geq n}\frac{H(\sigma,\tau)}{D(\sigma,\tau)}\right),$$

so there must exist $\forall n\in\mathbb{N}$ a point $(\sigma_n,\tau_n)\in\mathbb{R}^2$ such that $\sqrt{\sigma_n^2+\tau_n^2}\geq n$ and

$$\frac{H(\sigma_n,\tau_n)}{D(\sigma_n,\tau_n)} > h$$

By considering the functions $\omega_n\in W_0^{1,p(x)}(\Omega)$, $\zeta_n\in W_0^{1,q(x)}(\Omega)$ defined by

$$\omega_n(x) = \begin{cases} 0 & \text{if } x\in\Omega\setminus B(x^*,\gamma) \\ \sigma_n & \text{if } x\in B(x^*,\frac{\gamma}{2}) \\ \frac{2\sigma_n}{\gamma}(\gamma-|x-x^*|_N) & \text{if } x\in B(x^*,\gamma)\setminus B(x^*,\frac{\gamma}{2}) \end{cases}$$

$$\zeta_n(x) = \begin{cases} 0 & \text{if } x\in\Omega\setminus B(x^*,\gamma) \\ \tau_n & \text{if } x\in B(x^*,\frac{\gamma}{2}) \\ \frac{2\tau_n}{\gamma}(\gamma-|x-x^*|_N) & \text{if } x\in B(x^*,\gamma)\setminus B(x^*,\frac{\gamma}{2}) \end{cases}$$

one has $\forall n\in\mathbb{N}$

$$(\Phi+\Psi)(\omega_n,\zeta_n) =$$

$$= \int_\Omega \frac{1}{p(x)}|\nabla\omega_n|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla\zeta_n|^{q(x)}dx - \int_\Omega H(\omega_n(x),\zeta_n(x))dx =$$

$$= \int_{B(x^*,\gamma)\setminus B(x^*,\frac{\gamma}{2})} \frac{1}{p(x)}\left|\frac{2\sigma_n}{\gamma}\right|^{p(x)} dx + \int_{B(x^*,\gamma)\setminus B(x^*,\frac{\gamma}{2})} \frac{1}{q(x)}\left|\frac{2\tau_n}{\gamma}\right|^{q(x)} dx -$$

$$- \int_\Omega H(\omega_n(x),\zeta_n(x))dx \le \left[\Gamma_p D_p(\sigma_n) + \Gamma_q D_q(\tau_n)\right]\frac{\omega\gamma^N}{2^N} - \int_{B(x^*,\frac{\gamma}{2})} H(\sigma_n,\tau_n) <$$

$$< \frac{\omega\gamma^N}{2^N}\left[(\Gamma_p - h)D_p(\sigma_n) + (\Gamma_q - h)D_q(\tau_n)\right] < 0$$

Since $\{\sqrt{\sigma_n^2+\tau_n^2}\}_n$ is unbounded, at least one of the two sequences $\{\sigma_n\}_n$ or $\{\tau_n\}_n$ admits one divergent subsequence. Hence $\{\max(|\sigma_n|^{p^+},|\sigma_n|^{p^-})\}_n$ or $\{\max(|\tau_n|^{q^+},|\tau_n|^{q^-})\}_n$ admit one divergent subsequence, so the functional $\Phi+\Psi$ is not bounded from below. The conclusion (a) of theorem A assures that there is a sequence $\{x_n\}_n$ of critical points of $\Phi+\Psi$ such that $\lim_{n\to+\infty}\| x_n \|_X = +\infty$.

Theorem 3.2. Let us suppose there exist two real sequences $\{a_n\}_n$ and $\{b_n\}_n$ in $]0,+\infty[$ with $a_n < b_n$, $\forall n\in\mathbb{N}$, $\lim_{n\to+\infty} b_n = 0$ such that

$$\lim_{n\to+\infty}\frac{b_n}{a_n} = +\infty$$

$$\max_{S(a_n)} H = \max_{S'(b_n)} H > 0 \ \ \forall n\in\mathbb{N}$$

$$\max\left\{\frac{2^{p^+}(2^N-1)}{p^- D^{p^*}}, \frac{2^{q^+}(2^N-1)}{q^- D^{q^*}}\right\} < \limsup_{(\xi,\eta)\to(0,0)}\frac{H(\xi,\eta)}{D(\xi,\eta)} < +\infty$$

where $D=\sup_{x\in\Omega} d(x,\partial\Omega)$, $D^{p^*}=\min\{D^{p^-},D^{p^+}\}$, $D^{q^*}=\min\{D^{q^-},D^{q^+}\}$ and $D(\xi,\eta)=D_p(\xi)+D_q(\eta)$. Let us assume also that $\inf_{\mathbb{R}^2} H \ge 0$. Then the problem (P) admits a sequence of non-zero weak solutions which strongly converges to θ_X in X.

Proof. We fix, for each $n\in\mathbb{N}$, $(\xi_n,\eta_n)\in S(a_n)$ such that $\max_{S'(b_n)} H = H(\xi_n,\eta_n)$ and we put $\varepsilon_n = \delta b_n$ $\forall n\in\mathbb{N}$ with $\delta=\min(\alpha,\beta)$. Then $\{\varepsilon_n\}_n$ is real sequence of positive terms, infinitesimal and such that

$$\lim_{n\to+\infty}\frac{\varepsilon_n}{D(\xi_n,\eta_n)} = +\infty$$

By choosing a constant h satisfying

$$\max\left\{\frac{2^{p^+}(2^N-1)}{p^- D^{p^*}}, \frac{2^{q^+}(2^N-1)}{q^- D^{q^*}}\right\} < h < \limsup_{(\xi,\eta)\to(0,0)} \frac{H(\xi,\eta)}{D(\xi,\eta)}$$

and proceeding as in the proof of the theorem 3.1., one simply obtains that $\varphi(\varepsilon_n) < 1$ i.e. there exists a sequence $\{u_n, v_n\}_n \subset X$ with $\Psi(u_n, v_n) < \varepsilon_n$ and such that

$$\Phi(u_n, v_n) - \inf_{\Psi^{-1}(]-\infty,\varepsilon_n])} \Phi < \varepsilon_n - \Psi(u_n, v_n)$$

for $n \in \mathbb{N}$ large enough.
Now, we verify that (θ, θ) (the only global minimum of Ψ) is not a local minimum of $\Phi + \Psi$.
From $h < \limsup_{(\sigma,\tau)\to(0,0)} \frac{H(\sigma,\tau)}{D(\sigma,\tau)}$, it follows that

$$h < \inf_{n\in\mathbb{N}} \left(\sup_{\sqrt{\sigma^2+\tau^2}\leq\frac{1}{n}} \frac{H(\sigma,\tau)}{D(\sigma,\tau)} \right),$$

so there must exist $\forall n \in \mathbb{N}$ a point $(\sigma_n, \tau_n) \in \mathbb{R}^2$ such that $\sqrt{\sigma_n^2 + \tau_n^2} \leq \frac{1}{n}$ and

$$\frac{H(\sigma_n, \tau_n)}{D(\sigma_n, \tau_n)} = \frac{H(\sigma_n, \tau_n)}{|\sigma_n|^{p^-} + |\tau_n|^{q^-}} > h$$

By considering the functions $\omega_n \in W_0^{1,p(x)}(\Omega)$, $\zeta_n \in W_0^{1,q(x)}(\Omega)$ of the preceding theorem one obtains

$$(\Phi + \Psi)(\omega_n, \zeta_n) \leq$$

$$\leq \int_{B(x^*,\gamma)\setminus B(x^*,\frac{\gamma}{2})} \frac{1}{p(x)} \left|\frac{2\sigma_n}{\gamma}\right|^{p(x)} dx + \int_{B(x^*,\gamma)\setminus B(x^*,\frac{\gamma}{2})} \frac{1}{q(x)} \left|\frac{2\tau_n}{\gamma}\right|^{q(x)} dx -$$

$$- \int_{B(x^*,\frac{\gamma}{2})} H(\omega_n(x), \zeta_n(x))dx \leq \left[\Gamma_p |\sigma_n|^{p^-} + \Gamma_q |\tau_n|^{q^-}\right] \frac{\omega\gamma^N}{2^N} -$$

$$- \int_{B(x^*,\frac{\gamma}{2})} H(\sigma_n, \tau_n) < \frac{\omega\gamma^N}{2^N} \left[(\Gamma_p - h)|\sigma_n|^{p^-} + (\Gamma_q - h)|\tau_n|^{q^-}\right] < 0$$

Since $\Phi(\theta_X) + \Psi(\theta_X) = 0$, $\| (\omega_n, \zeta_n) \|_X \to 0$ (from Proposition 2.2., assertions (3) and (4)) and $\Phi(\omega_n, \zeta_n) + \Psi(\omega_n, \zeta_n) < 0 \ \forall n \in \mathbb{N}$, θ_X can't be a local minimum of $\Phi + \Psi$. The conclusion (b) of theorem A assures that there is a sequence $\{x_n\}_n$ of critical points of $\Phi + \Psi$ such that $\lim_{n\to+\infty} \Psi(x_n) = \inf_X \Psi = 0$ with $x_n \rightharpoonup 0$; indeed $\lim_{n\to+\infty} \| x_n \|_X = 0$.

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