LE MATEMATICHE
Vol. XLVII (1992) – Fasc. I, pp. 3-8

CERTAIN OPERATORS IN THE SPACE OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES OF SEVERAL COMPLEX VARIABLES

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In this paper, we consider the space X of all Entire functions represented by Dirichlet series equipped with various toplogies. The main result is conserned with finding certain continuous linear operators which are used to determine the proper bases in X.

Let X denote the space of all Entire functions defined by Dirichlet series of two complex variables. For the sake of simplicity, we consider the case of two complex variables, thought our results can be easily extended to any finite integer n. The topological aspect of this space has been studied in details in [2] and [5]. The aim of this paper is to characterize certain continuous linear operators on X and use them in the determination of proper base in X; the correspondant case of an Entire Taylor series has been studied by Kamthan see [7] and [8].

When $f \in X$, it means that $f : \mathbb{C}^2 \to \mathbb{C}$, where $\mathbb{C}$ is the complex plane equipped with the usual topology, such that

$$(1) \qquad f(s_1, s_2) = \sum_{m,n=0}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$$

Entrato in Redazione il 6 febbraio 1991.

where the coefficients $a_{m,n}$ are complex numbers,

$$\lambda_0 = \mu_0 = 0 \; ; \quad (\lambda_m)_{m \geq 1} \, , \; (\mu_n)_{n \geq 1}$$

are two sequences of real increasing numbers whose limits are infinity and further [2]

$$\limsup_{m+n \to \infty} \frac{\log(m+n)}{\lambda_m + \mu_n} = D < \infty \tag{2}$$

$$\limsup_{m+n \to \infty} \frac{\log |a_{m,n}|}{\lambda_m + \mu_n} = -\infty \, . \tag{3}$$

The topological aspect (in various directions) of this space has been studied in details by the author (see [4], [5]). In this paper we have considered two topologies in X namely:

(1) The topology τ_1 generated by the family of semi-norms (indeed, norms) $\{M(\sigma_1, \sigma_2), \sigma_1, \sigma_2 \text{ real}\}$ where

$$M(f; \sigma_1, \sigma_2) = M(\sigma_1, \sigma_2) = \sup_{-\infty < t_1, t_2 < \infty} |f(\sigma_1 + it_1, \sigma_2 + it_2)| \tag{4}$$

(2) The topology τ_2 generated by the family of semi-norms (indeed, norms) $\{\|f; \sigma_1, \sigma_2\|, \sigma_1, \sigma_2 \, , \text{ real}\}$ where

$$\|f; \sigma_1, \sigma_2\| = \sum_{m,n=0}^{\infty} |a_{m,n}| \exp(\lambda_m \sigma_1 + \mu_n \sigma_2) \, . \tag{5}$$

One has proved earlier (see [3]) that for some $\alpha > 0$

$$M(\sigma_1, \sigma_2) \leq \|f; \sigma_1, \sigma_2\| \leq C(\alpha) \, M(\sigma_1 + \alpha, \sigma_2 + \alpha) \tag{6}$$

$C(\alpha)$ being a constant depending on α only. Hence one finds that the topologies τ_1 and τ_2 are equivalent.

To facilitate the present work, we recall few things from the earlier work [6].

Definition 1. A sequence $\{f_{m,n} : m, n \geq 0\} \subset X$ is said to be a base for X, if for each $f \in X$, there exists a unique sequence $\{a_{m,n} : m, n \geq 0\} \subset \mathbb{C}$ such that

$$f = \sum_{m,n=0}^{\infty} a_{m,n} f_{m,n}$$

where the convergence of the double series is with respect to the topology on X.

Definition 2. A base $\{f_{m,n} : m, n \geq 0\} \subset X$ will be called a genuine base for X if the corresponding coefficients in the expansion of an f satisfy eq. (3).

Definition 3. A sequence $\{f_{m,n} : m, n \geq 0\} \subset X$ will be called an absolute base for X, if each $f \in X$ can be uniquely expressed as $\sum a_{m,n} f_{m,n}$ where the double series is absolutely convergent with respect to the topology on X.

Definition 4. A sequence $\{f_{m,n} : m, n \geq 0\} \subset X$ will be called proper base for X if it is a genuine as will as an absolute base for X.

We have earlier shown that if$\{f_{m,n} : m, n \geq 0\}$ is a proper base, then [6]

$$\limsup_{m+n\to\infty} \frac{\log M(f_{m,n}; \sigma_1, \sigma_2)}{\lambda_m + \mu_n} < \infty \tag{7}$$

for each σ_1, σ_2.

Theorem 1. *Let $\{\alpha_{m,n};\ m, n \geq 0\} \subset X$. Suppose T is a linear operator from X into X, such that $T(\delta_{m,n}) = \alpha_{m,n} \quad m, n \geq 0$, where $\delta_{m,n}(s_1, s_2) = \exp(\lambda_m s_1 + \mu_n s_2)$. Then if T is a continuous operator provided*

$$\limsup_{m+n\to\infty} \frac{\log\ M(\alpha_{m,n};\ \sigma_1, \sigma_2)}{\lambda_m + \mu_n} < \infty \tag{8}$$

for each $\sigma_1, \sigma_2 \geq 0$.

Conversely, if eq. (8) holds, then there esists a continuous linear operator $T : X \to X$ such that

$$T(\delta_{m,n}) = \alpha_{m,n} \qquad m, n \geq 0\,.$$

Proof. Suppose T is a continuous linear operator from X into X with $T(\delta_{m,n}) = \alpha_{m,n}$. Then for a given σ_1, σ_2 there exist a positive constant K and $\sigma_1^0 > 0$, $\sigma_2^0 > 0$ such that:

$$M(T\,\delta_{m,n}; \sigma_1, \sigma_2) = M(\alpha_{m,n}; \sigma_1, \sigma_2)$$

$$\leq KM(\delta_{m,n}, \sigma_1^0, \sigma_2^0)$$
$$= K \exp(\lambda_m + \mu_,)\sigma \quad , \quad \sigma = \max(\sigma_1^0, \sigma_2^0) .$$

Hence

$$\limsup_{m+n\to\infty} \frac{\log M\ (\alpha_{m,n}\,; \sigma_1, \sigma_2)}{\lambda_m + \mu_n} < \infty$$

thus eq. (8) follows.

Conversely, assume that eq. (8) is true. Let $\alpha \epsilon X$, then α is represented by

$$\alpha = \sum_{m,n=0}^{\infty} a_{m,n}\, \delta_{m,n}$$

where the coefficients $a_{m,n}$'s, satisfy eq. (3). Since eq.(8) holds, therefore there exists an $A = A(\sigma_1, \sigma_2)$, depending on σ_1, σ_2 such that

$$\frac{\log M\ (\alpha_{m,n}; \sigma_1, \sigma_2)}{\lambda_m + \mu_n} \leq A \qquad \text{for all} \quad m+n \geq N$$

or

$$M(\alpha_{m,n}; \sigma_1, \sigma_2) \quad \leq \quad \exp A\,(\lambda_m + \mu_n) \qquad \text{for all} \quad m+n \geq N$$

therefore, noticing that eq. (3) is already valid for the coefficients a_{mn} 's we find that $\sum a_{m,n}\, \alpha_{m,n}$ converges in X and so it represents an elements of X. Hence there is a natural transformation $T : X \to X$ such that:

$$T(\alpha) = \sum_{m,n=0}^{\infty} a_{m,n}\, \alpha_{m,n}$$

with

$$\alpha = \sum_{m,n=0}^{\infty} a_{m,n}\, \delta_{m,n} .$$

Clearly $T(\delta_{m,n}) = \alpha_{m,n}, m, n \geq 0$. We are now required to show that T is continuous on (X, τ_1). To do that it is enought to prove that T is continuous on (X, τ_2). The norms $\|\ldots, \sigma_1, \sigma_2\|$ are continuous on X, therefore given any two reals σ_1, σ_2, we find

$$\|T(\alpha), \sigma_1, \sigma_2\| = \| \lim_{M+N\to\infty} \sum_{m,n=0}^{M+N} a_{m,n} \quad \alpha_{m,n}, \sigma_1, \sigma_2\|$$

$$\leq \lim_{M+N\to\infty} \sum_{m,n,=0}^{M+N} |a_{m,n}| \, \|\alpha_{m,n};\ \sigma_1, \sigma_2\|$$

$$\leq \lim_{M+N\to\infty} \sum_{m,n=0}^{M+N} |a_{m,n}| \exp(\lambda_m + \mu_n)\, \sigma$$

where $\sigma = \sigma(\sigma_1, \sigma_2)$

$$= \|\, \alpha; \sigma, \sigma\|$$

Thus $T : (X, \| \ldots, \sigma, \sigma\|)$ into $(X, \| \ldots, \sigma_1, \sigma_2 \|)$ is continuous and as σ_1, σ_2 are arbitrary, we find that $T : X \to X$ is continuous.

Theorem 2. *If T is a linear operator on X into itself, such that T and T^{-1} are continuous. Then $\{ T(\delta_{m,n}); m, n \geq 0\}$ is a proper base in the closed subspace $T[X]$ of X.*

Conversely if $\{\alpha_{m,n} : m, n \geq 0\}$ is a proper base in a closed subspace Y of X, then there exists a continuous linear operator $T : X \to X$, such that $T(\delta_{m,n}) = \alpha_{m,n}$.

Proof. Suppose that T is the linear operator mentioned in the hypothesis. Then $T[X]$ is a closed subspace of X.
Let $T(\delta_{m,n}) = \alpha_{m,n}$, $m, n \geq 0$. Let $f \in T\,[X]$, then

$$T^{-1}(f) = \sum_{m,n=0}^{\infty} a_{m,n}\, \delta_{m,n}$$

where $a_{m,n}$'s satisfy eq. (3). Now

$$\sum_{m,n=0}^{M+N} a_{m,n}\, \delta_{m,n} \to T^{-1}(f) \qquad \text{in } X \text{ as } M + N \to \infty. \tag{9}$$

Now T is continuous and linear, and eq. (9) implies

$$f = \sum_{m,n=0}^{\infty} a_{m,n}\, \alpha_{m,n} \tag{10}$$

Since eq.(8) holds, then $\sum M(a_{m,n}\alpha_{m,n}, \sigma_1, \sigma_2)$ converges for every real σ_1, σ_2 and the representation of f in (10) is unique. Since T^{-1} is continuous. We conclude that $\{\alpha_{m,n}; m, n \geq 0\}$ is a proper base for $T[X]$.

Conversely, let $\{\alpha_{m,n}; m, n \geq 0\}$ be a proper base for a closed subspace

Y of X. Hence eq. (8) holds. Therefore by theorem 1., there exists a continuous linear operator T on X into itself, such that $T(\delta_{m,n}) = \alpha_{m,n},\ m, n \geq 0$. Now let $f \in X,\ f \neq 0$, then f is represented by eq. (1) whose coefficients $a_{m,n}$'s satisfy eq. (3). Thus

$$T(f) = \sum_{m,n=0}^{\infty} a_{m,n}\, \alpha_{m,n} \neq 0$$

Therefore T is one - to - one. Hence T is a continuous algebric isomorphism from X into Y. Now apply the well - known theorem of Banach ([1] p. 41) we see that T^{-1} exists and is continuous (since Y is complete [5]) and the proof of the theorem is now complete.

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