LE MATEMATICHE
Vol. XLVII (1992) – Fasc. I, pp. 9-12

NOTE ON GRAPHS COLOURING

DĂNUȚ MARCU

In this paper, we give the maximal number of $(k+r)$-colourings of a graph with n vertices and chromatic number k. Also, we obtain the maximal values for chromatic polynomial of a graph.

Graphs, considered here, are *finite, undirected* and *simple* (without loops or multiple edges), and [1,2] are followed for terminology and notation. Let $G = (V, E)$ be a graph with V the set of *vertices* and E the set of *edges*. A $(k+r)$-*colouring* of a graph G with n vertices and the chromatic number $\mathcal{Y}(G) = k$ is a partition of the vertex set into $k+r$ classes $(0 \leq r \leq n-k)$, such that two vertices belonging to the same class are not adjacent, the order of classes being indifferent.

Theorem 1. *The number $C(n,k,r)$ of $(k+r)$-colourings of a graph which is composed by a complete k-subgraph and $n-k$ isolated vertices is given by*

$$C(n,k,r) = \sum_{p=r}^{n-k} \binom{n-k}{p} S(p,r) k^{n-k-p} \qquad \textit{for } 1 \leq r \leq n-k,$$

and

$$C(n,k,0) = k^{n-k},$$

Entrato in Redazione l'8 maggio 1991.

where $S(p,r)$ is the Stirling number of the second kind, that is, the number of partitions of a p-set into r classes.

Proof. Indeed, the k vertices which form a complete subgraph belong to different classes of a colouring, and for any choice of $p \geq r$ isolated vertices from the $n-k$ isolated vertices, these p vertices generate $S(p,r)$ different partitions into r classes, and the remaining $n-k-p$ isolated vertices may be added in k^{n-k-p} different ways to k classes which contain the vertices of the complete subgraph. The formula follows, since one can generate all $(k+r)$-colourings of the graph, without repetitions. □

Theorem 2. *For $n \geq k$ and $0 \leq r \leq n-k$, the following hold:*

$$(a) \qquad C(n,k,r) = C(n-1,k,r-1) + (k+r)C(n-1,k,r)$$

for $1 \leq r \leq n-k$, *and*

$$C(n,k,0) = kC(n-1,k,0);$$

$$(b) \quad C(n,k,r) = \sum_{q=0}^{n-k-r} \binom{n-k}{q} C(n-1-q,k-1,r) \quad \textit{for } k \geq 2.$$

Proof. The relation (a) is obtained by numbering in two different ways the $(k+r)$-colourings of graph, which contain in a class a given isolated vertex. The relation (b) is obtained by numbering the $(k+r)$-colourings of graph, which contain in a class a vertex which belongs to the complete subgraph with k vertices. □

Theorem 3. *The maximal number of $(k+r)$-colourings of a graph $G=(V,E)$ with n vertices and the chromatic number k is equal to $C(n,k,r)$, and the single graph which has this maximal number of colourings is the graph composed by a complete k-subgraph and $n-k$ isolated vertices.*

Proof. The proof is made by induction on n. Obviously, for $n = 1,2$, the theorem is true. Suppose that the theorem is true for all graphs G with $n-1$ vertices and let $G-v$ be the subgraph obtained by deleting a vertex v. We denote by $C(k+r,G)$ the number of $(k+r)$-colourings of vertices of G. If $\mathcal{Y}(G-v) = \mathcal{Y}(G) = k$, then

$$C(k+r,G) \leq C(k+r-1,G-v) + (k+r)C(k+r,G-v) \leq$$
$$\leq C(n-1,k,r-1) + (k+r)C(n-1,k,r) = C(n,k,r)$$

for $r \geq 1$, and

$$C(k,G) \leq kC(k,G-v) \leq kC(n-1,k,0) = C(n,k,0)$$

for $r = 0$, the equality holding only if v is an isolated vertex (we have used the fact that v can not form, single, a class of a k-colouring of G for $r = 0$). If $\mathcal{Y}(G-v) = \mathcal{Y}(G) - 1$, then a minimal colouring of G is

$$\{v\}, I_1, I_2, \ldots, I_{k-1},$$

where $I_1, I_2, \ldots, I_{k-1}$ are independent sets of G, and there exist the vertices $v_{i_1} \in I_1,\ v_{i_2} \in I_2, \ldots, v_{i_{k-1}} \in I_{k-1}$ which are joined, every one, by an edge with v, since, otherwise, $\mathcal{Y}(G) = k-1$. In this case, we have

$$\begin{aligned} &C(k+r,G) \leq C(k+r-1,G-v)+ \\ &+\sum_{q\geq 1}\ \sum_{v_{j_1},\ldots,v_{j_q}\in V-\{v,v_{i_1},\ldots,v_{i_{k-1}}\}} C(k+r-1,G-v-v_{j_1}-\cdots-v_{j_q}) \\ &\leq C(n-1,k-1,r) + \sum_{q=1}^{n-k-r} \binom{n-k}{q} C(n-1-q,k-1,r) \\ &= C(n,k,r)\,, \end{aligned}$$

the equality holding only if the vertices from $V - \{v, v_{i_1}, \ldots, v_{i_{k-1}}\}$ are isolated (we have also used the fact that if $\{v\} \cup \{v_{j_1}, \ldots, v_{j_q}\}$ is a class of a $(k+r)$-colouring of G with $\mathcal{Y}(G) = k$ and $\mathcal{Y}(G-v) = k-1$, then $\mathcal{Y}(G-v-v_{j_1}-\ldots-v_{j_q}) = k-1$).
Obviously, $\mathcal{Y}(G) = k$ implies that $\{v, v_{i_1}, \ldots, v_{i_{k-1}}\}$ is a complete k-subgraph. □

Corollary 1. *If we denote by $P(G_n;\lambda)$ the chromatic polynomial of a graph G_n with n vertices, then*

$$\max_{\mathcal{Y}(G_n)=k} P(G_n;\lambda) = \lambda(\lambda-1)\ldots(\lambda-k+1)\lambda^{n-k}$$

for each $\lambda \in \mathbb{N}$ (the set of natural numbers), and the maximum is attained for each $\lambda \geq k$, only for the graph G_n with the chromatic number k and which is composed by a complete k-subgraph and $n-k$ isolated vertices and whose chromatic polynomial is

$$P(G_n;\lambda) = \lambda(\lambda-1)\ldots(\lambda-k+1)\lambda^{n-k}.$$

Proof. Obviously, for each $\lambda < \mathcal{Y}(G_n)$, we have $P(G_n;\lambda) = 0$. Thus, if $\lambda \geq \mathcal{Y}(G_n)$, then

$$P(G_n;\lambda) = \sum_{j=\mathcal{Y}(G_n)}^{\min(n,\lambda)} j!\binom{\lambda}{j} C(j,G_n) \leq$$

$$\leq \sum_{j=\mathcal{Y}(G_n)}^{\min(n,\lambda)} j!\binom{\lambda}{j} C(n,\mathcal{Y}(G_n), j-\mathcal{Y}(G_n)) = \lambda(\lambda-1)\ldots(\lambda-k+1)\lambda^{n-k}.$$

REFERENCES

[1] C. Berge, *Graphes et Hypergraphes,* Dunod, Paris,1970.

[2] C. Berge, *Principes de Combinatoire,* Dunod, Paris,1968.

Str. Pasului 3, Sect. 2,
70241-Bucharest, Romania.