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ON THE REGULARITY OF OPTIMAL CONTROL FOR A PARABOLIC SYSTEM OF ORDER $2m$

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An optimal control problem for a parabolic operator of order $2m$ with the boundary conditions containing the control is considered.
A regularity theorem for the parabolic problem and the regularity of the optimal control is proved.

Introduction.

In [1] M. GIURGIU studies an optimal control problem with quadratic cost functional for the equation $\Omega^p y(t,x) = 0$ where $\Omega = \partial_t - \partial_x^2$ and $p \in N$ with same of the boundary conditions given by means of the solution of a linear differential equation system that contains the control $u(t)$ with $u : [0,T] \to U$ ($T < +\infty$; $U = L^2(0,T;R^n)$). Furthermore in [1] it is proved that if data are continuous, then also the optimal control $\widetilde{u}$ is continuous in the closed interval $[0,T]$.

In [5] T.I. Seidman considers an optimal control problem with quadratic cost functional for the equation $\partial_t - Au = f$ where A is a second order uniformly elliptic operator, with the boundary condition containing the control Φ and states very strong regularity results, for the optimal control Φ_* in the open interval $]0,T[$.

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In the present paper, starting from the mentioned papers and making use of some results in [2], we have considered an optimal control problem with quadratic cost functional for the parabolic equation $\partial_t y = Ly$, where L is a strongly elliptic operator, having order $2m$ ($m \in N$) in $]0,T] \times R^{n-1} \times R^+$ and, following [1], the boundary conditions containing the control are given by linear operators β satisfying assigned conditions. Besides, a regularity theorem is proved for the parabolic problem; this results allows us to study the regularity of the optimal control in the closed interval $[0,T]$.

1. Setting of the problem.

Let X a Hilbert space; for every $\sigma \in N$, let us denote by:
-$C^{(0)}([0;T];X)$ the space of the functions f that are continuous in $[0,T]$, with values in X and norm:

$$|f|_{C^{(0)}([0,T];X)} = \max_{t\in[0,T]} |f|_X \,;$$

-$C^{(\sigma)}([0,T];X)$ the space of the functions f such that

$$\partial_t^s f(t) \in C^{(0)}([0,T];X) \quad \forall s \in \{0,\dots,\sigma\}\,,$$

with norm:

$$|f|_{C^{(\sigma)}([0,T];X)} = \sum_{s=0}^{\sigma} |\partial_t^s f|_{C^{(0)}([0,T];X)} \,;$$

-$L^2(0,T;X)$ the space of the functions f that are Bochner measurable on $]0,T[$, with values in X, such that

$$\int_0^T |f(t)|_X^2 \, dt < +\infty$$

with norm

$$|f|_{L^2(0,T;X)} = \left[\int_0^T |f(t)|_X^2 \, dt\right]^{\frac{1}{2}} ;$$

-$H^\sigma(0,T;X)$ is the space of the functions $f \in L^2(0,T;X)$, differentiable of order σ in $L^2(0,T;X)$, equipped with the norm:

$$|f|_{H^\sigma(0,T;X)} = \sum_{s=0}^{\sigma} |\partial_t^s f|_{L^2(0,T;X)} \,.$$

-$H^\sigma(\Omega)$ is the ordinary Sobolev space with $\Omega = R^{n-1}_{x'} \times R^+_{x_n}$ for every $\sigma \in N$.
-$H^\sigma_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ with respect to the norm in $H^\sigma(\Omega)$.

Let us consider the following parabolic problem:

$$(1.1) \qquad \begin{cases} \partial_t y = Ly, \quad (t,x) = (t,x',x_n)) \in]0,T] \times \Omega \\ y(0,x) = x^p y_0(x), \; x \in \Omega \\ \partial^{j-1}_{xn} y(t,(x',0)) = \gamma_j(x') t^q (\beta u)_j, \\ \qquad (t,x') \in]0,T] \times R^{n-1}_{x'}, j \in \{1,\dots,m\}. \end{cases}$$

where:

i_1) $L = L(x,\partial_x) = (-1)^m \sum_{|\alpha|=m} \partial^\alpha_x a_\alpha(x) \partial^\alpha_x \qquad (\partial_x = \partial/\partial x)$

is a strongly elliptic operator in $\overline{\Omega}$, with real coefficient. We also assume that, for every multiindex β, the functions $\partial^\beta_x a_\alpha(x)$ are convergent for $|x| \to +\infty$ and:

$$\sum_{|\alpha|=m} \int_\Omega a_\alpha(x) \partial^\alpha_x \phi(x) \overline{\partial^\alpha_x \phi(x)}\, dx \le 0 \qquad \forall \phi \in H^m_0(\Omega)$$

i_2) p is nonnegative real and $q > \frac{1}{2}$;

i_3) $\gamma(x') = [\gamma_j(x')]$ is a $1 \times m$ matrix with values in $\mathcal{S}(R^{n-1}_{x'})$;

i_4) $\beta = [\beta_{i,j}]$ is a $m \times k$ matrix, whose elements $\beta_{i,j}$ are continuous linear operators: $L^2(0,T) \to H^2(0,T)$, and their adjoint $\beta^*_{i,j}$ are also linear and continuous: $L^2(0,T) \to H^1(0,T)$;

i_5) $(1+x_n^2)^{p/2} y_0 \in L^2(\Omega)$;

i_6) the control $u = u(t)$ belongs to $L^2(0,T;R^k)$.

Such hypotheses ensure (see [6]) that the problem (1.1) has a unique solution y_u in $C^{(0)}([0,T];L^2(\Omega)) \cap L^2(0,T;H^m_0(\Omega))$, depending of the data with continuity.

The cost $J(u)$ is the functional:

$$(1.2) \qquad J(u) =$$

$$= \int_0^T dt \int_\Omega dx \int_\Omega K_1(t,x,\varepsilon)[(y_u(t,x) - y_T(t,x))(y_u(t,\varepsilon) - y_T(t,\varepsilon))]\, d\varepsilon$$

$$+ \int_\Omega dx \int_\Omega K_2(x,\varepsilon)[(y_u(T,x) - \omega_T(x))(y_u(T,\varepsilon) - \omega_T(\varepsilon))]\, d\varepsilon +$$

$$+ \int_0^T < E(t)u(t), u(t) > dt + \int_0^T < H(t)\beta u(t), \beta u(t) > dt\,,$$

where $<,>$ denotes the scalar product and:

j_1) the kernels:

$$K_1(t,x,\varepsilon) \in C^{(0)}([0,T] \times \Omega \times \Omega) \cap C^{(1)}([0,T]; L^2(\Omega \times \Omega)),$$

$$K_2(x,\varepsilon) \in C^{(0)}(\Omega \times \Omega) \cap H^{2m}(\Omega \times \Omega) \cap H_0^m(\Omega \times \Omega),$$

are positive semi-definite and symmetric in x and ε.

j_2) $E(t)$ is a $k \times k$ simmetric positive definite matrix, whose elements are in $C^{(1)}[0,T]$; $H(t)$ is a $m \times m$ symmetric positive semi-definite matrix, whose elements are in $C^{(0)}[0,T]$.

j_3) the *target state* $y_T(t,x)$ and the *target trajectory* $\omega_T(x)$ are in

$$C^{(0)}([0,T]; L^2(\Omega)) \quad \text{and} \quad L^2(\Omega)$$

respectively.

The present optimal control problem consists in determining a function $\widetilde{u} \in L^2(0,T;R^k)$ (*the optimal control*) such that $J(\widetilde{u}) \leq J(u)$ for every $u \in L^2(0,T;R^k)$.

The following lemma (see [2]) holds

1.1. *In the hypotheses i_1) to i_6) and j_1) to j_3) the functional $J(u)$ is strictly convex, Gateaux differentiable, coercive and lower semicontinuous in $L^2(0,T;R^k)$.*

It follows that (see [2], [4]) :

1.2. *In the hypotheses of lemma 1.1 there exist a unique optimal control. Furthermore the control $\widetilde{u} \in L^2(0,T;R^k)$ is optimal if and only if:*

$$\begin{aligned}(1.3) \qquad & \int_0^T dt \int_\Omega dx \int_\Omega K_1(t,x,\varepsilon)(y_u(t,x) - y_{\tilde{u}}(t,x))(y_{\tilde{u}}(t,\varepsilon)- \\ & -y_T(t,\varepsilon))\, d\varepsilon + \int_\Omega dx \int_\Omega K_2(x,\varepsilon)(y_u(T,x) - y_{\tilde{u}}(T,x))(y_{\tilde{u}}(T,\varepsilon)- \\ & -\omega_T(\varepsilon))\, d\varepsilon + \int_0^T < E(t)\widetilde{u}(t), u(t) - \widetilde{u}(t) > dt + \\ & + \int_0^T < H(t)\beta\widetilde{u}(t), \beta(u - \widetilde{u})(t) > dt = 0\end{aligned}$$

for every $u \in L^2(0,T;R^k)$.

2. A regularity theorem for the parabolic problem.

Let us consider the problem:

$$(2.1)\qquad \begin{cases} \partial_t y(t,x) = Ly(t,x) + f(t,x) & (t,x) \in]0,T] \times \Omega \\ y(0,x) = y_0(x) \\ \partial_{x_n}^{j-1} y(t,x',0) = 0 & j \in \{1,\dots,m\} \end{cases}$$

Let E denote the space of the couples $(\rho(t,x), \theta(x))$ such that:

$$\rho(t,x) \in H^1(0,T;L^2(\Omega))$$

$$\theta(x) \in H_0^m(\Omega)$$

$$L\theta(x) \in L^2(\Omega)$$

and τ operator:

$$\tau : (\rho,\theta) \in E \longrightarrow \tau(\rho,\theta) = \rho(0,x) + L\theta(x) \in L^2(\Omega)\,.$$

Besides, U denote the space $C^{(0)}([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^m(\Omega))$. Let us observe that from [2], (see th.1.1) it follows that problem (2.1) is uniquely solvable in U; let us denote the solution by $Y[f,y_0]$.
By the same theorem one gets

$$(2.2)\qquad (f,y_0) \in E$$

then

$$(2.3)\qquad Y[f,y_0] \in C^{(0)}([0,T];H^{2m}(\Omega))$$

$$(2.4)\qquad \partial_t Y[f,y_0] \in U$$

$$(2.5)\qquad \partial_t Y[f,y_0] - Y[\partial_t f, \tau(f,y_0)].$$

Let us now prove the following regularity theorem.

Theorem 2.1. *If a $\sigma \in N$ exists, such that*

$$(2.6)\qquad \begin{cases} \partial_t^i L^j f(t,x) \in L^2(0,T;L^2(\Omega)) & \forall i,j \in N_0 \quad i+j \leq \sigma \\ \partial_t^i L^j f(t,x) \in C^{(0)}([0,T];H_0^m(\Omega)) & \forall i,j \in N_0 \quad i+j \leq \sigma - 1 \end{cases}$$

$$(2.7)\qquad \begin{cases} L^j y_0(x) \in L^2(\Omega) & \forall j \in N_0 \quad j \leq \sigma \\ L^j y_0(x) \in H_0^m(\Omega) & \forall j \in N_0 \quad j \leq \sigma - 1 \end{cases}$$

then

$$(2.8)\qquad \partial_t^s Y[f,y_0] \in C^{(0)}([0,T];H^{2m}(\Omega)) \qquad \forall s \leq \sigma - 1$$

$$(2.9)\qquad \partial_t^s Y[f,y_0] \in U \qquad \forall s \leq \sigma.$$

Proof. Let now observe that the hypotheses imply (2.2), and therefore, (2.3) and (2.4), i.e. the conclusion for $\sigma - 1$. If $\sigma > 1$, the hypotheses imply (2.2) and (2.5), too, where the couple $(\partial_t f, \tau(f,y_0))$ belongs to E. Relations (2.3) and (2.4) hold also with $\partial_t Y[f,y_0]$ instead of $Y[f,y_0]$ and this implies the conclusion for $\sigma - 2$. It is easy to see that (2.6) and (2.7) allow to apply such a procedure exactly $\sigma - 1$ times. The theorem follows.

Let us now consider the problem

$$(2.10)\qquad \begin{cases} \partial_t y = Ly & (t,x) \in \,]0,T] \times \Omega \\ y(0,x) = x_n^p\, y_0(x) & x \in \Omega \\ \partial_{x_n}^{j-1} y(t,(x',0)) = t^q v_j(t,x') & \\ \qquad j \in \{1,\dots,m\}\ (t,x') \in \,]0,T] \times R^{n-1} & \end{cases}$$

where ρ and q are nonnegative reals. The following corollary holds.

Theorem 2.2. *If there exists $\sigma \in N$ such that :*

$$(2.11)\qquad \rho \geq 2m\sigma$$

$$(2.12)\qquad q > \sigma + \frac{1}{2}$$

$$(2.13)\qquad v_j(t,x') \in H^{\sigma+1}(0,T;S(R^{n-1}))$$

(2.14) $$(1+x_n^2)^{p/2} y_0(x) \in H^{2m}(\Omega)$$

then the problem (2.10) admits a unique solution that belongs to

$$C^{(\sigma)}([0,T]; L^2(\Omega) \cap C^{(\sigma-1)}([0,T]; H^{2m}(\Omega)) .$$

Proof. By putting the following

$$\overline{y}(t,x) = t^q \sum_{j=1}^{m} x_n^{j-1} \frac{e^{-x_n^2}}{(j-1)!} v_j(t,x')$$

(2.15)
$$(\partial_t - L)\bar{y}(t,x) = t^{q-1} g(t,x)$$

(2.16) $$y = y^* + \bar{y}$$

the problem (2.10) is reduced to

(2.17) $$\begin{cases} \partial_t y^* = L y^* - t^{q-1} g(t,x) \\ y^*(0,x) = x_n^p \, y_0(x) \\ \partial_{xn}^{j-1} y^*(t,(x',0)) = 0 \qquad j \in \{1,\dots,m\} \end{cases}$$

the hypotheses (2.11) - (2.14) allow to apply th. 2.1 to the problem. We have

(2.18) $$y^* \in C^{(\sigma)}([0,T]; L^2(\Omega)) \cap C^{(\sigma-1)}([0,T]; H^{2m}(\Omega)).$$

The mentioned hypotheses also guarantee that

$$\overline{y}(t,x) \in C^{(\sigma)}([0,T]; L^2(\Omega)) \cap C^{(\sigma-1)}([0,T]; H^{2m}(\Omega)) ;$$

then the conclusion from (2.16).

Remark. If p is an integer, the result holds again, provided that (2.11) is replaced by

(2.19) $$p \geq 2m\sigma - m .$$

3. Adjoint state and implicit representation of the optimal control.

In order to represent the optimal control, let us put

$$(3.1)\qquad \begin{aligned} f(t,x) &= -\int_\Omega K_1(t,x,\varepsilon)(y_{\tilde u}(t,\varepsilon) - y_T(t,\varepsilon))\, d\varepsilon \\ p_T(x) &= -\int_\Omega K_2(x,\varepsilon)(y_{\tilde u}(T,\varepsilon) - \omega_T(t,\varepsilon))\, d\varepsilon \end{aligned}$$

and consider the problem

$$(3.2)\qquad \begin{cases} -\partial_t p(t,x) = Lp(t,x) + f(t,x) & a.e. \quad (t,x) \in [0,T[\times\Omega \\ p(T,x) = p_T(x) & a.e. \quad x \in \Omega \\ \partial_{x_n}^{j-1} p(t,(x',0)) = 0 & j = 1,\dots,m \end{cases}$$

where $y_{\tilde u}(t,x)$ is the solution of the problem (1.1) corresponding to the optimal control $\widetilde{u}$. By corollary 2.2, for $\sigma = 1$, $y_{\tilde u}(t,x)$ belongs to

$$C^{(1)}([0,T]; L^2(\Omega)) \cap C^{(0)}([0,T]; H^{2m}(\Omega))\,.$$

This property is inherited by the function $f(t,x)$, introduced in (3.1), in virtue of the hypotheses j_1) and j_3).
From th. 2.1 it follows that the problem (3.2) has a unique solution $p_{\tilde u}$ in $C^{(1)}([0,T]; L^2(\Omega)) \cap C^{(0)}([0,T]; H^{2m}(\Omega))$ and in $H^{(1)}(0,T; H_0^m(\Omega))$. Let us now consider the following differential operators of order $2m - j$

$$Q_j(x',\partial_x) = \sum_{|\alpha|=m} (-1)^{m-j} \partial_x^{\alpha-j} a_\alpha(x',0)\partial_x^\alpha \qquad j = 1,\dots,m.$$

Applying the notation:

$$(3.3)\qquad w_j(t,x') = \gamma_j(x')Q_j(x',\partial_x)p_{\tilde u}(t,(x',0)) \qquad j = 1,\dots,m$$

we get $w_j(t,x') \in C^{(0)}([0,T]; L^1(R^{n-1}))$, for every $j \in \{1,\dots,m\}$.
With the further notations:

$$(3.4)\qquad w_j(t) = (-1)^j \int_{R^{n-1}} w_j(t,x')dx' \quad , \quad w(t) = \begin{bmatrix} w_1(t) \\ \vdots \\ w_m(t) \end{bmatrix}$$

we are now able to say that $w(t) \in C^{(0)}([0,T]; R^m)$.
The following proposition holds

Theorem 3.1. *Under the same hypotheses of th.1.2 the optimal control satisfies the functional equation:*

$$\widetilde{u}(t) = E^{-1}(t)\big[-\beta^*(t^q w(t) + H(t)\beta\widetilde{u})\big]. \tag{3.5}$$

Proof. Let us recall that, if it results:

$$p(t,x)\,,\, y(t,x) \in C^{(0)}([0,T]; H^{2m}(\Omega)) \cap C^1([0,T]; L^2(\Omega))$$

$$\partial_{x_n}^{j-1} p(t,x',0) = 0 \qquad \forall\, j \in \{1,\dots,m\}$$

then the Green's formula holds

$$\int_\Omega dx \int_0^T p(t,x)(\partial_t - L(x,\partial_x)y(t,x)) + y(t,x)(\partial_t + L(x,\partial_x)) \cdot \tag{3.6}$$

$$\cdot\, p(t,x)dt = p(T,x)y(T,x) - p(0,x)y(0,x)$$

$$-\sum_{j=1}^m (-1)^{j-1} \int_0^T dt \int_{R^{n-1}} \partial_{x_n}^{j-1} y(t,(x',0))Q_j(x',\partial_x)p(t,(x',0))\,dx'.$$

So the formula (3.6) applies to the couple $(p_{\tilde u}, y_u - y_{\tilde u})$. By (3.1), (3.3) and (3.4) and problems (1.1) and (3.1), we get

$$\int_\Omega dx \int_0^T dt\,(y_u(t,x) - y_{\tilde u}(t,x)) \int_\Omega d\varepsilon\; K_1(t,x,\varepsilon)(y_{\tilde u}(t,\varepsilon) - y_T(t,\varepsilon)) =$$

$$= -\int_\Omega dx \int_\Omega K_2(x,\varepsilon)(y_{\tilde u}(T,\varepsilon) - \omega_T(\varepsilon))(y_u(T,x) - y_{\tilde u}(T,x))\,d\varepsilon$$

$$+\sum_{j=1}^m \int_0^T t^q((\beta u)_j(t) - (\beta\widetilde{u})_j(t))w_j(t)\,dt.$$

Therefore, by using the optimality necessary and sufficient condition (1.3), we obtain, for every $u \in L^2(0,T;R^k)$

$$\int_0^T < E(t)\widetilde{u}(t), u(t) - \widetilde{u}(t) > dt +$$

$$+\int_0^T < H(t)\beta\widetilde{u}(t),\; \beta(u - \widetilde{u})(t) > dt +$$

$$+\sum_{j=1}^{m}\int_0^T t^q((\beta u)_j(t)-(\beta\widetilde{u})_j(t))w_j(t)\,dt=0.$$

Thus, for every $u\in L^2(0,T;R^k)$

$$\int_0^T < u(t)-\widetilde{u}(t), E(t)\widetilde{u}(t)+\beta^*(t^q w)+\beta^*(H(t)\beta\widetilde{u}(t)) > dt=0$$

and, $u(t)$ being arbitrary, the conclusion follows.

4. Regularity of the optimal control.

In this section we shall use (3.5) in order to study the regularity of $\widetilde{u}(t)$. The results are expressed in the following theorem

Theorem 4.1. *In the hypotheses $i_1)\to i_6)$ and $j_1)\to j_3)$, if there is $\sigma\in N$ such that:*

$i'_2)$ $p\geq 2m\sigma$ *and* $q>\sigma+\frac{1}{2}$

$i'_4)$ $\forall s\in\{0,\dots,\sigma\}$ *the operators* $\beta_{i,j}$ *and* $\beta^*_{i,j}$ *are linear and continuous, respectively:* $H^s(0,T)\to H^{s+2}(0,T)$ *and* $H^s(0,T)\to H^{s+1}(0,T)$*;*

$i'_5)$ $(1+x_n^2)^{p/2}y_0\in H^{2m}(\Omega)\quad p\geq 2m\sigma$*;*

$j'_1)$ *for every* $i,j\in N_0$ *it is:*

$$\partial_t^i L^j(x,\partial_x)K_1(t,x,\varepsilon)\in C^{(0)}([0,T];L^2(\Omega\times\Omega))\qquad i+j\leq\sigma$$

$$\partial_t^i L^j(x,\partial_x)K_1(t,x,\varepsilon)\in C^{(0)}([0,T];H_0^m(\Omega\times\Omega))\qquad i+j\leq\sigma-1$$

$$L^j(x,\partial_x)K_2(x,\varepsilon)\in L^2(\Omega\times\Omega)\qquad j\leq\sigma$$

$$L^j(x,\partial_x)K_2(x,\varepsilon)\in H_0^m(\Omega\times\Omega)\qquad j\leq\sigma-1;$$

$j'_2)$ *the elements of the matrices* $E(t)$ *and* $H(t)$ *belong to* $C^{(\sigma)}([0,T])$ *and* $C^{(\sigma-1)}([0,T)]$ *respectively;*

$j'_3)$ *it is:* $y_T(t,\varepsilon)\in C^{(\sigma)}([0,T];L^2(\Omega))$ *and* $\omega_T(\varepsilon)\in L^2(\Omega)$*, then the optimal control* $\widetilde{u}(t)$ *belongs to* $H^{(\sigma)}(0,T)$*.*

Proof. Let us recall the functional equation of the optimal control

$$\widetilde{u}(t)=E^{-1}(t)\big[-\beta^*(t^q w(t)+H(t)\beta\widetilde{u})\big]. \tag{3.5}$$

As $\widetilde{u}(t)\in L^2(0,T)$, by $j_2)$ and $i_4)$, it is $H(t)\beta\widetilde{u}\in C^{(0)}[0,T]$; besides, recalling Section 3, $w(t)\in C^{(0)}[0,T]$ from which, by $i_4)$ and $j_2)$, we get $\widetilde{u}\in H^1(0,T)$. So, the conclusion holds for $\sigma=1$.

Let us suppose $\sigma > 1$.
As $\widetilde{u} \in H^1(0,T)$ by i'_4) the functions $v_j(t,x) = \gamma_j(x')(\beta\widetilde{u})_j$ belong to $H^3(0,T;S(R^{n-1}))\ \forall j \in \{1,\dots,m\}$; this implies

$$\partial_t^i v_j(t,x') \in H^2(0,T;S(R^{n-1}))$$

for every $i \leq 1$. As $y_0(x)$ fulfils i'_6), by i'_2) and corollary 2.2, we get

$$y_{\tilde{u}} \in C^{(2)}([0,T];L^2(\Omega)) \cap C^{(1)}([0,T];H^{2m}(\Omega))\,.$$

This, together with hypotesis j'_1), implies by 2.1

$$p_{\tilde{u}} \in C^{(1)}([0,T];H^{2m}(\Omega))$$

and so $w(t) \in C^{(1)}[0,T]$.
Proceeding in a similar way, we have $H(t)\beta\widetilde{u} \in C^{(1)}[0,T]$, so that, by (3.5), $\widetilde{u}$ belongs to $H^2(0,T)$, i.e. the conclusion for $\sigma = 2$.
In virtue of the hypotheses, the procedure can be applied $\sigma - 1$ times.
Therefore the theorem follows.

REFERENCES

[1] M. Giurgiu, *A non-homogeneous feedback solution of a linear quadratic problem for boundary control of p-caloric equation,* $p \geq 1$, Rev. Roum: Math. Pures et Appl. Tome XXI n. 4, pp. 401-427, Bucarest 1976.

[2] S. Giuga, *Su un problema di controllo ottimale relativo ad un operatore parabolico di ordine* $2m$, Rend. Acc. Sc. Fis. Mat. IV vol. LII/2 Napoli (1985).

[3] K. Yosida, *Functional Analysis,* Springer N.Y. (1966).

[4] J.L. Lions, *Contrôle optimal de systèmes gouvernès par des èquations aux dèrivèes partielles,* Dunod Gauthier-Villars, 1968.

[5] T.I. Seidman, *Regularity of optimal boundary controls for parabolic equations I Analyticity,* Siam J. Control and Optimization n. 3 (1982).

[6] H. Tanabe, *Equation of Evolution (English translation),* Pitman, London (1979).

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