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POTENTIAL THEORY FOR STATIONARY SCHRÖDINGER OPERATORS: A SURVEY OF RESULTS obtained with non-probabilistic methods

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In this paper we deal with a uniformly elliptic operator of the kind: $Lu \equiv Au + Vu$, where the principal part A is in divergence form, and V is a function assumed in a "Kato class". This operator has been studied in different contexts, especially using probabilistic techniques. The aim of the present work is to give a unified and simplified presentation of the results obtained with non probabilistic methods for the operator L on a bounded Lipschitz domain. These results regard: continuity of the solutions of $Lu = 0$; Harnack inequality; estimates on the Green's function and L-harmonic measure; boundary behavior of positive solutions of $Lu = 0$, in particular a "Fatou's theorem".

Summary.

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0. Introduction.

Object and plan of the work. In this work, we are interested to study uniformly elliptic operators with principal part in divergence form and a term of order zero, that is :

$$Lu \equiv Au + Vu \equiv -(a_{ij}(x)u_{x_i})_{x_j} + V(x) \cdot u$$

where L is supposed defined on a bounded domain of $\Re^n$. The operator A (principal part operator) has been extensively studied, under the only assumptions of measurability and boundedness of coefficients and uniform ellipticity:

$$\lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2$$

for some positive λ, every $\xi \in \mathbb{R}^n$, a.e. x.

Among the most important results which have been established for the operator A we recall the works [11], [21], [19], [20], [18], [5]. We shall discuss later these results. It is generally true that, under suitable assumptions on the function V, similar theorems to those proved for A also hold for L. A quite natural assumption on V, from the standpoint of variational theory of elliptic equations on bounded domains, is that $V \in \mathcal{L}^p(\Omega)$ for some $p > n/2$. (See for instance [25]). In many works of the 80's (see [2], [24], [30], [6], [9]) the operator L has been studied under weaker assumption on V; namely, a theory for the operator L can be developped when V is assumed in a Kato-Stummel class, properly containing $\mathcal{L}^p(\Omega)$ for $p > n/2$. (We shall define this class in section 1). Some reasons for this choice of the space where V is assumed will be discussed later in this introduction. Now we present a table of some results obtained for A and subsequently for L, which we shall deal with in this paper. (The content of the theorems will be expounded in the following).

Kind of results.	*Who obtained it for the operator $\mathcal{L} = A \ldots$*	*$\ldots$ and who for $\mathcal{L} = L$.*
Continuity of solutions of $\mathcal{L}u = 0$.	De Giorgi '57–Nash '58–Moser '60 (see [11], [21], [19])	Chiarenza–Fabes–Garofalo '86 (see [6])
Harnack inequality.	Moser '61 (see [20])	the above work*
Estimate on the Green's function; regularity of boundary points.	Littman–Stampacchia–Weinberger '63 (see [18])	Cranston–Fabes–Zhao '86 for Lipschitz domains (see [9])*
Boundary behavior of positive solutions of $\mathcal{L}u = 0$; Fatou's theorem (for Lipschitz domains).	Caffarelli–Fabes–Mortola–Salsa '83 (see [5])	the above work

* partial results in the same direction previously obtained in [2], [30] and others.

We note that some of these results, namely the continuity estimate and Harnack inequality, have been extended to operators sum of squares of vector fields plus a potential in a Kato class. (See [8], [7]).

Both the work [9] and most of the previous ones on the operator L, such as [2], [24], [30] are essentially probabilistic. On the other hand, [6] follows typically analytic methods, based on a real variable approch. Different analytical proofs of similar results can be found also in [15], where Harnack inequality is discussed, and [23], where a continuity estimate is stated, to get Harnack inequality. Both the works consider operators with principal part $A = -\Delta$. A first aim of the present paper is to give a unified and, as far as possible, self-contained exposition of the above results for the operator L, obtained in a nonprobabilistic way. In [3] I have already shown how one can get the results of [9] without any probabilistic formalism: in particular, the proof of the basic estimate on the Green's function (see thm. 4.1) appears quite simplified. (The meaning of this simplification will be better explained in section 7, where a comparison between analytical and probabilistic approch is drawn). Therefore the present paper is mostly based on the two nonprobabilistic works [6] and [3] ([1]).

Coupling together the results of these papers allows further simplifications, which are the main contribution of this work; let me explain them here. The strategy used in [6] to obtain a Harnack inequality for positive solutions of $Lu = 0$ and the continuity of any solution of $Lu = 0$ is the following:

1^{st}step. Estimate from above on $\sup |u| +$ estimate from below on $\inf u =$ Harnack inequality.

2^{nd}step. Harnack inequality $\Rightarrow$ Continuity of the solutions.

The line we will follow here is:

1^{st}step. Estimate from above on $\sup |u| \Rightarrow$ Continuity of the solutions.

2^{nd}step. Once that continuity of the solutions is know, Harnack inequality follows from the estimate on L-harmonic measure, a result which is established in order to study boundary behavior of positive solutions.

So I prove the continuity result in a more direct way than [6], and simplify the proof of Harnack inequality, bypassing the "estimate from below" which, in [6], involves a "reverse Hölder inequality" for the Green's function and some properties of the A_p classes of weight functions.

The plan of the works is the following. Section 1 contains introductory material: some definitions and known results, a brief outline of the properties of Kato class, the variational formulation of Dirichlet's problem for L, existence of the Green's function for L. Section 2 presents some local estimates proved in [6] (which represent, in that work, "the first half" of the proof of Harnack inequality). In section 3 we derive from these facts continuity of the solutions of $Lu = 0$, and obtain an estimate for the local modulus of continuity of u (i.e. we get in other way the continuity results of [6]). The next two section are based on [3]. Section 4 deals with the Green's function G_L: we present an analytical proof of the estimate comparing the Green's functions for A and L; then we discuss some regularity properties of G_L. In section 5 we study Dirichlet's problem for L when the datum is a (continuous) function defined only on the boundary. This leads to introduce the notion of L-harmonic measure; then we prove that harmonic measures for A and L are comparable. From this fact we derive, in section 6, Harnack inequality and some results about boundary behavior of positive solutions of $Lu = 0$, i.e. we transfer to the operator L properties which, by [5], are known to hold for the operator A. In the last section we make some remarks about the assumptions of this work and compare probabilistic "language" and methods with analytic ones.

([1]) The proofs of the results taken from these works are here in general omitted; only a brief sketch of them is given, to make understandable the general line of the work.

Motivation. What follows is a brief discussion of some of the reason for choosing the Kato class $K(\Omega)$ as a suitable class for V. I start by recalling the phisical context from which the study of our operator arises.

In quantum mechanics the evolution of the state of a system is described by the Schrödinger equation, which, by a suitable choice of units, can be written as:

$$i\frac{\partial\psi}{\partial t} = H\psi$$

where H is the Hamiltonian operator of the system and the unknown function ψ is a probability distribution. For istance, for a single particle of mass one, it is:

$$H = -\frac{1}{2}\Delta + V$$

where V is the potential energy of the particle. The eigenfunctions of H in $\mathcal{L}^2$ represent the stationary states of the system; this is the reason why H, and by extension our operator L, is usually called *stationary Schrödinger operator.*

The basic mathematical property which, in quantum theory, the operator H is requested to satisfy (as any other operator representing an "observable") is selfadjointness, in the Hilbert space sense. To prove selfadjointness of a Hamiltonian operator is a delicate problem which has been treated in many works, and solved under assumptions on V of increasing generality. (See for instance [16], [17]). It is in this frame that $K(\Omega)$ naturally appears as a class of "admissible potentials" for a Schrödinger operator.
Kato-Stummel classes first appear in literature in [28]; their properties are then studied in [22] (where they are not explicitely defined). The single class which we will simply refer to as "the Kato class" appears in [17], where selfadjointness of L is proved for this class of potentials, and is extensively studied, for instance, in [2], [24].

In section 1 I shall give the definition of $K(\Omega)$ and discuss a few properties of this class which will be useful in the following. Here I want to point out just an example:

if V is a central potential having at the origin a singularity of the kind:

$$V(r) = r^{-2}|\log r|^{-1-\xi} \quad \text{for } \epsilon > 0$$

then if B is a small ball centred at the origin, V lies in $K(B)$, while it does not belong to any $\mathcal{L}^p(B)$, for $p > n/2$.

However, to be able to handle potentials with strong singularities is not the only reason (or the most important one) to choice the Kato class as a suitable one. One can say that this class is *natural* under different regards. In [1], for istance, properties of exponential decay in $\mathbb{R}^n$ or, more generally, on unbounded domains, are established for the eigenfunctions of some Schrödinger-type operators, by assuming V in the Kato class. In [2] it is proved that the condition $V \in K(\Omega)$ is, in a certain sense, *necessary* and sufficient in order to a "strong Harnack inequality" may hold for the operator $H = -\frac{1}{2}\Delta + V$. In [24] $\mathcal{L}^p$ properties of the *Scrödinger semigroup* relative to the same operator H are discussed. The present work itself gives one more example of the naturalness of assuming V in $K(\Omega)$, in connection to the use of the Green's function for studying local properties of the solutions of $Lu = 0$.

Finally, just a remark about the principal part operator. It has to be said that a motivation from quantum mechanics to replace $-\frac{1}{2}\Delta$ with A appears, up to now, rather improbable. However, thanks to the results of [18], [5], and theorem 1.6 (taken from [9]), this generalization is not troublesome, from a mathematical point of view.

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1. Kato class, Dirichlet's problem and Green's function.

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$ $(n \geq 3)$. This means that there exists a pair of positive numbers r_0 and M such that for every $z \in \partial\Omega$, local coordinates can be selected so that $B(z, r_0) \cap \partial\Omega$ is the graph of a Lipschitz function φ with $|D\varphi| \leq M$. The constants r_0 and M determine what will be called the Lipschitz charachter of Ω. The operator A is supposed to have bounded, measurable, real valued coefficients a_{ij}. We also suppose $a_{ij} = a_{ji}$ and A uniformly elliptic. So there is a positive constant λ such that

$$\lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n, \text{ for a.e. } x \in \Omega.$$

Let us now define the Kato class $K(\Omega)$.

$$K(\Omega) = \left\{ f \in \mathcal{L}^1_{loc}(\Omega) : \lim_{r \to 0^+} \sup_{x \in \mathbb{R}^n} \int_{\Omega \cap B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}} dy = 0 \right\}.$$

If $V \in K(\Omega)$, put:

$$\eta(r) = \sup_{x \in \mathbb{R}^n} \int_{\Omega \cap B(x,r)} \frac{|V(y)|}{|x-y|^{n-2}} dy.$$

We shall call η "the Kato norm of V". If we write $V = V^+ - V^-$, with V^+, V^- nonnegative, we shall denote by η^- the Kato norm of the negative part V^-. Sometimes it will be useful to handle η^- instead of η. (Clearly, $\eta^-(r) \leq \eta(r)$ for any r). Now, all the quantitative informations we need about L are contained in the number λ and the functions η, η^-.

Note that, if $V \in \mathcal{L}^p(\Omega)$ with $p > n/2$, then by Hölder's inequality $V \in K(\Omega)$ and $\eta(r) \leq c\,\|V\|_p \cdot r^\alpha$ with c, α only depending on n and p. So our assumption generalizes the case which is studied in standard variational approach. If $V \in K(\Omega)$, one can verify that:

(i) $V \in \mathcal{L}^1(\Omega)$;

(ii) $\eta(r)$ is finite for every r, monotone and nondecreasing;

(iii) $\|V\|_1 \leq d^{n-2}\eta(d)$ where $d = $diam.$\Omega$;

(iv) $\sup_\Omega \int_\Omega \frac{|V(y)|}{|x-y|^{n-2}} dy \leq \eta(d)$;

(v) η is bounded and definitively constant, $\eta(r) \leq \eta(2d)$ for every r.

Note that, if $f \in \mathcal{L}^1_{loc}(\Omega)$ and $\eta(r) < \infty$ for some r, then properties (i)-(v) hold, but f must not necessarily belong to $K(\Omega)$ (a counterexample is given in [2]). So the crucial property in defining $K(\Omega)$ is that $\eta(r) \to 0$. To visualize, in some particular cases, what kind of singularities are admitted for a Kato potential, we mention the following criterion:

Theorem 1.1. *(See [2]). Let $\Omega \subseteq B(0,R)$ and V be a central potential, $V(x) = f(|x|)$. Then $V \in K(\Omega)$ if and only if:*

$$\int_0^R r\,|f(r)|\,dr < \infty.$$

(Compare with the example given in the introduction).

A fundamental result, due to Schechter (see [22], p.138) is the following:

Theorem 1.2. *If $V \in K(\Omega)$, there exist a constant $k = k(n)$ and, for every $\delta > 0$, a constant $c_\delta = c(\delta, n)$ such that for all $\varphi \in {H_0}^{1,2}(\Omega)$:*

$$\int_\Omega |V|\varphi^2\,dx \leq k\eta(\delta)\,\|\varphi\|^2_{H^{1,2}} + c_\delta\eta(1)\,\|\varphi\|^2_{\mathcal{L}^2}.$$

Let us considere now the bilinear form associated to L:

$$\mathbf{a}(u,v) = \int_\Omega (a_{ij}u_{x_i}v_{x_j} + V\,u\,v)\,dx.$$

Since Ω is Lipschitz, there exists a linear continuous "extension operator":

$$T : H^{1,2}(\Omega) \to {H_0}^{1,2}(\Sigma)$$

(where Σ is a sphere containing Ω) with $Tu/_\Omega = u$ for every $u \in H^{1,2}(\Omega)$. Then for every u, $v \in H^{1,2}(\Omega)$ one has:

$$\left| \int_\Omega Vuv \right| \le \int_\Sigma |VTuTv| \le \quad \text{(by theorem 1.2)}$$
$$c(n) \cdot \eta(2d) \cdot \|Tu\|_{H_0^{1,2}(\Sigma)} \cdot \|Tv\|_{H_0^{1,2}(\Sigma)} \le$$
$$\le \|T\|^2 c(n) \cdot \eta(2d) \cdot \|u\|_{H^{1,2}(\Omega)} \cdot \|v\|_{H^{1,2}(\Omega)}.$$

Hence $\mathbf{a}$ is bilinear and continuous on $H^{1,2}(\Omega)$. Moreover, by ellipticity, for every $u \in H_0^{1,2}(\Omega)$ one has:

$$\mathbf{a}(u,u) \ge \lambda^{-1} \int_\Omega |Du|^2 \, dx - c(n) \cdot \eta^-(2d) \cdot \|u\|^2_{H^{1,2}(\Omega)} \, .$$

Hence there exists a constant $c_1(n,\lambda)$such that if:

$$c_1(n,\lambda) \cdot \eta^-(2d) \le 1 \tag{1.1}$$

then:

$$\mathbf{a}(u,u) \ge c_0 \|u\|^2_{H^{1,2}(\Omega)}$$

for some positive c_0, i.e. the bilinear form $\mathbf{a}$ is coercive on $H_0^{1,2}(\Omega)$.

We recall now some standard terminology about weak solutions.

We say that $u \in H^{1,2}(\Omega)$ is a *solution* of:

$$\begin{cases} Lu = T & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

for $T \in H^{-1,2}(\Omega)$, $g \in H^{1,2}(\Omega)$, if:

$$\begin{cases} \mathbf{a}(u,\varphi) = \langle \varphi, T \rangle & \forall \varphi \in H_0^{1,2}(\Omega); \\ u - g \in H_0^{1,2}(\Omega). \end{cases}$$

We say that $u \in H^{1,2}_{loc}(\Omega)$ is a *local solution* of $Lu = 0$ in Ω if:

$$\mathbf{a}(u,\varphi) = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

We say that $u \in H^{1,2}(\Omega)$ is a *supersolution* of $Lu = 0$ in Ω if:

$$\mathbf{a}(u,\varphi) \ge 0 \quad \forall \varphi \in H_0^{1,2}(\Omega) \text{ such that } \varphi \ge 0 \text{ in } \Omega.$$

We say that u is a it subsolution if $-u$ is a supersolution.

Under assumption (1.1), by Lax-Milgram's lemma the following holds:

Theorem 1.3. *The problem (1.2), for $T \in H^{-1,2}(\Omega)$, $g \in H^{1,2}(\Omega)$ assigned, is well posed. The constants in the continuous dependence estimate depend on n, λ, η, r_0, M. (If $g \equiv 0$, the constant only depends on n, λ, η).*

Under the same assumption, let us state also a maximum principle.

Theorem 1.4. *Suppose that $u \in H^{1,2}(\Omega)$ is a supersolution for L, and $u \geq 0$ on $\partial\Omega$ in sense $H^{1,2}$ (i.e. $u^- \in H_0^{1,2}(\Omega)$); then $u \geq 0$ a.e. in Ω.*

Proof. Since u is a supersolution and u^- is a nonnegative test function:

$$0 \leq \mathbf{a}(u, u^-) = -\mathbf{a}(u^-, u^-) \leq -c_0 \|u^-\|^2_{H^{1,2}}$$

with $c_0 > 0$, by coerciveness. Hence $u^- \equiv 0$ in Ω, i.e. $u \geq 0$ a.e.

Now we recall some basic facts about the Green's function G for A in Ω.

Theorem 1.5. *(See [18]). There exists a function $G(x, y)$ such that for every $f \in \mathcal{L}^p(\Omega)$, $p > n/2$, the solution of:*

$$\begin{cases} Au = f \textit{ in } \Omega \\ u \in H_0^{1,2}(\Omega) \end{cases}$$

is given by:

$$u(x) = \int_\Omega G(x, y) f(y)\, dy. \tag{1.3}$$

Moreover G is nonnegative, symmetric and satisfies the following estimate:

$$G(x, y) \leq \frac{c_2(n, \lambda)}{|x - y|^{n-2}} \quad \textit{for every } x, y \in \Omega. \tag{1.4}$$

In particular:

$$\sup_{x \in \Omega} \|G(x, \cdot)\|_q \leq c(n, \lambda, d, q) \quad \textit{for all } q < \frac{n}{n-2} \tag{1.5}$$

Next theorems 1.6 - 1.7 are taken from [9]. We stress that both of them are obtained in a purely analytical way. (No probabilistic argument enters the proof).

Theorem 1.6. *("3G Theorem"). There exists a constant c_3 depending on n, λ, η and the Lipschitz charachter of Ω such that:*

$$\frac{G(x,y)\,G(y,z)}{G(x,z)} \leq c_3 \cdot \left\{ \frac{1}{|x-y|^{n-2}} + \frac{1}{|y-z|^{n-2}} \right\} \tag{1.6}$$

for all $x, y, z \in \Omega$.

Theorem 1.7. *For any $z \in \overline{\Omega}$, $w \in \Omega$, $w \neq z$:*

$$\lim_{\substack{x,y\to z\\ x,y\in\Omega}} \frac{G(x,w)\,G(w,y)}{G(x,y)} = 0.$$

If $z, z' \in \partial\Omega$, $w \in \Omega$, the limit:

$$\lim_{\substack{x\to z,\\ y\to z'}} \frac{G(x,w)G(w,y)}{G(x,y)} = K(z,w,z')$$

exists and is a continuous function of (z, z') on $\partial\Omega \times \partial\Omega$.

Now, put $c = \max(c_1, c_2, 2c_3)$ where c_1, c_2, c_3 are as in (1.1), (1.4), (1.6). Note that $c = c(\lambda, n, r_0, M)$. Put:

$$\delta = c \cdot \eta(2d) \tag{1.7.a}$$

It is also useful to define the number:

$$\delta^- = c \cdot \eta^-(2d). \tag{1.7.b}$$

Clearly, $\delta^- \leq \delta$. Henceforth we shall suppose the Kato norm of V so small to have:

$$\delta < \frac{1}{2}. \tag{1.8}$$

Particulary, theorems 1.3 - 1.4 hold under this assumption.

Remark 1.8. Note that, whatever is $V \in K(\Omega)$, condition (1.8) is fulfilled if we restrict to a sufficiently small ball B_r. Then our assumption does not imply any qualitative restriction about those results which are of *local* nature, such as continuity of solutions and Harnack inequality (see sections 3,6): these theorems will hold for *any* $V \in K(\Omega)$ on sufficiently small balls and, in consequence, on compact subsets of Ω (with constants depending on the compact set). On the other hand condition (1.8) is necessary to give sense to "global" concepts, such as "solution of Dirichlet's problem" and therefore is requested in order that some results a "Fatou's theorem" (thm. 6.13) may hold.

Now we discuss existence of the Green's function G_L for L in Ω. (In the following, we will always indicate by G, G_L the Green's functions for A and L, respectively). This will follow from the next theorem, which is taken from [6]:

Theorem 1.9. *Let u be the solution of:*

$$\begin{cases} Lu = f \quad \text{in } \Omega \quad \text{with } f \in \mathcal{L}^p(\Omega),\ p > n/2 \\ u \in H_0^{1,2}(\Omega) \end{cases} \tag{1.9}$$

Then $u \in \mathcal{L}^\infty(\Omega)$, and:

$$\|u\|_\infty \leq c\|f\|_p \quad \text{with } c = c(n, \lambda, d, \delta). \tag{1.10}$$

The theorem is proved in [6] in the case of regular coefficients and potential. The general case can be obtained from this using a standard "mollification tecnique". An explanation of this method, which will be useful in the following, too, can be found for instance in [18].

As a consequence of theorem 1.9 we have that, for each fixed $x \in \Omega$, the linear functional on $\mathcal{L}^p(\Omega)$ $(p > n/2)$ which to every f associates $u(x)$, where u is the solution of (1.9), is continuous. So there exists a function $G_L(x, y)$, with:

$$\sup_x \|G_L(x, \cdot)\|_{\mathcal{L}^q(\Omega)} \leq c(n, \lambda, d, \delta) \quad \text{for all } q < \frac{n}{n-2} \tag{1.11}$$

such that the solution of (1.9) is given by:

$$u(x) = \int_\Omega G_L(x, y) f(y)\, dy \tag{1.12}$$

We call G_L the Green's function for L and Ω. Note that G_L is symmetric (by the symmetry of the bilinear form associated to L) and nonnegative (by the maximum principle, thm. 1.4). Moreover, $G_L(x, \cdot)$ solves the equation $Lu = \delta_x$ in distributional sense (where δ_x is the Diraç mass concentrated at x), so if a_{ij} and V are smooth in Ω, $G_L(x, \cdot)$ is smooth in $\Omega \setminus \{x\}$ and solves the equation $Lu = 0$ in $\Omega \setminus \{x\}$. More properties of G_L will be proved in section 4.

2. Local estimates for solutions of $Lu = 0$.

We will need in the following two basic local estimates for solution of $Lu = 0$. Namely, we need bounds for the $\mathcal{L}^2$-norm of the gradient of u and for the $\mathcal{L}^\infty$-norm of u, on a ball, both in terms of the $\mathcal{L}^2$-norms of u on a larger ball. All the results in this section are taken from [6].

Theorem 2.1. *("Caccioppoli's inequality"). Suppose that u satisfies $Lu = 0$ in Ω. If $0 < s < t$, B_s, B_t denote concentric balls of radii s, t and $B_t \subseteq \Omega$, then:*

$$\int_{B_s} |Du|^2\, dx \le \frac{c}{(t-s)^2} \int_{B_t} u^2\, dx$$

with $c = c(n, \lambda, \eta)$.

The proof can be found in [6], and follows from the definition of weak solution, by standard "test function techniques". Thm 1.2 is also employed, to estimate integrals involving the potential V.

Remark 2.2. By a covering argument, theorem 2.1 extends to compact subsets of Ω: if $\Omega_1 \Subset \Omega_2 \Subset \Omega$ and u is a solution of $Lu = 0$ in Ω, then:

$$\int_{\Omega_1} |Du|^2\, dx \le c(\Omega_1, \Omega_2, n, \lambda) \cdot \int_{\Omega_2} u^2\, dx\,.$$

Particularly, we will use theorem 2.1 in the following form:
let C be a spherical shell contained in $B_t \setminus B_s$ $(1/2 \le s < t \le 1)$, $C \equiv B_{t-2\epsilon} \setminus B_{s+2\epsilon}$, with ϵ of the order of $(t-s)$, and $C' \equiv B_{t-\epsilon} \setminus B_{s+\epsilon}$. Then:

$$\int_C |Du|^2\, dx \le \frac{c}{(t-s)^2} \int_{C'} u^2\, dx\,.$$

Lemma 2.3. *Let u be a solution of $Lu = 0$ in Ω, and suppose that $B_2 \subseteq \Omega$. Then:*

$$\left(\int_{B_{1/2}} u^2\, dx\right)^{1/2} \le c \cdot \int_{B_1} |u|\, dx$$

with $c = c(n, \lambda, \eta)$.

The proof is in [6], and makes use of an argument which can be found in [13] and relies on an idea od Dahlberg-Kenig, consisting in getting the desired result from the following estimate:

$$I(s) \le c \left\{ \frac{I(t)}{t-s} \right\}^{\frac{n}{n+2}}$$

where:

$$I(s) = \left(\int_{B_s} u^2\, dx\right)^{1/2} \quad \text{for } \frac{1}{2} \le s \le 1\,.$$

This estimate, in turn, follows from Sobolev inequality and thm. 2.1.

Remark 2.4. By dilatation, from lemma 2.3 it follows:

$$\fint_{B_{r/2}} u^2\,dx \le c \fint_{B_r} |u|\,dx$$

(with c independent from r) if $Lu = 0$ in $\Omega \supseteq B_{2r}$.

Then (see remark 2.2) we can obtain the following version of lemma 2.3, for spherical shells:

$$\left(\int_{B_{t-2\xi}\setminus B_{s+2\xi}} u^2\,dx\right)^{1/2} \le \frac{c}{(t-s)^{n/2}} \int_{B_{t-\xi}\setminus B_{s+\xi}} |u|\,dx$$

$(\frac{1}{2} \le s < t \le 1)$ if $Lu = 0$ in $\Omega \supseteq B_2$, and ξ of the order of $(t-s)$.

Theorem 2.5. *Let u be a solution of $Lu = 0$ in Ω, $\Omega \supseteq B_2$. Then, for $\frac{1}{2} \le s < t \le 1$, we have:*

$$\|u\|_{\mathcal{L}^\infty(B_s)} \le \frac{c}{(t-s)^\alpha} \cdot \left(\int_{B_t} u^2\,dx\right)^{1/2}$$

with $c(n,\lambda,\eta)$, $\alpha = \alpha(n)$.

The proof, in the case of smooth coefficients and potential, can be found in [6]. (The general case then follows by mollification). In the regular case, representation formulas by means of Green functions and a suitable choice of test functions are the main tools of the proof. Thms. 2.1 and 2.3, in the case of spherical shells, are employed for the estimates.

3. Continuity of the solution of $Lu = 0$.

A famous result of De Giorgi-Nash-Moser (see [11], [21], [19], [20]) states that any local solution of $Au = 0$ in Ω is locally Hölder continuous. We recall here a precise statement of this fact, with will be used in this section:

Theorem 3.1. *(See [20]). Let u be a solution of $Au = 0$ in $B_r(x_0)$. Then:*

$$|u(x) - u(x_0)| \le c(n,\lambda) \cdot \sup_{B_r(x_0)} |u| \cdot \left(\frac{|x-x_0|}{r}\right)^\alpha \tag{3.1}$$

for every $x \in B_r(x_0)$, with $\alpha = \alpha(n,\lambda)$.

Here we want to show how from this fact continuity of solutions of $Lu = 0$ can be proved, using the local boundedness of solutions stated in thm. 2.5. Let u be a solution of $Lu = 0$ in $B_{2r}(x_0)$. Then, one has:

$$u = v + w \quad \text{in } B_r(x_o),$$

with:

$$(3.2) \qquad \begin{cases} Av = 0 & \text{in } B_r(x_0) \\ v = u & \text{on } \partial B_r(x_0) \end{cases} \qquad \begin{cases} Aw = -Vu & \text{in } B_r(x_0) \\ w = 0 & \text{on } \partial B_r(x_0). \end{cases}$$

Since u is bounded in $B_r(x_0)$, v is bounded, too, and by thm. 3.1 v is continuous in $B_r(x_0)$ and satisfies (3.1). Moreover, $Vu \in K(B_r(x_o))$. Then we say that:

Lemma 3.2.

$$(3.3) \qquad w(x) = -\int_{B_r(x_0)} G(x,y)\, V(y)\, u(y)\, dy$$

(where G is the Green's function for A in $B_r(x_0)$) and w is continuous.

Proof. The fact that representation formula (3.3) holds can be seen at the following way. Put:

$$f = -Vu, \quad f_m(x) = \begin{cases} f(x) & \text{if } |f(x)| \le m \\ 0 & \text{otherwise} \end{cases}.$$

Since $f_m \in \mathcal{L}^\infty(\Omega)$, the solution of:

$$\begin{cases} Aw_m = f_m & \text{in } B_{2r}(x_0) \\ w_m = 0 & \text{on } \partial B_{2r}(x_0) \end{cases}$$

is:

$$w_m(x) = \int_{B_r(x_0)} G(x,y)\, f_m(y)\, dy$$

and:

$$Vertw_m\|_\infty \le \int_{B_r(x_0)} G(x,y)|f(y)|dy \le \text{(by thm. 1.5)}$$

$$\le c \int_{B_r(x_0)} \frac{|f(y)|}{|x-y|^{n-2}} dy \le \text{const.}$$

Hence, by Caccioppoli's inequality, w_m is locally bounded in $H^{1,2}$, so there exists a subsequence w_m converging (locally) to w, weakly in $H^{1,2}$ and a.e.. Since:

$$\int_{B_r(x_0)} G(x,y) f_m(y)\, dy \to \int_{B_r(x_0)} G(x,y) f(y)\, dy$$

it follows (3.3).

Now, observe that if $f \in K(\Omega)$ the function:

$$w(x) = \int_{B_r(x_0)} G(x,y) f(y)\, dy$$

is continuous. In fact ([2]), if we put:

$$w_m(x) = \int_{\{y \in B_r(x_0): |x-y| > \frac{1}{m}\}} G(x,y) f(y)\, dy$$

by definition of Kato class, we see that $w_m \to w$ uniformly. Moreover, since $G(x,\cdot) \in C(B_{2r}(x_0) \setminus \{x\})$ and (1.4) holds, one gets that w_m is continuous, so that w is continuous, too.

Up to this time, we have proved the continuity of solutions of $Lu = 0$ only in a qualitative way. Now we want to, determine a (local) modulus of continuity for u, wich depends on u only through its local supremum. Namely, we have the following:

Theorem 3.3. *Let u be a solution of $Lu = 0$ in $B_{2r}(x_0)$. Then for every $x \in B_r(x_0)$:*

$$\text{(3.4)} \qquad |u(x) - u(x_0)| \le c(n,\lambda) \cdot \sup_{B_r(x_0)} |u| \cdot \left\{ \left(\frac{|x-x_0|}{r} \right)^{\alpha} (1 + \eta(r)) + \eta\left(3r^{\frac{1}{2}} |x-x_0|^{\frac{1}{2}}\right) \right\}$$

with $\alpha = \alpha(n,\lambda)$.

Note: estimate (3.4) is, up to constants, identical to the continuity estimate which is found in [6] as a consequence of Harnack inequality.

Proof. Here we keep the notations of (3.2) - (3.3). Then, for every $x \in B_r(x_0)$:

$$\text{(3.5)} \qquad |w(x) - w(x_0)| \le \le \sup_{B_{r(x_0)}} |u| \int_{B_{r(x_0)}} |G(x,y) - G(x_0,y)| \cdot |V(y)|\, dy =$$

([2]) The following argument is taken from [2], thm. 4.15.

$$= \sup_{B_r(x_0)} |u| \cdot \left\{ \int_{B_{r(x_0)} \cap \{y: |y-x_0| \leq 2r^{1/2}|x-x_0|^{1/2}\}} |G(x,y) - G(x_0,y)| \cdot |V(y)|\, dy + \right.$$

$$\left. + \int_{B_{r(x_0)} \cap \{y: |y-x_0| > 2r^{1/2}|x-x_0|^{1/2}\}} |G|, (x,y) - G(x_0,y)| \cdot |V(y)|\, dy \right\} =$$

$$= \sup_{B_r(x_0)} |u| \cdot \{I + II\}.$$

Now, by (1.4):

$$I \leq c|, (n,\lambda) \left\{ \int_{|y-x_0| \leq 2r^{1/2}|x-x_0|^{1/2}} \frac{|V(y)|}{|x_0 - y|^{n-2}} dy \quad + \right. \tag{3.6}$$

$$\left. + \int_{|y-x| \leq |x-x_0| + 2r^{1/2}} \frac{|V(y)|}{|x-y|^{n-2}} dy \right\} \leq$$

(since $|x - x_0| \leq r^{1/2}|x - x_0|^{1/2}$)

$$\leq c(n,\lambda)\, \eta(3r^{1/2}|x - x_0|1/2).$$

Now we estimate II.

Put $\rho = \frac{1}{2}|y - x_0|$. If $|y - x_0| > 2r^{1/2}|x - x_0|^{1/2}$, then $|x - x_0| < \rho$. Now we apply thm. 3.1 to $G(\cdot, y)$, which is a solution of $Au = 0$ in $B_\rho(x_0)$:

$$|G(x,y) - G(x_0,y)| \leq c(n,\lambda) \cdot \sup G(w,y)_{w \in B_\rho(x_0)} \cdot \left(\frac{|x - x_0|}{\rho}\right)^\alpha.$$

But:

$$G(w,y) \leq \frac{c(n,\lambda)}{|w-y|^{n-2}} \leq (\text{for } w \in B_\rho(x_0)) \quad \frac{c}{\rho^{n-2}} \leq$$

$$\leq \frac{c(n,\lambda)}{|x_0 - y|^{n-2}}$$

while, since $|y - x_0| > 2r^{1/2}|x - x_0|^{1/2}$:

$$\left(\frac{|x - x_0|}{\rho}\right)^\alpha \leq \left(\frac{|x - x_0|}{r}\right)^{\alpha/2}.$$

Then:

$$II \leq c(n,\lambda) \cdot \left(\frac{|x - x_0|}{r}\right)^{\alpha/2} \cdot \int_{B_r(x_0)} \frac{|V(y)|}{|x_0 - y|^{n-2}} dy \leq \tag{3.7}$$

$$\leq c(n,\lambda) \cdot \eta(r) \cdot \left(\frac{|x - x_0|}{r}\right)^{\alpha/2}.$$

From (3.5), (3.6), (3.7) the theorem follows.

Remark 3.4. To have a control for the local modulus of continuity of u in terms of the local supremum of u (and therefore, by thm. 2.5, in terms of the local $\mathcal{L}^2$-norm of u) is important for the following reason. Let u_m be a sequence of local solutions of $Lu = 0$ in Ω such that:

$$\|u_m\|_{\mathcal{L}^2(\Omega')} \leq c(\Omega') \quad \text{for every} \quad \Omega' \Subset \Omega.$$

Then, by theorems 2.5 and 3.3 the family $\{u_m\}$ is locally (equi) bounded and equicontinuous, so there exists a subsequence converging to some $u \in \mathcal{C}(\Omega)$ uniformly on every compact subset of Ω. Moreover, by thm. 2.1, the function u is also the limit of a subsequence weakly in $H^{1,2}_{loc}(\Omega)$. These facts will be implicitely used in sections 5-6, to apply the arguments contained in [5].

Remark 3.5. While solutions of $Au = 0$ are locally Hölder continuous, solutions of $Lu = 0$ have been proved to be simply continuous. This result cannot be improved if V is assumed only in $K(\Omega)$. On the other hand, if $V \in \mathcal{L}^p(\Omega)$, $p > n/2$, it is known from [25] that a local Hölder continuity estimate holds. However it is possible to obtain this result under a weaker assumption: if V is assumed in suitable *Morrey spaces* (contained in $K(\Omega)$), then solutions are still locally Hölder continuous. This result is contained in [12]. Namely, the following holds. Let:

$$\mathcal{L}^{1,\lambda}(\Omega) \equiv \left\{ f \in \mathcal{L}^1(\Omega) : \|f\|_{1,\lambda} \equiv \sup_{\substack{x \in \mathcal{R}^n \\ r>0}} \frac{1}{r^\lambda} \int_{\Omega \cap B_r(x)} |f(y)|dy < \infty \right\},$$

for any fixed $\lambda > 0$. Then if $V \in \mathcal{L}^{1,\lambda}(\Omega)$ for some $\lambda > n-2$ and u is a local solution of $Lu = 0$ in Ω, u is locally Hölder continuous in Ω.

Note that: $\mathcal{L}^{1,\lambda}(\Omega) \subseteq K(\Omega)$ if $\lambda > n-2$; if $V \in \mathcal{L}^p(\Omega)$ for some $p > n/2$, then $V \in \mathcal{L}^{1,\lambda}(\Omega)$ for some $\lambda > n-2$, but being in $\mathcal{L}^{1,\lambda}(\Omega)$ does not imply any extra integrability property. (See [12] for these facts).

Other sufficient conditions to assure local Hölder continuity of the solutions are discussed in [23], see thm. 3.4.

4. Properties of the Green's function.

The first result we present in this section is an estimate comparing G_L and G. This will be the basis for all the following development of potential theory. (Section 5-6). The result is due to [9]; the proof we present here is an analytical one, as appears in [3]. We stress that it does not depend on any argument or result contained in sections 2-3: the only nontrivial ingredient of the proof is the "3 Green theorem" (thm. 1.6).

Theorem 4.1. *If δ is defined as in (1.7. a) and (1.8) holds, then:*

$$(4.1)\qquad \left(\frac{1-2\delta}{1-\delta}\right)G(x,y) \le G_L(x,y) \le \frac{1}{1-\delta}G(x,y) \quad \textit{for a.e. } x,y \in \Omega.$$

Proof. Using representation formulas (1.3) - (1.12) one can find the following identity:

$$(4.2)\qquad \begin{aligned} G_L(x,y) &= \\ &= G(x,y) - \int_\Omega G_L(x,w)V(w)\,G(w,y)\,dw \quad \text{for a.e. } x,y, \in \Omega. \end{aligned}$$

Now, let consider the space $\mathcal{B}$ defined by:

$$\mathcal{B} = \Big\{ f : \Omega \times \Omega \to \mathbb{R},\ f \text{ measurable such that } \|f\|_{\mathcal{B}} \equiv$$

$$\equiv \sup_{x\in\Omega} \int_\Omega |f(x,y)|dy < +\infty \Big\}.$$

$\mathcal{B}$ is a Banach space. If we define the operator T as:

$$Tf(x,y) \equiv \int_\Omega f(w,y)V(w)\,G(x,w)\,dw$$

one can verify that T is a well defined, linear continuous operator from $\mathcal{B}$ to $\mathcal{B}$, with $\|T\|_{\mathcal{L}(\mathcal{B})} \le \delta$.
Let us consider the integral equation:

$$(4.3)\qquad f + Tf = G$$

where the unknown function f is sought in $\mathcal{B}$.Then $(I+T)$ can be inverted by Neumann series:

$$(4.4)\qquad (I+T)^{-1} = \sum_0^\infty (-)^n T^n$$

where the series converges in $\mathcal{L}(\mathcal{B})$. Since, by (4.2), the solution of (4.3) is G_L, (4.4) gives:

$$(4.5)\qquad G_L = \sum_0^\infty (-)^n T^n G$$

where the series converges in $\mathcal{B}$. By thm. 1.6 we have:

$$|TG(x,y)| = \left| \int_\Omega G(w,y) V(w) G(x,w)\, dw \right| \leq$$

$$\leq G(x,y) \cdot \int_\Omega \frac{G(x,w)\, G(w,y)}{G(x,y)} |V(w)|\, dw \leq$$

$$\leq G(x,y) \cdot c_3 \left\{ \int_\Omega \frac{|V(w)|}{|y-w|^{n-2}}\, dw + \int_\Omega \frac{|V(w)|}{|x-w|^{n-2}}\, dw \right\} \leq$$

$$\leq G(x,y) \cdot c_3 \cdot 2\eta(d) \leq \delta \cdot G(x,y).$$

By iteration:

$$|T^n G(x,y)| \leq \delta^n \cdot G(x,y). \tag{4.6}$$

Since convergence in $\mathcal{B}$ implies convergence a.e. of a subsequence, it follows from (4.5) - (4.6) that:

$$G_L(x,y) \leq \frac{1}{1-\delta} G(x,y) \quad \text{for a.e. } x, y \in \Omega, \tag{4.7}$$

and so we have the right hand inequality in (4.1). On the other hand, again from (4.2), we have:

$$G_L(x,y) = G(x,y) \cdot \left\{ 1 - \int_\Omega \frac{G_L(w,y)\, G(x,w)}{G(x,y)} V(w)\, dw \right\} \geq$$

$$\geq \text{ (by (4.7))}$$

$$\geq G(x,y) \cdot \left\{ 1 - \frac{1}{1-\delta} \int_\Omega \frac{G(w,y)\, G(x,w)}{G(x,y)} |V(w)|\, dw \right\} \geq$$

$$\geq \text{ (by thm.1.6)}$$

$$\geq G(x,y) \cdot \left\{ 1 - \frac{1}{1-\delta} \cdot c_3 \cdot 2\eta(d) \right\} \geq \left(\frac{1-2\delta}{1-\delta} \right) G(x,y)$$

and the proof is complete.

Let us see some consequences of thm. 4.1. Combining this fact with results in [18], we have:

Theorem 4.2. *Let Σ be a bounded Lipschitz domain which can be mapped smoothly onto a sphere, and let G_L, g be the Green's functions for L and $-\Delta$, respectively, in Σ. Then, for any compact subset C of Σ, there exists a constant K only depending on C, Σ and δ, such that:*

$$k^{-1} \cdot g(x,y) \leq G_L(x,y) \leq k \cdot g(x,y) \quad \text{for a.e. } x, y \in C.$$

Corollary 4.3.

$$G_L(x,y) \leq \frac{c(\delta)}{|x-y|^{n-2}} \quad \text{for a.e. } x, y \in \Omega.$$

Remark 4.4. By the maximum principle (thm. 1.4) the following property holds. Let $V_1, V_2 \in K(\Omega)$ (both V_i satisfying our assumption (1.8)), and let G_{L_1}, G_{L_2} be the Green's functions for $A + V_1$, $A + V_2$, respectively. If $V_1 \leq V_2$, then $G_{L_1} \leq G_{L_2}$.

Now, let us write $V = V^+ - V^-$ let η^- be the Kato norm of V^-, η^- satisfying (1.8), and $V^+ \in K(\Omega)$. Then the bilinear form associated to L is still coercive, and there exists the Green's function G_L. Hence, by the above remark, $G_L \leq G_{A-V^-}$, while G_{A-V^-} clearly satisfies thm. 4.1. So it is still true that:

$$G_L(x,y) \leq \frac{1}{1-\delta} G(x,y).$$

It is the estimate from below for G_L that, with our techique of proof, cannot be stated without the assumption that V^+ is small, too.

Now we want to discuss some regularity properties for the Green's function G_L. We remember that, by our definition of G_L, all we know about it, in terms of function spaces, is that:

$$G_L \in \mathcal{L}^\infty(\Omega, \mathcal{L}^q(\Omega)) \quad \text{for } q < \frac{n}{n-2}.$$

On the other hand, if V and the coeeficients a_{ij} are smooth, then $G_L(x,\cdot)$ is smooth in $\Omega \setminus \{x\}$, and solves the equation $Lu = 0$ in $\Omega \setminus \{x\}$. Now we state the following result:

Theorem 4.5. *For a.e. $x \in \Omega$, $G_L(x,\cdot)$ belongs to:*

$$H^{1,2}_{loc}(\Omega \setminus \{x\}), \quad H^{1,p'}(\Omega) \quad \text{for } p' < \frac{n}{n-1}, \; C(\Omega \setminus \{x\}).$$

Moreover, $G_L(x,\cdot)$ is a local solution of $Lu = 0$ in $\Omega \setminus \{x\}$.

To obtain these results we will follow the line of [18], seeing G_L as the limit, in suitable spaces, of a sequence of Green's function for approximating operators. This will be possible by the results of sections 2-3, and using a notion of "very weak solution", introduced for the operator A in [18], which we are going to explain here. We start with the following:

Theorem 4.6. *Let u be the solution of:*

$$(4.8)\qquad \begin{cases} Lu = \sum_1^n (f_i)_{x_i} & \text{with } f_i \in \mathcal{L}^p(\Omega),\ p > n \\ u \in H_0^{1,2}(\Omega). \end{cases}$$

Then $u \in C(\overline{\Omega})$ and $\|u\|_\infty \leq c\,\|f_i\|_p$, with $c = c\,(n, \lambda, p, |\Omega|, \delta)$.

This result holds for the operator A, and is due to Stampacchia. (See [18] or [26]). We can prove it for the operator L, with the same technique used in thm. 3.2: first, we suppose V and a_{ij} smooth, so that u is actually bounded. Then we write $u = v + w$ with v satisfying thm. 4.6 for the operator A, and:

$$w(x) = -\int_\Omega G\,(x,y)V(y)\,u\,(y)\,dy,$$

so that $\|w\|_\infty \leq \delta\|u|_\infty$. Then estimate (4.8) follows. By mollification, (4.8) holds in the general case, too, so that u is bounded even thought coefficients are discontinuous. Then w is continuous, and u is continuous.

Now we give the following:

Definition 4.7. *(See [18]). For a measure μ of bounded variation on Ω, we say that $u \in \mathcal{L}^1(\Omega)$ is a "very weak solution" of $Lu = \mu$ (vanishing at the boundary of Ω), if it satisfies:*

$$\int_\Omega u \cdot L\varphi\,dx = \int_\Omega \varphi\,d\mu$$

for every $\varphi \in H_0^{1,2}(\Omega) \cap C(\bar{\Omega})$ such that $L\varphi \in C(\overline{\Omega})$.

Uniqueness of the very weak solution in $\mathcal{L}^1(\Omega)$ is easily seen. On the other hand thm. 4.6 allows us to apply the "duality method" used in [18] to assure that a very weak solution always exists. Namely, the following can be proved:

Theorem 4.8. *For any measure μ of bounded variation, a unique very weak solution of $Lu = \mu$ exists, and lies in $H_0^{1,p'}(\Omega)$ for any $p' < \frac{n}{n-1}$; moreover u satisfies:*

$$(4.9)\qquad \|u\|_{H_0^{1,p'}(\Omega)} \leq c\,(n, \lambda, p', |\Omega|, \delta) \cdot \int_\Omega |d\mu|$$

and u is assigned by the integral (a.e. converging):

$$(4.10)\qquad u(x) = \int_\Omega G_L(x,y)\,d\mu(y).$$

Finally, $G_L(x,\cdot)$ *is the very weak solution of* $Lu = \delta_x$.

Now we turn to the proof of thm. 4.5.
We regularize V and a_{ij} by mollifiers, and consider the sequence G^s_L of the Green's functions of regularized operators. By thm. 4.8,

$$\|G^s_L(x,\cdot)\|_{H_0^{1,p'}(\Omega)} \leq \text{constant} \quad (\text{for any } p' < \frac{n}{n-1}).$$

From this it follows that (a subsequence) $G^s_L(x,\cdot)$ converges to $G_L(x,\cdot)$ weakly in $H_0^{1,p'}(\Omega)$. Particulary, $G_L(x,\cdot) \in H_0^{1,p'}(\Omega)$.
By thms. 2.1 - 2.5, $G^s_L(x,\cdot)$ is bounded in $H^{1,2}(\Omega')$ for every $\Omega' \subset \Omega \setminus \{x\}$, so that a subsequence converges (to $G_L(x,\cdot)$) in $H^{1,2}(\Omega')$. Then $G_L(x,\cdot) \in H^{1,2}_{loc}(\Omega \setminus \{x\})$ and is a local solution of $Lu = 0$ in $\Omega \setminus \{x\}$. Therefore, by thm. 4.2, $G_L(x,\cdot) \in C(\Omega \setminus \{x\})$. So thm. 4.5 is proved.

5. Dirichlet's problem with continuous boundary data. L-harmonic measure.

In order to define the basic concept of L-harmonic measure and develop a potential theory for L, we need a sharper version of maximum principle.

Theorem 5.1. *(See [3]). If u is a supersolution (subsolution) for L in Ω, then (respectively):*

$$\min_{\Omega} u \geq \frac{1}{1-\delta} \min_{\partial\Omega} u^- \tag{5.1.a}$$

$$\max_{\Omega} u \leq \frac{1}{1-\delta} \max_{\partial\Omega} u^+ \tag{5.1.b}$$

If u is a solution of $Lu = 0$ in Ω, then:

$$\max_{\Omega} |u| \leq \frac{1}{1-\delta} \max_{\partial\Omega} |u| \,. \tag{5.2}$$

Moreover, the number δ can be replaced by δ^- (see (1.7 b)), so that if V is nonnegative one can take the constant in (5.1) - (5.2) equal to one.

The proof makes use of thm. 4.1 and the maximum principle, thm. 1.4. Replacement of δ with δ^- is possible by remark 4.4.

Remark 5.2. If V assumes also negative values, one cannot expect a "standard" maximum principle to be true:

$$\max_{\Omega} |u| \leq \max_{\partial\Omega} |u| . \tag{5.2}$$

To see this, is sufficient to consider Ω the unit ball, $A = -\Delta$, V a small negative constant: then the solution u with boundary value 1 is a positive function asuming a strong maximum at the origin. We are now intersted to solve a Dirichlet's problem:

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases} \tag{5.3}$$

where f is a continuous function defined only on $\partial\Omega$. If f is the trace of a function $\tilde{f} \in C^1(\overline{\Omega})$, clearly we can consider the variational solution of:

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = \tilde{f} & \text{on } \partial\Omega \end{cases} \tag{5.4}$$

In the general case, f can be approximated with smooth functions defined on $\mathbb{R}^n$, and the corresponding "approximating problems" can be solved. Thms. 2.1 and 5.1 enable us to repeat the same argument which is found in [18], to get the following result:

Theorem 5.3. *There exists a linear mapping $\mathcal{J}$ wich to any continuous function f defined on $\partial\Omega$ associates a function $u \in H^{1,2}_{loc}(\Omega) \cap C(\Omega)$ which is a local solution of $Lu = 0$ in Ω, and satisfies the maximum principle (5.2). Moreover, if f is the trace of a function $\tilde{f} \in C^1(\bar{\Omega})$, $u = \mathcal{J} f$ coincides with the variational solution of (5.3).*

Remark 5.4. If we do not assume that Ω is Lipschitz, we can still prove the previous theorem, except for the last conclusion: it is no more true that if f is the trace of a function $\tilde{f} \in C^1(\bar{\Omega})$, then u solves the problem in variational sense.

Theorem 5.3. allows us to give the following definitions.

Definition 5.5. *A point $w \in \partial\Omega$ is said to be regular for L iff for every $h \in C(\Omega)$ one has:*

$$\lim_{\substack{x \to w \\ x \in \Omega}} \mathcal{J} h(x) = h(w) .$$

Definition 5.6. *For a fixed $x \in \Omega$, let us consider the functional $f \to \mathcal{J} f(x)$ defined on $C(\partial\Omega)$. By thm. 5.3 this is a linear continuous functional; hence by*

Riesz' theorem there exists a positive regular Borel measure $w_L{}^x$ representing it:

$$\mathcal{J}f(x) = \int_{\partial\Omega} f(y)dw_L^x(y)\,.$$

We call w_L^x the L-harmonic measure evaluated at x. We shall indicate with w_A^x the Borel measure obtained by the same construction for the operator A (A-harmonic measure evaluated at x).

The main result we are going to prove now is the following: there exist constants c_1, c_2 such that for any Borel set $E \subseteq \partial\Omega$ and $x \in \Omega$:

$$c_1 \cdot w_A^x(E) \le w_L^x(E) \le c_2 \cdot w_A^x(E)\,. \tag{5.5}$$

From this fact and known results for the operator A (namely: Harnack inequality, see [20], and potential theoretical results, see [5]) analogue results for the operator L will follow. (See section 6). We shall obtain (5.5) from the estimate of thm. 4.1 involving the Green's functions for A and L. The link between harmonic measures and Green's functions is the "kernel function" for A, a notion studied in [5].

Definition 5.7. *Fix $x \in \Omega$, $z \in \partial\Omega$. A function $K_A^x(\cdot, z)$ defined in Ω is called a kernel function at z for the operator A, normalized at x, if:*

(i) $K_A^x(\cdot, z)$ *is a solution of* $Au = 0$ *in* Ω;

(ii) $K_A^x(\cdot, z) \in \mathcal{C}(\overline{\Omega} \setminus \{z\})$ *and:* $\lim\limits_{\substack{w \to z' \neq z \\ z' \in \partial\Omega}} K_A^x(w, z) = 0$;

(iii) $K_A^x(w, z) > 0$ *for each* $w \in \Omega$ *and* $K_A^x(w, z) = 1$;

For x and z fixed, there exists one and only one kernel function $K_A^x(\cdot, z)$ and it is:

$$K_A^x(w, z) = \frac{dw_A^w}{dw_A^x(z)} \tag{5.6}$$

(Radon-Nikodym derivative of the A-harmonic measures). (See [5]).

It can be proved also that, for any $w \in \Omega$, $z \in \partial\Omega$, there exists:

$$\lim_{\substack{y \to z \in \partial\Omega \\ y \in \Omega}} \frac{G(w, y)}{G(x, y)} = K_A^x(w, z)\,. \tag{5.7}$$

The following two theorems are taken from [3]; these results are proved in a probabilistic way in [9].

Theorem 5.8. *For $x \in \Omega$, $z \in \partial\Omega$, there exists:*

$$F(x,z) = \lim_{\substack{y \to z \\ y \in \Omega}} \frac{G_L(x,y)}{G(x,y)}. \tag{5.8}$$

Moreover F is continuous on $\Omega \times \partial\Omega$ and:

$$F(x,z) = 1 - \int_\Omega K_A^x(w,z) V(w)\, G_L(x,w)\, dw. \tag{5.9}$$

The proof consists in showing that in the equality:

$$\frac{G_L(x,y)}{G(x,y)} = 1 - \int_\Omega \frac{G(w,y)}{G(x,y)} \cdot G_L(x,w) V(w)\, dw \tag{5.10}$$

the limit for $y \to z$ can be taken under the integral sign. This can be done by Lebesgue's theorem, using thms. 4.1, 1.6 and the definition of Kato class.

Theorem 5.9. *(Comparision between harmonic measures).*
For each $x \in \Omega$, $z \in \partial\Omega$, one has:

$$d\omega_L^x(z) = F(x,z)\, d\omega_A^x(z). \tag{5.11}$$

Moreover, there exist constants c_1, c_2 depending on δ such that (5.5) holds, for every Borel set $E \subseteq \partial\Omega$, $x \in \Omega$.

Proof. Let $f \in C(\partial\Omega)$. By (5.9):

$$\int_{\partial\Omega} f(z)\, F(x,z)\, d\omega_A^x(z) = \int_{\partial\Omega} f(z)\, d\omega_A^x(z) - \tag{5.12}$$

$$- \int_{\partial\Omega} f(z)\, d\omega_A^x(z) \int_\Omega K_A^x(w,z) V(w) G_L(x,w)\, dw =$$

(by Fubini and (5.6))

$$= \int_{\partial\Omega} f(z)\, d\omega_A^x(z) -$$

$$- \int_\Omega V(w) G_L(x,w) dw \int_{\partial\Omega} f(z) d\omega_A^w(z).$$

Now, let v, u be the solutions of:

$$\begin{cases} Lv = 0 & \text{in } \Omega \\ v = f & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} Au = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Then (5.12) becomes:

$$\int_{\partial\Omega} f(z)\, F(x,z)\, dw_A^x(z) = u(x) -$$

$$-\int_\Omega V(w)\, G_L(x,w)\, u(w)\, dw = v(x) = \int_{\partial\Omega} f(z)\, dw_L^x(z).$$

Since this is true for every $f \in C(\partial\Omega)$, it follows (5.11). Now, by thm. 5.8, (4.1) implies:

$$\frac{1-2\delta}{1-\delta} \le f(x,z) \le \frac{1}{1-\delta} \quad \text{for all} \quad x \in \Omega,\ z \in \partial\Omega.$$

So (5.5) follows from (5.11).

6. Potential theory for L.

Now we obtain from thm. 5.9 a Harnack inequality for L (i.e. the main result of [6]).

Theorem 6.1. *(Harnack inequality).*
Let u be a positive solution of $Lu = 0$ in $\Omega \supseteq B_{2r}(x_0)$. Then:

$$\max_{B_r(x_0)} u \le c \cdot \min_{B_r(x_0)} u$$

with $c = c(n, \lambda, \delta)$.

Proof. Since u is a solution in $B_{2r}(x_0)$, by thm. 4.2 $u \in C(\overline{B_r(x_0)})$; then if w_L^x is the L-harmonic measure relative to $B_r(x_0)$, for every $x_1, x_2 \in B_r(x_0)$ one has:

$$u(x_1) = \int_{\partial B_r(x_0)} u(z)\, dw_L^{x_1}(z) \le \quad \text{(by thm. 5.9)}$$

$$\le \frac{1}{1-\delta} \int_{\partial B_r(x_0)} u(z)\, dw_A^{x_1}(z) \le$$

(by the Harnack principle for A, see [20])

$$\le \frac{c(n,\lambda)}{1-\delta} \int_{\partial B_r(x_0)} u(z)\, dw_A^{x_2}(z) \le \quad \text{(by thm. 5.9)}$$

$$\leq \frac{c(n,\lambda)}{1-2\delta} \int_{\partial B_r(x_0)} u(z)\, dw_L^{x_2}(z) = c\,(,n,\lambda,\delta)\, u\,(x_2)\,.$$

Remark 6.2. By the Harnack principle we have that, for any $x_1, x_2 \in \Omega$, the L-harmonic measures $w_L^{x_1}$, $w_L^{x_2}$ are mutually absolutely continuous. So it is not ambiguous to speak of a set "of L-harmonic measure zero".

Now we derive for L three theorems which are proved in [5] for A ([3]).

Theorem 6.3. *(Boundary Harnack Principle).*
Let $z_0 \in \partial\Omega$, $r > 0$, $x_r \in \Omega$ such that $|x_r - z_0| = r$ and dist $(x_r, \partial\Omega) \simeq r$ (i.e. the distance is of the same order of r). If v is a positive solution of $Lv = 0$ in Ω, vanishing continuously on $\partial\Omega \cap B_{2r}(z_0)$, then:

$$\sup_{B_r(z_0)} v \leq c \cdot v(x_r) \tag{6.1}$$

for some constant $c = c\,(n, \lambda, \delta, r_0, M)$.

Proof. The proof is similar to the previous one: one considers the region $\Omega' = \Omega \cap B_{2r}(z_0)$. By thm. 4.1 and the assumptions of this theorem, $v \in C(\overline{\Omega'})$, then by representing v in Ω' by the L-harmonic measure of Ω' and using the analogue result proved in [5] one gets the theorem.

Similarly one proves the following:

Theorem 6.4. *(Comparison Principle).*
Let u, v be positive solutions of $Lw = 0$ in Ω, vanishing continuously on $\partial\Omega \cap B_{2r}(z_0)$, let z_0, r and x_r be as in thm. 6.3. Then:

$$\sup_{B_r(z_0)} \frac{u}{v} \leq c \cdot \frac{u}{v}(x_r) \tag{6.2}$$

for some constant $c = c\,(n, \lambda, \delta, r_0, M)$.

Theorem 6.5. *(Comparison between solutions of L and A),*
Let u, v be positive solutions of $Lv = 0$, $Au = 0$ in Ω, vanishing continuously on $\partial\Omega \cap B_{2r}(z_0)$, let z_0, r and x_r be as in thm. 6.3. Then (6.2) holds.

Now we point out a consequence of Comparison Principle:

([3]) All the results in the following of this section have been firstly obtained in [9]. The present proofs are taken from [3].

Theorem 6.6. *Let u, v be positive solutions of $Lw = 0$ in Ω, vanishing continuously on $\partial\Omega \cap B_{2r}(z_0)$, let z_0 and r be as in thm. 6.3. Then the quotient u/v can be extended as a Hölder continuous function on $\overline{\Omega} \cap B_{2r}(z_0)$.*

The technique used in [20] to prove Hölder continuity of the solutions from Harnack inequality is here employed to prove Hölder continuity of the quotient of solutions by means of the Comparison Principle (thm. 6.4).

Remark 6.7. In the proofs of thms. 6.1, 6.3 - 6.6 we have applied our estimate on L-harmonic measures only on sets of the size of B_r. Therefore the assumption (1.8) is not necessary here (see also remark 1.8): whatever is $V \in K(\Omega)$, we can say that there exists R_0, depending on n, λ, η, such that for every $r \leq R_0$ the statements of the previous theorems hold.

Once one knows these results, one can repeat the arguments contained in section 2 of [5]: existence and uniqueness of the kernel function for the operator L can be proved. Moreover, from the formula:

$$K_L^x(w,z) = \frac{d\omega_L{}^w}{d\omega_L^x}(z)$$

it follows, by (5.14), that:

$$K_L^x(w,z) = \frac{F(w,z)\, d\omega_A^w(z)}{F(x,z)\, d\omega_A^x(z)} = \frac{F(w,z)\, K_A^x(w,z)}{F(x,z)}.$$

Hence we have that $K_L^x(w,\cdot) \in C(\partial\Omega)$ and:

$$c_1 \cdot K_A^x(w,z) \leq K_L^x(w,z) \leq c_2 \cdot K_A^x(w,z)$$

for some constants c_1, c_2 depending on δ, any x, $w \in \Omega$, $z \in \partial\Omega$. Now we are interested in stating regularity of boundary points for L. This fact holds for A by [18], since Ω is Lipschitz; to transfer this property to L we have to sharpen our study of the quotient $F = G_L/G$.

Lemma 6.8. *The function F can be extended continuously to $\partial\Omega \times \partial\Omega$. For $z, z' \in \partial\Omega$ it is:*

$$F(z,z') = 1 - \int_\Omega F(w,z')\, K(z,w,z')\, V(w)\, dw \tag{6.4}$$

where K is defined as in thm. 1.7.
(Note that, since $K(z,w,z) \equiv 0, F(z,z) = 1$).

This result is a consequence of thms. 5.8 and 1.7, and can be proved in a similar way to that of thm. 5.8 (see [3]).

Theorem 6.9. *(Regularity of boundary points).*
For every $f \in C(\partial\Omega)$, $z_0 \in \partial\Omega$:

$$\lim_{\substack{x \to z_0 \\ x \in \Omega}} \int_{\partial\Omega} f(z)\, dw_L^x(z) = f(z_0).$$

Proof. Let $\{x_m\}$ be a sequence in Ω converging to z_0, $g_m(z) = f(z) \cdot F(x_m, z)$. Then $g_m \in C(\partial\Omega)$ and $\|g_m\|_\infty \le c \cdot \|f\|_\infty$. By (5.14):

$$\int_{\partial\Omega} f(z)\, dw_L^{x_m}(z) = \int_{\partial\Omega} g_m(z)\, dw_A^{x_m}(z) =$$

$$= \int_{\partial\Omega} \Big(g_m - f \cdot F(z_0, \cdot)\Big)(z)\, dw_A^{x_m}(z) + \int_{\partial\Omega} f(z)\, F(z_0, z)\, dw_A^{x_m}(z).$$

The first term tends to zero by uniform continuity of $F(\cdot,\cdot)$ on $\partial\Omega \times \partial\Omega$, while the second term, by regularity of boundary points for A (see [18]) converges to $f(z_0)$, $F(z_0, z_0) = f(z_0)$ (by (6.4)). So we are done.

Remark 6.10. By remark 5.4, the notion of regular point makes sense even for a non-Lipschitz domain Ω. It is an open question to study regularity of boundary points for L on a general bounded domain. In particular, since we do not assume that V is nonnegative, the technique employed in [18] or [25], based on the notion of "barrier function", seems not applicable to get a characterization of regular points for L ("Wiener test").

From the facts we have stated up to this point, the arguments contained in section 4 of [5] can be repeated for the operator L, and a "Fatou's theorem" for L can be stated. However, we want to recall here the main steps in the proof of this result.

Let Σ be the unit ball in $\mathbb{R}^n$, K_L the kernel function for L in Σ evaluated at the origin. The first step is the following:

Theorem 6.11. *Let u be a nonnegative solution of $Lu = 0$ in Σ. Then there exists a finite positive Borel measure ν on $\partial\Sigma$ such that:*

$$u(x) = \int_{\partial\Sigma} K_L(x, z)\, d\nu(z).$$

This results depends on: properties of the Kernel function; Harnack inequality; thm. 3.3 (i.e.an estimate on the modulus of continuity of solutions independent of regularity of coefficients). See [5] for the proof.

Second step. Let z_0 be a point on $\partial\Sigma$, $\Gamma(z_0)$ a cone of vertex z_0, contained in Σ; u^* the nontangential maximal function of u in $\Gamma(z_0)$:

$$u^*(z_0) = \sup_{x\in\Gamma(z_0)} u(x).$$

Let $\mathcal{M}_w(\nu)$ be the Hardy-Littlewood maximal function of the measure ν with respect to the L-harmonic measure $w \equiv w_L^0$:

$$\mathcal{M}_w(\nu)(z_0) = \sup_r \frac{\nu(\Delta_r(z_0))}{w(\Delta_r(z_0))}$$

where $z_0 \in \partial\Omega$ and the sup is taken among all the sets $\Delta_r(z_0) = B_r(z_0) \cap \partial\Omega$. Then:

Theorem 6.12. *Let ν be a finite Borel measure on $\partial\Sigma$, and let u be defined by ν as in (6.6). Then:*

$$u^*(z_0) \leq c \cdot \mathcal{M}_w(\nu)(z_0)$$

for some constant $c = c(n, \lambda, \delta, r_0, M)$, any $z_0 \in \partial\Omega$.

This fact is based on Harnack inequality, comparison principle (thm. 6.4) and its consequences. (See [5]). Now, using theorems 6.11 - 6.12, properties of the kernel function and arguments of real analysis ([4]), to handle $\mathcal{M}_w(\nu)$, one can prove the following "Fatou's theorem":

Theorem 6.13. *(Existence of nontangential boundary limits).*
Let u be a nonnegative solution of $Lu = 0$ in Σ. Then almost everywhere on $\partial\Sigma$ with respect to the L-harmonic measure w, the nontagential limit of u exists. This means that (for a.e. $z_0 \in \partial\Omega$) for every cone $\Gamma(z_0)$, there exists:

$$\lim_{\substack{x\to z_0\\ x\in\Gamma(z_0)}} u(x).$$

If ν is as in (6.6), let us consider the Lebesgue decomposition of ν with respect to w:

$$d\nu = d\nu_s + f dw$$

(i.e $d\nu_s$ is a singular measure and f is locally integrable, with respect to dw). Then the limit is given by f. Moreover, if f is bounded, then $d\nu_s \equiv 0$ and the following representation formula holds:

$$u(x) = \int_{\partial\Sigma} K_L(x,z) f(z)\, dw(z) = \int_{\partial\Sigma} f(z)\, dw_L^x(z). \tag{6.7}$$

As in [5], thm. 6.13 still holds when Σ is replaced by a bounded Lipschitz starshaped domain.

([4]) See for instance [27], chps. 1 - 2.

Remark 6.14. A consequence of thm. 6.13 is that we can solve Dirichlet's problem for L when the datum is assigned in $\mathcal{L}^1(\partial\Omega, dw)$. Namely, one can prove that if u is assigned by (6.7), where $f \in \mathcal{L}^1(\partial\Omega, dw)$, then u is a local solution of $Lu = 0$ in Σ, and (a.e. with respect to w) has nontangential limit equal to f. We also say that u is a solution of Dirichelet's problem:

$$\begin{cases} Lu = 0 & \text{in } \Sigma \\ u = f & \text{on } \partial\Sigma \end{cases}$$

"in the sense of nontangential convergence". Note that this solution is not in general unique.

7. Analytic and Probabilistic approach: a comparison.

Here we want to present, in an unformal way, the probabilistic standpoint in the study of the Schrödinger operator. Then we will compare the probabilistic approach of [9] with our one in deriving the results collected in this paper.

Let us consider a particle moving of "brownian motion" in a domain Ω: the particle starts at a point x and follows a "random path" in Ω. (For instance, small particles suspended in water move of brownian motion under the action of molecular bombardments). This phenomenum can be rigorously modelized as a "stochastic process" $X(t, \omega)$ ([5]): for every random path ω, $X(t)$ denotes the position in Ω of the particle, at the time t. We call "first time exit" τ_Ω the (first) time at which the particle hits the boundary of Ω:

$$\tau_\Omega = \inf \left\{ t : X(t) \notin \Omega \right\}.$$

Clearly, also τ_Ω is a "random variable". Now, suppose that f is a continuous function defined on $\partial\Omega$, which we interpret as a *payoff*: if the particle starts at $x \in \Omega$ and hits $\partial\Omega$ for the first time at the point $z = X(\tau_\Omega)$, then we have winnings (or losses) equal to $f(z)$ (according to the sign of f, we will have gain or loss). Therefore one can ask what is the *expected winning* for a particle starting at $x \in \Omega$. What one can prove is that this value equals the value at x of the solution u of Dirichelet's problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

([5]) For these general notes on brownian motion, some references are: [29], chap. 31, and [4], chaps.12, 15.

In symbols:

$$u(x) = E^x[X(f(\tau_\Omega))].$$

(Henceforth, E^x will indicate the expected value of a random variable corresponding to a brownian motion starting at x). We note that the presence in (7.1) of the laplacian is a consequence of the assumption, in modelizing the physical process, of a homogeneous and isotropic medium. In the general case the operator A can replace $-\Delta$. So one has:

$$E^x[X(f(\tau_\Omega))] = \int_{\partial\Omega} f(z)\, dw_A^x(z). \tag{7.2}$$

Similarly, if $g \in C(\overline{\Omega})$ is a "payoff" defined in Ω and we integrate winnings and losses along the path of a particle starting at x until it exits Ω, then one can prove that the expected winning equals $u(x)$, where u is the solution of:

$$\begin{cases} Au = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.3}$$

In symbols:

$$E^x \int_0^{\tau_\Omega} g(X(t))\, dt = \int_\Omega G(x,y)\, g(y)\, dy. \tag{7.4}$$

From (7.2) - (7.4) we also have the following probabilistic interpretation for the integral of the Green's function and for the A-harmonic measure:

$$E^x(\tau_\Omega) = \int_\Omega G(x,y)\, dy$$

$$P^x[X(\tau_\Omega) \in B] = w_A^x(B)$$

for any Borel subset B of $\partial\Omega$ (The first term in the last formula means: "the probability that a particle starting at x hits the boundary in a point of $B \subseteq \partial\Omega$").

Now we come to the Schrödinger operator. First of all, the condition that $V \in K(\Omega)$ can be expressed in a probabilistic way. In fact the condition:

$$\lim_{r\downarrow 0} \sup_x \int_{\Omega\cap B_r(x)} \frac{|V(y)|}{|x-y|^{n-2}}\, dy = 0$$

can be proved to be equivalent to the following:

$$\lim_{t\downarrow 0} \sup_x E^x \int_0^t |V(X(s))|\, ds = 0. \tag{7.5}$$

The relationship between brownian motion idea and the Schrödinger operator is given by the Feynmann-Kac formula, which we now state. Let us define the random variable:

$$e_V(\tau_\Omega) = \exp - \int_0^{\tau_\Omega} V(X(s))\, ds.$$

Then the solution u of problem:

$$\begin{cases} Au + Vu = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

is given by:

$$u(x) = \int_{\partial\Omega} f(z)\, dw_L^x(z) = E^x[f(X(\tau_\Omega)) \cdot e_V(\tau_\Omega)]. \tag{7.6}$$

(We will specify later conditions under which this formula holds).

To estimate the size of V in $K(\Omega)$, two basic quantities are defined in [9]: the *gauge* and the *conditional gauge*. The gauge of (A, Ω, V) is, by definition, the function:

$$F(x) = E^x[e_V(\tau_\Omega)] \qquad (\text{for } x \in \Omega).$$

To introduce the conditional gauge, a slight modification must be done in our model of brownian motion. Henceforth, we think that the particle moving in Ω has a (random) "path lifetime" τ_Ω, that is *either* the particle is "killed" at the first time τ_Ω when it hits the boundary, *or* it "dies" at some point $y \in \Omega$ (before hitting the boundary) at the time τ_Ω. In both cases still call τ_Ω the "first time exit".

So one can speak of the "conditional expectation E_y^x (of a random variable) for a particle starting at x and conditioned to converge at y at the path lifetime τ_Ω". $(x, y \in \Omega)$. Then the conditional gauge of (A, Ω, V) is, by definition, the function:

$$F(x, y) = E_y^x[e_V(\tau_\Omega)] \qquad (\text{for } x, y \in \Omega).$$

Now we can describe the line followed in [9] to obtain potential theoretical results for L in probabilistic way.

The assumptions made in [9] on V are the following:

(i) $V \in K(\Omega)$, i.e. (7.5) holds;

(ii) $F(x)$ is not identically infinite.

Under the assumption $F(\cdot) \not\equiv \infty$ it is proved in [9] that (7.6) (Feynman-Kac formula) and the following identities hold:

$$dw_L^x(z) = F(x,z)\, dw_A^x(z) \tag{7.7.a}$$

$$G_L(x,y) = F(x,y)\, G(x,y). \tag{7.7.b}$$

The central result in [9] is the "Conditional Gauge Theorem", stating that if $F(\cdot) \not\equiv \infty$, then:

$$0 < c_1 \le F(x,y) \le c_2 < +\infty$$

for some constants c_1, c_2, any $x, y \in \Omega$. From this fact and (7.7) it follows comparability of G_L and G, dw_L^x and dw_A^x. So, in the probabilistic approach, the bounds for the Green's function of L and the L-harmonic measure are obtained as a consequence of the Conditional Gauge Theorem, which is a quite difficult result, based both on the analytic estimates of thms. 1.6 - 1.7, and on involved probabilistic techniques. We recall that in our approach (7.7.b) is just the *definition* of F, which is proved to be bounded from above and from below directly by thm. 1.6. We also note that, putting $f \equiv 1$ in (7.6), we read:

$$F(x) = w_L^x(\partial\Omega).$$

So the gauge is certainly finite (and bounded) if the L-harmonic measure simply *exists* (and if the Feynmann-Kac formula holds). On the other side, we recall that, to get existence of w_L^x, we had to assume (1.8), which is a more restrictive condition than the finiteness of F, since this last condition only limits the size of V^-, while (1.8) limits $|V|$. (However, as we noted in remarks 1.8, 6.7, some results of ours still hold when only V^- satisfies (1.8)).

A. Appendix: the operator with a drift term.

We conclude with some notes on a different operator to that studied in this paper, which may be (and has been) studied in a similar way. Let:

$$Lu \equiv -\frac{1}{2}\Delta u + \underline{b} \cdot Du$$

(where $\underline{b}$ is an n-vector of functions defined on Ω). In the language of brownion motion the term $\underline{b}$ is called "drift" and appears when the position of a particle starting at x has expected value different from x. In [10] it is proved that the

Green's function of L and the L-harmonic measure are comparable with those of the principal part $-\frac{1}{2}\Delta$. The key analytical estimate which is employed to get this result is the following one (where G is the Green's function for $-\frac{1}{2}\Delta$):

$$(A.1) \qquad \frac{G(x,w)\,|D_y G(w,y)|}{G(x,y)} \leq c \cdot \left\{ |x-y|^{1-n} + |w-y|^{1-n} \right\}.$$

This holds if Ω is a $C^{1,1}$ domain. In this case, the assumptions requested on $\underline{b}$ to obtain the mentioned results are:

$$|\underline{b}|^2 \in K(\Omega)$$

and:

$$|\underline{b}| \in K_{n-1}(\Omega)$$

i.e.:

$$\sup_x \int_{\Omega \cap B_r(x)} |\underline{b}(y)| \cdot |x-y|^{1-n}\, dy \to 0 \quad \text{for } r \to 0.$$

When Ω is only a Lipschitz domain, a nice estimate as (A.1) lacks, and the assumptions which have to be made on $\underline{b}$ are much more involved.

Even in this case the estimate on the Green's function of L could be obtained in a purely analytical way, using the bound (A.1) and bypassing all the probabilistic machinery. However, to follow this line we should assume that $\underline{b}$ is small enough, in the classes involved, and in this case this request would seem rather innatural, since in the study of a complete elliptic operator:

$$\mathcal{L}u \equiv -(a_{ij}u_{x_i} + d_j u)_{x_j} + b_i u_{x_i} + cu$$

no smallness condition is usually imposed on the term (b_i). (See for instance [14], [25]).

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