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GRAPH INTERSECTION PROPERTY

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A pair (X, Y) of topological spaces X and Y is said to have the graph intersection property provided that for each continuous function $g: X \rightarrow Y$, if a connected subset of $X \times Y$ projects onto the whole Y, then it intersects the graph of g. Various relations between this and other known properties related to mapping theory are studied.

In particular, it is proved that: 1) if a space X is completely regular and a space Y is an arcwise connected metric continuum distinct from an arc, then the pair (X, Y) has the graph intersection property if and only if X is hereditarily disconnected; 2) if a connected space Y is fixed, then the graph intersection property holds for every pair (X, Y) if and only if there is a closed linear order on Y with minimal and maximal elements. Related results are obtained.

Introduction.

Given two topological spaces X and Y, the pair (X, Y) is said to have the *graph intersection property* provided that for each continuous function $g: X \rightarrow Y$, if a connected subset of $X \times Y$ projects onto the whole Y, then it intersects the graph of g. Therefore, if the space Y is not connected, then each pair (X, Y) has the property. We omit this trivial case from our further considerations, and we assume that the discussed space Y is connected. Taking, in particular, $X = Y$ and the diagonal $D = \{(x, x) : x \in X\}$ of the cartesian

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square $X \times X$ as the connected subset mentioned in the definition, we see that the graph intersection property turns into the (well known) fixed point property.

Certainly, the basic problem in the area is

PROBLEM A. What pairs of topological spaces have the graph intersection property?

Two other problems are special cases of the above one. They are the following.

PROBLEM B. Given a topological space X, characterize all topological spaces Y such that the pair (X, Y) has the graph intersection property.

PROBLEM C. Given a topological space Y, characterize all topological spaces X such that the pair (X, Y) has the graph intersection property.

One can say that it is too early to work on Problem A (or problems of this kind in general) because contemporary mathematics is (or rather mathematicians are) not able to solve a very partial case of it: what topological spaces have the fixed point property? However, as it can be seen from further parts of this paper, the graph intersection property is also related to several other well known properties or notions pertained to mapping theory as for example disconnection of the cartesian square by its diagonal, linear ordering, or the (metric) concept of the span of a space. Because of its various connections with other concepts, the graph intersection property seems to be a property interesting enough to pay some attention to it. Obviously the authors are not able to solve the above mentioned problems in full.

The obtained results can be divided into three groups. After some basic observations, in the first of them it is shown that a connected space Y admits a closed linear order with minimal and maximal elements if and only if for each space X the pair (X, Y) has the graph intersection property (Theorem 14). A structural property (called component intersection arc property) concerning a connected space Y is studied which is related to the graph intersection property. There are two main results of the second group. The former one says that if a continuum X is arcwise connected and a space Y has the component intersection arc property, then the pair (X, Y) has the graph intersection property (Theorem 25). The latter result asserts that if a space X is completely regular, and a connected space Y does not have the component intersection arc property, then the pair (X, Y) has the graph intersection property if and only if X is hereditarily disconnected (Theorem 27). The third group of results is devoted to relations between the graph intersection property and the concept of semispan.

Preliminaries.

The term *space* always means a topological space, and the term *mapping* means a continuous function. If a subset A of a space is given, we denote by cl A the closure of A, and by bd A the boundary of A in the space. We use the term *continuum* for a (nondegenerate) compact connected metric space. A continuum homeomorphic to the closed unit interval $[0,1]$ is called an *arc*. We use the symbol ab to denote an arc with end points a and b. A *simple triod* means the union of three arcs ap, bp and cp such that every two and all three of them have the point p as the only common point. A *simple closed curve* is defined as the union of two arcs having their end points as the only points of their intersections. We use the concept of a *completely regular space* (called also a *Tychonoff space)* in the sense defined in [2], p. 61.

Basic observations.

Given the product $X \times Y$ of two topological spaces X and Y, we denote by $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ the natural *projections*, i.e., $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ for each $(x,y) \in X \times Y$. For a function $g : X \to Y$ we let $\Gamma(g)$ to denote the *graph* of g, i.e.,

$$\Gamma(g) = \{(x, g(x)) \in X \times Y : x \in X\}.$$

Let us accept the following definition.

Definition 1. The pair (X,Y) of topological spaces X and Y is said to have the *graph intersection property* (shortly $(X,Y) \in GIP$) provided that for every mapping $g : X \to Y$ and every connected subset K of $X \times Y$ such that

$$\pi_2(K) = Y, \tag{2}$$

one has

$$K \cap \Gamma(g) \neq \emptyset. \tag{3}$$

We start with several easy observations.

Observation 4. *If the spaces X and X' are homeomorphic, as well as the spaces Y and Y', then $(X,Y) \in GIP$ if and only if $(X',Y') \in GIP$.*

Observation 5. *If the spaces X and Y have the property that every mapping from X to Y is constant, then $(X,Y) \in GIP$.*

In fact, if $g : X \to Y$ is constant, then $\Gamma(g)$ is of the form $X \times \{y_0\}$ for some $y_0 \in Y$, hence for each subset K of $X \times Y$ condition (2) implies (3).

Observation 6. *If the space Y is not connected, then*

$$(X,Y) \in GIP \quad \textit{for each space} \quad X. \tag{7}$$

So, we omit this trivial case from our further considerations, and we assume that the discussed space Y is connected.

Recall that a topological space is said to be *hereditarily disconnected* provided it does not contain any connected subset of cardinality greater than one, i.e. if all (connected) components of the space are singletons. For these spaces we have the next easy observation.

Observation 8. *If a space X is hereditarily disconnected, then*

$$(X,Y) \in GIP \quad \textit{for each space} \quad Y. \tag{9}$$

Indeed, if a connected subset K of the product $X \times Y$ satisfies (2), then $K = \{x_0\} \times Y$ for some $x_0 \in X$. Thus for every mapping $g : X \to Y$ we have $(x_0, g(x_0)) \in K \cap \Gamma(g)$, whence (3) follows.

For further results related to the graph intersection property in case the space X is hereditarily disconnected see Theorem 27.

A proof of the following observation is contained in the proof of Proposition 12 below.

Observation 10. *For an arbitrary topological space X we have*

$$(X, [0,1]) \in GIP. \tag{11}$$

It is natural to ask what topological spaces Y can be substituted in place of [0,1] to get the conclusion $(X, Y) \in GIP$ for each space X. This problem is symmetric (i.e. the roles of X and Y are reversed) to the one of (9) of Observation 8. To present an answer to it, a concept of the linear ordering is useful.

Linear order.

We say that a relation $\leq$ on a set X is a *linear order* provided that for each points x, y, z of X the following conditions are satisfied:

(a) $x \leq x$;

(b) if $x \leq y$ and $y \leq z$, then $x \leq z$;

(c) if $x \leq y$ and $y \leq x$, then $x = y$;

(d) either $x \leq y$ or $y \leq x$.

An element x_0 of a linearly ordered set X is said to be *maximal (minimal)* if $x_0 \leq x$ (if $x \leq x_0$) implies $x_0 = x$ for every $x \in X$.

The following result generalizes Observation 10.

Proposition 12. *If a connected space Y admits a closed linear order with minimal and maximal elements, then*

$$(X,Y) \in GIP \quad \text{for each space} \quad X. \tag{7}$$

Proof. Denote by $\leq$ the linear order on Y, and by y_0 and y_1 the minimal and maximal elements of Y with respect to $\leq$. Further, let X, g and K have the same meaning as in Definition 1. Suppose on the contrary that $K \cap \Gamma(g) = \emptyset$, and put

$$K_0 = \{(x,y) \in K : y \leq g(x) \quad \text{and} \quad y \neq g(x)\},$$

$$K_1 = \{(x,y) \in K : g(x) \leq y \quad \text{and} \quad y \neq g(x)\}.$$

Obviously K_0 and K_1 are disjoint and open in K; and $K_0 \neq \emptyset \neq K_1$ since there are points x_0 and x_1 of X such that $(x_0, y_0) \in K_0$ and $(x_1, y_1) \in K_1$. Moreover $K_0 \cup K_1 = K$, contrary to connectedness of K.
The proof is complete.

Remark 13. With regard to Proposition 12 let us mention that if a topological space Y with its topology $\mathscr{T}$ admits a closed linear order $\leq$, then this order $\leq$ induces an order topology $\mathscr{T}(\leq)$ which is weaker than $\mathscr{T}$, i.e., $\mathscr{T}(\leq) \subset \mathscr{T}$ (compare Section 7 of [1], p. 43 - 44). Therefore the reader can easily reformulate Proposition 12 in terms of linearly ordered topological spaces.

Theorem 14. *For a connected space Y the following conditions are equivalent:*
(i) there is a closed linear order on Y with minimal and maximal elements;
(ii) $(X,Y) \in GIP$ *for each space* X;
(iii) $(Y,Y) \in GIP$.

Proof. The implication *(i)* $\to$ *(ii)* is just Proposition 12. Obviously *(ii)* implies *(iii)*. To complete the circle of implications, assume *(iii)*. Put $K = Y \times Y \setminus \{(y,y) : y \in Y\}$. Thus $\pi_2(K) = Y$. It is shown in [1], Theorem I, p. 40, that a connected space Y admits a closed linear order if and only if the set K is not connected. Equivalently, if Y does not admit such an order, the set K is connected, and taking the identity on Y as $g : Y \to Y$ we see that $\Gamma(g) = \{(y,y) : y \in Y\}$ is disjoint with K, thus *(iii)* does not hold.

Therefore we can assume that Y admits a closed linear order. Suppose on the contrary that the latter part of the conclusion does not hold. Three cases are possible. 1°. There is a minimal, while there is no maximal element in Y with respect to the considered order. 2°. There is a maximal, while there is no minimal element. 3°. Neither minimal nor maximal element does exist in Y. Assume 1°, and let m stand for the minimal element of $(Y, <)$. For a fixed point $x \in Y \setminus \{m\}$

put $K(x) = \{(x, y) \in Y \times Y : y < x\}$, and observe that by connectedness of Y the sets $K(x)$ are connected. Since for each $x \in Y \setminus \{m\})$ the set $K(x)$ intersects the (connected) set $(Y \setminus \{m\}) \times \{m\}$, the union.

$$K = \cup\{K(x) : x \in Y \setminus \{m\}\} = \{(x, y) \in Y \times Y : y < x\}$$

is also connected. Of course we have $\pi_2(K) = Y$. Taking again the identity on Y as $g : Y \to Y$ we have $K \cap \Gamma(g) = \emptyset$, a contradiction to *(iii)*.

Since the cases 2° and 3° can be treated similarly, we conclude that *(i)* follows. Thus the proof is complete.

If we additionally assume compactness of the considered space Y, then one more condition can be joined to the three above of Theorem 14.

Corollary 15. *Let a space Y be a continuum. Then every of the conditions (i), (ii) and (iii) of Theorem 14 is equivalent to*
(iv) Y is an arc.

Proof. Surely each arc satisfies *(i)*. Conversely, if *(i)* is assumed, then an order preserving homeomorphism between Y and the closed unit interval $[0, 1]$ can be constructed in a standard way.

Component intersection arc property.

Now we shall investigate a structural property which is closely related to existence of a linear order. Using this new property we give partial answers to Problems A, B and C formulated in the Introduction.

We need some auxiliary notation first. Let points a and b of a space Y be given. Then we denote by $C(a; b)$ the component of the set $Y \setminus \{a\}$ containing the point b.

Proposition 16. *If p and q are distinct points of a connected space Y, then $C(p; q) \cup C(q; p) = Y$.*

Proof. It is shown in [4], §46, III, Theorem 6, p.140 that if a space Y is connected, then every finite system $\mathscr{S}$ (containing at least two elements) of disjoint connected subsets of Y contains at least two elements P and Q such that there exists a connected set disjoint from P (respectively from Q) which contains all the elements of $\mathscr{S}$ other than P (respectively other than Q). Let $r \in Y$, and put $\mathscr{S} = \{\{p\}, \{q\}, \{r\}\}$. Then we conclude that either $r \in C(p; q)$ or $r \in C(q; p)$, so the proof is complete.

We say that a space Y has the *component intersection arc property* (shortly $Y \in CIAP$) provided it is connected, and

$$(17) \qquad C(p;q) \cap C(q;p) = pq \setminus \{p,q\} \quad \text{for each arc} \quad pq \subset Y.$$

Remark 18. Note that the inclusion $pq \setminus \{p,q\} \subset C(p;q) \cap C(q;p)$ holds true for each arc $pq \subset Y$.

Proposition 19. *If $Y \in CIAP$, then for each arc $pq \subset Y$ the set $C(p;q)$ is open.*

Proof. Note that $C(p;q) = (C(p;q) \setminus pq) \cup (pq \setminus \{p\})$. So, it suffices to show that the former member of this union is open, while the latter one is contained in the interior of $C(p;q)$. In fact, it follows from (17) that $\operatorname{cl} C(q;p) = C(q;p) \cup \{q\}$, whence by Proposition 16 and (17) again we have $C(p;q) \setminus pq = Y \setminus \operatorname{cl} C(q;p)$, and thus $C(p;q) \setminus pq$ is open. Further, take a point $x \in pq \setminus \{p\}$. Let $q_1 \in px \setminus \{p,x\} \subset pq$. Applying Proposition 16 and (17) once more we conclude as previously that $\operatorname{cl} C(q_1;p) = C(q_1;p) \cup \{q_1\}$, whence

$$\begin{aligned} x \in Y \setminus \operatorname{cl} C(q_1;p) &= C(p;q_1) \setminus \operatorname{cl} C(q_1;p) \\ &= C(p;q) \setminus \operatorname{cl} C(q_1;p) \subset \operatorname{int} C(p;q). \end{aligned}$$

Thus $pq \setminus \{p\} \subset \operatorname{int} C(p;q)$, and the proof is complete.

As an immediate consequence of Proposition 19 and of (17) we get

Corollary 20. *If $Y \in CIAP$, then for each arc $pq \subset Y$ the set $pq \setminus \{p,q\}$ is open.*

Proposition 21. *If $Y \in CIAP$, then*

(22) for each arc $pq \subset Y$ each point $r \in pq \setminus \{p,q\}$ disconnects Y into exactly two open connected subsets, one of which contains p and the other contains q.

Proof. We show that the two subsets are $C(r;p)$ and $C(r;q)$. Indeed, these sets are surely connected; they are open by Proposition 19. Further, since

$$C(r;p) = C(q;p) \setminus rq \quad \text{and} \quad C(r;q) = C(p;q) \setminus pr,$$

we infer from (17) and Proposition 16 that $Y \setminus \{r\} = C(r;p) \cup C(r;q)$. Finally,

$$\begin{aligned} C(r;p) \cap C(r;q) &= (C(q;p) \setminus rq) \cap (C(p;q) \setminus pr) \\ &\subset (C(q;p) \cap C(p;q)) \setminus (pr \cup rq) = \emptyset \end{aligned}$$

by (17). Thus the proof is complete.

Proposition 23. *If a nondegenerate arcwise connected continuum A is contained in a space Y such that $Y \in CIAP$, then A is an arc.*

Proof. By the theorem of R.L. Moore (cf. e.g. [4], §47, IV, Theorem 5, p.177) the continuum A contains at least two points, say p and q, which do not separate A. Thus the sets $A \setminus \{p\}$ and $A \setminus \{q\}$ are connected, and we have $q \in A \setminus \{p\}$ and $p \in A \setminus \{q\}$, whence it follows that

$$A \setminus \{p\} \subset C(p;q) \quad \text{and} \quad A \setminus \{q\} \subset C(q;p),$$

and consequently we have

$$(A \setminus \{p\}) \cap (A \setminus \{q\}) \subset C(p;q) \cap C(q;p). \tag{24}$$

Since A is arcwise connected, there is an arc pq in A. Since $Y \in CIAP$, we see that (24) and (17) imply that $A \setminus \{p,q\} \subset pq \setminus \{p,q\}$.
Taking the closure of both members of this inclusion we get $A \subset pq$, so the conclusion follows.

We shall use Propositions 21 and 23 to show the following result.

Theorem 25. *Let a continuum X be arcwise connected, and let $Y \in CIAP$. Then $(X,Y) \in GIP$.*

Proof. Take a connected subset K of $X \times Y$ such that $\pi_2(K) = Y$, and let $g : X \to Y$ be a mapping. If g is constant, then obviously $K \cap \Gamma(g) \neq \emptyset$. If g is not constant, the image $g(X)$ of X is an arcwise connected nondegenerate subcontinuum of Y. Since $Y \in CIAP$, we conclude by Proposition 23 that $g(X)$ is an arc. Denote it by pq, and suppose on the contrary that $(X,Y) \notin GIP$. Thus there is a connected subset K of $X \times Y$ with $\pi_2(K) = Y$ and $K \cap \Gamma(g) = \emptyset$. Put

$$\begin{array}{rcllll} K(p) & = & \{(x,y) \in K : g(x) \neq p & \text{and} & y \in C(g(x);p)\} \\ & \cup & \{(x,y) \in K : g(x) = p & \text{and} & y \notin \operatorname{cl} C(p;q))\}, \end{array}$$

and

$$\begin{array}{rcllll} K(q) & = & \{(x,y) \in K : g(x) \neq q & \text{and} & y \in C(g(x);q)\} \\ & \cup & \{(x,y) \in K : g(x) = q & \text{and} & y \notin \operatorname{cl} C(q;p)\}. \end{array}$$

It is evident from the above definitions that $K(p) \cup K(q) = K$ and $K(p) \cap K(q) = \emptyset$. Further, it follows from Proposition 21 that $K(p)$ and $K(q)$ are both open in K. Thus K is not connected, a contradiction. The proof is finished.

Remarks 26. We show now that no assumption of Theorem 25 can be omitted.

1) Compactness of the space X is essential, because taking the real line $\mathbb{R}$ as X and as Y simultaneously, we see that X is a noncompact arcwise connected space, $Y \in CIAP$, while $(X,Y) \notin GIP$ as it can be seen from an example of a translation $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x + a$ for $a \neq 0$, whose graph $\Gamma(g)$ is disjoint with the diagonal $K = \{(x,x) : x \in \mathbb{R}\}$. 2) Arcwise connectedness of the continuum X is also a necessary assumption. For example, let X and Y be the pseudo-arc P ([4], §48, X, 3, p. 224). Then X is a continuum which is not arcwise connected, and since P is a hereditarily indecomposable continuum, it contains no arc, whence it follows that $Y = P \in CIAP$ (vacuously). However, it follows from the equivalence *(iii)* $\Leftrightarrow$ *(iv)* of Corollary 15 that $(P, P) \notin GIP$.

3) Also the assumption concerning Y, viz. that $Y \in CIAP$ or, since Y is assumed to be connected by Observation 6 that Y satisfies condition (17) is indispensable in Theorem 25. Namely taking $X = Y$ as the unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ we see that X is an arcwise connected continuum, and that Y is a connected space for which (17) does not hold (and thus $Y \notin CIAP$). The antipodal mapping $g : S^1 \to S^1$ defined by $g((x,y)) = (-x,-y)$ is fixed point free, hence taking K as the diagonal of $S^1 \times S^1$ we see that K is connected, $\pi_2(K) = S^1$ and $K \cap \Gamma(g) = \emptyset$, and therefore $(S^1, S^1) \notin GIP$.

The phenomenon observed in 3) of Remarks 26 has its roots in a much more general result that is connected with lack of component intersection arc property for connected spaces Y. Namely $CIAP$ is so close to existence of a linear ordering that its absence changes the situation drastically. This is described in the next theorem.

Theorem 27. *Let a space X be completely regular, and let a space Y be connected without having the component intersection arc property. Then*

(28) $\quad (X,Y) \in GIP$ *if and only if X is hereditarily disconnected.*

Proof. One implication is shown in Observation 8. To see the other one, suppose on the contrary that there exists a nondegenerate component. A of the space X, and let a and b be distinct points of A. Since the space Y is connected and $Y \notin CIAP$, condition (17) does not hold, i.e., (compare Remark 18) there is an arc pq in Y such that

$$C(p;q) \cap C(q;p) \setminus pq \neq \emptyset.$$

Take a point $z \in C(p;q) \cap C(q;p) \setminus pq$. Since X is completely regular, there is a mapping $g : X \to pq$ such that $g(a) = p$ and $g(b) = q$. By connectedness of

A it follows that the partial mapping $g \mid A : A \to pq$ is a surjection. Put

$$K = (A \times \{z\}) \cup (\{a\} \times C(p; q)) \cup (\{b\} \times C(q; p)).$$

It follows from this definition that K is connected and $K \cap \Gamma(g) = \emptyset$, and Proposition 16 implies that $\pi_2(K) = Y$. Thus $(X, Y) \notin GIP$ and so the proof is complete.

Note that if a connected space Y contains either a simple triod or a simple closed curve, then there is an arc pq in Y for which

$$C(p; q) \cap C(q; p) \setminus pq \neq \emptyset,$$

i.e., condition (17) does not hold. Therefore we have a corollary.

Corollary 29. *Let a space X be completely regular, and let a connected space Y contain either a simple triod or a simple closed curve. Then*

(28) $\quad (X, Y) \in GIP$ *if and only if X is hereditarily disconnected.*

As a consequence of Theorem 27 and of Proposition 23 we have the following result.

Corollary 30. *Let a space X be completely regular. If an arcwise connected continuum Y is not an arc, then*

(28) $\quad (X, Y) \in GIP$ *if and only if X is hereditarily disconnected.*

Recall that a space Y is said to be *indecomposable* provided it is connected and it is not the union of two closed connected sets different from Y. It is known that no closed connected subset of an indecomposable space is its separator ([4], §48, V, Theorem 1, p. 207) and that a connected space is indecomposable if and only if every closed connected proper subset of it is nowhere dense ([4], §48, V, Theorem 2, p. 207). Therefore if an indecomposable space Y contains an arc pq, then $Y \setminus pq$ is a connected dense subset of Y, and thus condition (17) is not satisfied by virtue of Corollary 20. So we conclude the next two corollaries.

Corollary 31. *If an indecomposable space Y contains an arc, then $Y \notin CIAP$.*

Corollary 32. *Let a space X be completely regular, and let an indecomposable space Y contain an arc. Then*

(28) $\quad (X, Y) \in GIP$ *if and only if X is hereditarily disconnected.*

Remark 33. Indecomposable continua are known each proper subcontinuum of which is an arc. Such are, e.g. the simplest indecomposable continuum ([4]), §48, V, Example 1, Fig.4, p.204 and 205) as well as an indecomposable solenoid (i.e. the inverse limit space of an inverse sequence of circles with open bonding mappings, infinitely many of whose are not homeomorphisms).These continua do not have $CIAP$ according to Corollary 31. On the other hand indecomposable continua are known that contain no arc. Such is, e.g., the pseudo-arc P, and we already know that $P \in CIAP$ and $(P, P) \notin GIP$ (see 2 of Remarks 26). Thus it seems to be natural to modify Problem A asking for what continua X and connected spaces Y we have $(X, Y) \in GIP$? The following proposition gives a partial answer to this question.

Proposition 34. *If there exists a noncostant mapping $g : X \to Y$ from a continuum X into an indecomposable space Y, then $(X, Y) \notin GIP$.*

Proof. Since g is noncostant, there are two distinct points a and b of X such that $g(a) \neq g(b)$. Let V_a and V_b be disjoint open subsets of Y containing $g(a)$ and $g(b)$ respectively. Let X_0 be the component of $X \setminus f^{-1}(V_b)$ containing the point a. Since X_0 meets the boundary of $f^{-1}(V_b)$ ([4], §47, III, p.172), it is a nondegenerate subcontinuum of X such that $g(X_0)$ is a proper nondegenerate subcontinuum of Y. Let $c \in X_0 \setminus \{a\}$ such that $g(c) \neq g(a)$. Put

$$K = (X_0 \times (Y \setminus g(X_0))) \cup (\{a\} \times (Y \setminus \{g(a)\}) \cup (\{c\} \times (Y \setminus \{g(c)\})$$

and observe that $\pi_2(K) = Y$ and that $K \cap \Gamma(g) = \emptyset$. Further, since no closed connected subset of an indecomposable space is its separator ([4], §48, V, Theorem 1, p. 207), the differences $Y \setminus g(X_0)$, $Y \setminus \{g(a)\}$ and $Y \setminus \{g(c)\}$ are connected, whence it follows that K is connected. Thus the proof is complete.

Relations to span.

The graph intersection property, when applied to metric spaces, is closely related to another known property, namely to having the surjective semispan zero. The concept of the surjective semispan has been introduced by A. Lelek in [5], p. 35.

Let (Y, d) be a connected metric space. The *surjective semispan* $\sigma_0^*(Y)$ of Y is defined as the least upper bound of the set of all nonnegative real numbers α such that there exists a connected subset K_α of $Y \times Y$ with $\pi_2(K_\alpha) = Y$ satisfyng the inequality $\alpha \leq d(y_1, y_2)$ for all $y_1, y_2 \in K_\alpha$. Various results concerning this concept and similarly defined notions of semispan and span, obtained by several authors (see e.g. [3], references therein and related papers) show that among many properties of metric spaces expressed in terms of surjective semispan the most important is if this number is or is not equal to zero. In this direction we have the following two observations.

Observation 35. *For each connected metric space Y condition*

$$(Y,Y)\in GIP \tag{36}$$

implies

$$\sigma_0^*(Y)=0. \tag{37}$$

Proof. Assume $\sigma_0^*(Y)=\alpha>0$ and note by the definition of $\sigma_0^*(Y)$ that there is a connected subset K_α of $Y\times Y$ with $\pi_2(K_\alpha)=Y$ and $d(y_1,y_2)\geq\alpha$ for all $y_1,y_2\in K_\alpha$. Taking $g=id:Y\to Y$ we see that $\Gamma(g)$ is the diagonal $\{(y,y):y\in Y\}$ of $Y\times Y$, whence $K_\alpha\cap\Gamma(g)=\emptyset$, and so $(Y,Y)\notin GIP$.

Observation 38. *Let a connected metric space Y be given. If there are a space X, a mapping $g:X\to Y$ and a connected subset K of $X\times Y$ with*

$$\pi_2(K)=Y \quad \textit{and} \quad K\cap\Gamma(g)=\emptyset$$

such that K is compact, then Y is a continuum and $\sigma_0^*(Y)>0$.

Proof. The former conclusion follows from $\pi_2(K)=Y$ by compactness of K and continuity of π_2. To see the latter one, put

$$C=\{(g(x),y)\in Y\times Y:(x,y)\in K\}, \quad D=\{(y,y):y\in Y\},$$

and observe that $C=(g\pi_1\times\pi_2)(K)$, whence it follows that C is compact, $\pi_2(C)=\pi_2(K)=Y$, and $C\cap D=\emptyset$ because of $K\cap\Gamma(g)=\emptyset$. Therefore there is an $\alpha>0$ such that $d(c_1,c_2)\geq\alpha$ for all $(c_1,c_2)\in C$. Thus $\sigma_0^*(Y)\geq\alpha>0$ as needed, which finishes the proof.

Remark 39. It follows from Observation 38 that the class of connected metric spaces Y such that

$$(X,Y)\in GIP \quad \text{for each space } X \tag{7}$$

is smaller than the class of spaces Y with $\sigma_0^*(Y)=0$ just because we do not require in the definition of the graph intersection property that the set $K\subset X\times Y$ is compact.

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