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$\mathscr{L}^{2,\lambda+2}$-REGULARITY OF THE SOLUTIONS TO NON HOMOGENEOUS PARABOLIC SYSTEMS IN DIVERGENCE FORM

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In this paper we study the $\mathscr{L}^{2,\lambda+2}$-regularity $(0 < \lambda < n+2)$ of the solution v to the Cauchy-Dirichlet problem:

$$\begin{cases} -\sum\limits_{i,j=1}^{n} D_i(A_{ij}(X)D_j v) + \dfrac{\partial v}{\partial t} = -\sum\limits_{i=1}^{n} D_i f^i + f^0 \text{ in } Q \\ v = u \text{ on the parabolic boundary } \Gamma_Q \text{ of } Q \end{cases}$$

under the assumptions

$$f_i \in L^{2,\lambda}(Q, \mathbb{R}^N),\ 0 < \lambda < n+2, \quad i = 0, 1, \dots, n,$$

$$u \in \mathscr{L}^{2,\lambda+2}(Q, \mathbb{R}^N) \cap H_{-T}^{*0,\frac{1}{2},(\lambda)}(Q, \mathbb{R}^N)$$

$$D_i u \in L^{2,\lambda}(Q, \mathbb{R}^N) \quad i = 1, 2, \dots, n.$$

For such a solution the following estimate holds:

$$\|v\|_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)} \le c\Big\{ \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + [u]^2_{H_{-T}^{*0,\frac{1}{2},(\lambda)}(Q,\mathbb{R}^N)} + $$

$$+ \|u\|_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n} \|f^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \Big\}.$$

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1. Introduction.

Let Ω a bounded open set of $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$ (for istance of class C^2), T a real positive number, Q the cylinder $\Omega \times (-T, 0)$ and $X = (x, t)$ a point of Q. Let N be integer > 1, $(\cdot|\cdot)_k$ and $\|\cdot\|_k$ are the scalar product and the norm in R^k.We shall omit the index whenever no ambiguity can arise.

Let us consider in Q the following Cauchy-Dirichlet problem (with non homogeneus data)

$$(1.1) \qquad -\sum_{i,j=1}^{n} D_i(A_{ij}(X)D_j v) + \frac{\partial v}{\partial t} = -\sum_{i=1}^{n} D_i f^i + f^0 \quad \text{in } Q$$

$$(1.2) \qquad v = u \quad \text{on} \quad \Gamma_Q \ (^1)$$

where

$$(1.3) \qquad u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap H^{*\frac{1}{2}}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N)), \ (^2)$$

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(1) $\Gamma_Q = \Big[\Omega \times \{-T\}\Big] \cup \Big[\partial\Omega \times (-T, 0)\Big]$

(2) $H^{\frac{1}{2}}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ $\Big[H^{\frac{1}{2}}_0(-T, 0, L^2(\Omega, \mathbb{R}^N))\Big]$ is the Banach space of those functions $u \in H^{\frac{1}{2}}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ such that

$$\int_{-T}^{0} dt \int_\Omega \frac{\|u(X)\|^2}{t+T}\, dx < +\infty \quad \left[\int_{-T}^{0} dt \int_\Omega \frac{\|u(X)\|^2}{t}\, dx < +\infty\right]$$

with norm

$$\|u\|_{H^{\frac{1}{2}}_{-T}(-T,0,L^2(\Omega,\mathbb{R}^N))} = \left\{\|u\|^2_{H^{\frac{1}{2}}(-T,0,L^2(\Omega,\mathbb{R}^N))} + \int_{-T}^{0} dt \int_\Omega \frac{\|u(X)\|^2}{t+T}\, dx\right\}^{1/2}$$

$$\left[\|u\|_{H^{\frac{1}{2}}_0(-T,0,L^2(\Omega,\mathbb{R}^N))} = \left\{\|u\|^2_{H^{\frac{1}{2}}(-T,0,L^2(\Omega,\mathbb{R}^N))} + \int_{-T}^{0} dt \int_\Omega \frac{\|u(X)\|^2}{t}\, dx\right\}^{1/2}\right].$$

(1.4) $$f^i \in L^2(Q, \mathbb{R}^N) \qquad i = 0, 1, \dots, n$$

and $A_{ij}(X)$, $i, j = 1, 2, \dots, n$ are $N \times N$ matrices satisfying the conditions

(1.5) $$A_{ij} \in C^0(\overline{Q}, \mathbb{R}^N) \qquad i, j = 1, 2, \dots, n$$

(1.6) there exists $\nu > 0$ such that:

$$\sum_{i,j=1}^{n} (A_{ij}(X)\, p^j \,|\, p^i) \geq \nu \sum_{i=1}^{n} \|p^i\|^2, \quad \forall (X, p) \in \overline{Q} \times \mathbb{R}^{nN}\ (^3).$$

Kaplan [4] established the existence and the uniqueness of the solution v to the problem (1.1), (1.2) in the following theorem:

Theorem 1.1. *If assumptions (1.3) - (1.6) hold, the problem (1.1), (1.2) admits a unique solution v in $L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap H^{*\frac{1}{2}}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$, in the sense that*

$$w = u - v \in L^2(-T, 0, H^1_0(\Omega, \mathbb{R}^N)) \cap H^{\frac{1}{2}}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$$

and

(1.7) $$\int_Q \sum_{i,j=1}^{n} (A_{ij} D_j w | D_i \varphi)\, dX + \widetilde{B}(w, \varphi) = \widetilde{B} < u, \varphi > +$$

$$+ \int_Q \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} A_{ij} D_j u - f^i \,|\, D_i \varphi \Big)\, dX - \int_Q (f^0 | \varphi)\, dX,$$

$H^{*\frac{1}{2}}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ is a Banach space of those $u \in H^{\frac{1}{2}}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ for which there exist $h \in L^2(\Omega, \mathbb{R}^N)$ such that $\displaystyle\int_{-T}^{0} dt \int_\Omega \frac{\|u(x,t) - h(x)\|^2}{t+T}\, dx < +\infty$ with norm

$$\|u\|_{H^{*\frac{1}{2}}_{-T}(-T,0,L^2(\Omega,\mathbb{R}^N))} = \Big\{ \|u\|^2_{H^{\frac{1}{2}}(-T,0,L^2(\Omega,\mathbb{R}^N))} +$$

$$+ \int_{-T}^{0} dt \int_\Omega \frac{\|u(X) - u(x,-T)\|^2}{t+T}\, dx \Big\}^{\frac{1}{2}}$$

where $u(x, -T) = h(x)$ (Cfr [6]).

(^{3}) $p = (p^1 | \dots | p^n)$, $p^i \in \mathbb{R}^N$ denotes a vector of $\mathbb{R}^{nN}$.

$\forall \varphi \in L^2(-T, 0, H_0^1(\Omega, \mathbb{R}^N)) \cap H_0^{\frac{1}{2}}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ (4).

For such a solution the following estimate holds

$$\int_Q \sum_{i=1}^n \|D_i v\|^2\, dX + \|v\|^2_{H^{*\frac{1}{2}}_{-T}(-T,0,L^2(\Omega,\mathbb{R}^N))} \leq \tag{1.8}$$

$$\leq C \left\{ \|u\|^2_{H^{*\frac{1}{2}}(-T,0,L^2(\Omega,\mathbb{R}^N))} + \int_Q \sum_{i=1}^n \|D_i u\|^2\, dX + \right.$$

$$\left. + \int_Q \sum_{i=0}^n \|f^i\|^2\, dX \right\}$$

M. Marino - A. Maugeri [6] established the $L^{2,\lambda}$-regularity $(0 < \lambda < n+2)$ of the spatial derivatives of the solution v to the problem (1.1), (1.2).

In this paper we study the $\mathscr{L}^{2,\lambda+2}$-regularity $(0 < \lambda < n+2)$, from which the Hölder continuity of v follows for $\lambda > n$, of the solution v to the same problem. To achieve this result we need before demostrating some preliminary lemmas that have interest in themselves and that we list in sect. n. 2.

2. Preliminary lemmas.

By $I(X^0,\sigma), Q(X^0,\sigma), Q^+(X^0,\sigma), M(X^0,\sigma), M^+(X^0,\sigma)$ we denote the cylinders of $\mathbb{R}^n_x \times \mathbb{R}_t$:

$$I(X^0,\sigma) = B(x^0,\sigma) \times (t^0 - \sigma^2, t^0 + \sigma^2)$$

$$Q(X^0,\sigma) = B(x^0,\sigma) \times (t^0 - \sigma^2, t^0)$$

(4) $\widetilde{B} < f, g >$ is the sesquilinear continuous form that extends $\int_Q \left(\frac{\partial f}{\partial t} \,|\, g\right) dX$ to $H^{\frac{1}{2}}_{-T} \times H_0^{\frac{1}{2}}$ (see, for more details, [4] e [5]).

Let us recall that for every $\varphi \in H^1(-T^0, 0, L^2(\Omega, \mathbb{R}^N))$, $\varphi(x, 0)$ in Ω it results:

$$\widetilde{B} < w, \varphi >= -\int_Q \left(w \,|\, \frac{\partial \varphi}{\partial t}\right) dX$$

$$\widetilde{B} < u, \varphi >= -\int_Q \left(u \,|\, \frac{\partial \varphi}{\partial t}\right) dX - \int_\Omega (u(x,-T) \,|\, \varphi(x,-T))\, dx\,.$$

$$Q^+(X^0,\sigma) = B^+(x^0,\sigma) \times (t^0-\sigma^2,t^0)$$
$$M(X^0,\sigma) = B(x^0,\sigma) \times (t^0,t^0+\sigma^2)$$
$$M^+(X^0,\sigma) = B^+(x^0,\sigma) \times (t^0,t^0+\sigma^2)$$

where $X^0 = (x^0,t^0) \in \mathbb{R}^n_x \times \mathbb{R}_t, \sigma > 0, B(x^0,\sigma) = \{x \in \mathbb{R}^n : \|x-x^0\| < \sigma\}$; $B^+(x^0,\sigma) = B(x^0,\sigma) \cap \{x \in \mathbb{R}^n : x_n > x^0_n\}$.

We list in this section a few lemmas concerning the solutions to the system

$$(2.1) \qquad -\sum_{i,j=1}^{n} D_i(A^0_{ij}D_jw) + \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_iF^i + F^0$$

with $A^0_{i,j}$, $i,j = 1,2,\dots,n$, $N \times N$ constant matrices satisfying the strong ellipticity condition

(2.2) there exists $\nu > 0$ such that:

$$\sum_{i,j=1}^{n} (A^0_{ij}p^j \,|\, p^i) \geq \nu \sum_{i=1}^{n} \|p^i\|^2, \qquad \forall p \in \mathbb{R}^{nN}.$$

Lemma 2.1. *If $u \in L^2(-T,0,H^1(B(x^0,\sigma),\mathbb{R}^N)) \cap H^{*\frac{1}{2}}_{-T}(-T,0,L^2(B(x^0,\sigma),\mathbb{R}^N))$, and $F^i \in L^2(B(x^0,\sigma) \times (-T,0),\mathbb{R}^N)$, $i = 0,1,\dots,n$ if $w \in L^2(-T,0,H^1(B(x^0,\sigma),\mathbb{R}^N)) \cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(B(x^0,\sigma),\mathbb{R}^N))$ is solution in $Q(X^0,\sigma) \subset B(x^0,\sigma) \times (-T,0)$, $X^0 = (x^0,t^0)$, to the system (2.1), in the sense that*

$$(2.3) \quad \int_{Q(X^0,\sigma)} \sum_{i,j=1}^{n} (A^0_{ij}D_jw|D_i\varphi)\,dX + \widetilde{B} < w,\varphi >= \widetilde{B} < u,\varphi > +$$
$$+ \int_{Q(X^0,\sigma)} \left\{ \sum_{i=1}^{n} (F^i \,|\, D_i\varphi) + (F^0 \,|\, \varphi) \right\} dX$$

$\forall \varphi \in L^2(t^0-\sigma^2,t^0,H^1_0(B(x^0,\sigma),\mathbb{R}^N)) \cap \left[H^{\frac{1}{2}}_{t^0-\sigma^2} \cap H^{\frac{1}{2}}_{t^0}\right](t^0-\sigma^2,t^0,L^2(B(x^0,\sigma),\mathbb{R}^N))$, then there exists a positive constant $c = c(\nu)$ such that $\forall \rho \in (0,\sigma)$ it results

$$(2.4) \qquad \int_{Q(X^0,\rho)} \|w - w_{Q(X^0,\rho)}\|^2\,dX \leq$$

$$\leq c(\nu)\left\{\left(\frac{\rho}{\sigma}\right)^{n+4}\int_{Q(X^0,\sigma)}\|w-w_{Q(X^0,\sigma)}\|^2\,dX+\right.$$

$$+\int_{Q(X^0,\sigma)}\|u-u_{Q(X^0,\sigma)}\|^2\,dX+\rho^2\left[\int_{Q(X^0,\sigma)}\sum_{i=1}^{n}\|D_iu\|^2\,dX+\right.$$

$$\left.\left.+\int_{Q(X^0,\sigma)}\sum_{i=0}^{n}\|F^i\|^2\,dX\right]\right\}\ (^5)$$

Proof. This result is obtained with the same technique of the lemma 2.1 of [6], using a well known estimate (see [2] Lemma 2. II) that assures $\forall\rho\in(0,\sigma)$

$$\int_{Q(X^0,\rho)}\|v-v_{Q(X^0,\rho)}\|^2\,dX\leq$$

$$\leq c(\nu)\left\{\left(\frac{\rho}{\sigma}\right)^{n+4}\int_{Q(X^0,\sigma)}\|v-v_{Q(X^0,\sigma)}\|^2\,dX+\right.$$

$$\left.+\rho^2\int_{Q(X^0,\sigma)}\sum_{i=0}^{n}\|f^i\|^2\,dX\right\}$$

Lemma 2.2. *If* $u\in L^2(-T,0,H^1(B(x^0,\sigma),\mathbb{R}^N))\cap \overset{*}{H}{}^{\frac{1}{2}}_{-T}(-T,0,L^2(B(x^0,\sigma),\mathbb{R}^N))$, *and* $F^i\in L^2(B(x^0,\sigma)\times(-T,0),\mathbb{R}^N)$, $i=0,1,\cdots,n$, *if* $w\in L^2(-T,0,H^1(B(x^0,\sigma),\mathbb{R}^N))\cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(B(x^0,\sigma),\mathbb{R}^N))$ *is a solution in* $M(X^0,\sigma)\subset B(x^0,\sigma)\times(-T,0)$, $X^0=(x^0,t^0)$, $t^0=-T$ *to the system (2.1)* (^{6}), *then there exists a positive constant* $c=c(\nu)$ *such that* $\forall\rho\in(0,\sigma)$

$$\int_{M(X^0,\rho)}\|w-w_{M(X^0,\rho)}\|^2\,dX\leq \tag{2.6}$$

$$\leq c(\nu)\left\{\left(\frac{\rho}{\sigma}\right)^{n+4}\int_{M(X^0,\sigma)}\|w-w_{M(X^0,\sigma)}\|^2\,dX+\right.$$

(^{5}) If $u(x,t)\in L^1(E)$, E is an open non–empty set of Q, then

$$u_E=\frac{1}{misE}\int_E u(x,t)\,dx\,dt$$

(^{6}) In the sense considered in the statement of Lemma 2.1.

$$+\sigma^2\Big[\|u\|^2_{H^{*\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B(x^0,\sigma),\mathbb{R}^N))}+$$

$$+\int_{M(X^0,\sigma)}\sum_{i=1}^n\|D_iu\|^2\,dX+\int_{M(X^0,\sigma)}\sum_{i=0}^n\|F^i\|^2\,dX\Big]\Big\}.$$

Proof. In $M(X^0,\sigma)$ let us decompose w in the sum w_1+w_2 where $w_2\in L^2(t^0,t^0+\sigma^2,H^1(B(x^0,\sigma),\mathbb{R}^N))\cap H^{\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B(x^0,\sigma),\mathbb{R}^N))$ is solution (6) in $M(X^0,\sigma)$ to the system

$$-\sum_{i=1}^n D_i\Big(A^0_{ij}D_jw_2\Big)+\frac{\partial w_2}{\partial t}=0 \tag{2.7}$$

and $w_1=u-v_1$ with $v_1\in L^2(t^0,t^0+\sigma^2;H^1(B(x^0,\sigma),\mathbb{R}^N))\cap H^{*\frac{1}{2}}_{t^0}(t^0,t^0+$ $+\sigma^2,L^2(B(x^0,\sigma),\mathbb{R}^N))$ solution (7) to the C. D. problem

$$-\sum_{i=1}^n D_i\Big(A^0_{ij}D_jv_1\Big)+\frac{\partial v_1}{\partial t}= \tag{2.8}$$

$$=-\sum_{i=1}^n D_i\Big(\sum_{j=1}^n A^0_{ij}D_ju-F^i\Big)-F^0\quad\text{in}\quad M(X^0,\sigma)$$

$$v_1=u\qquad\text{on}\qquad \Gamma_{M(X^0,\sigma)}\ ^{(8)} \tag{2.9}$$

Fixed $\rho\in(0,\sigma)$ and set $X^*=(x^0,t^0+\rho^2)$, let us consider the extention $w_2(x,t)$ of the function w_2 to the cylinder $B(x^0,\sigma)\times(t^0+\rho^2-\sigma^2,t^0+\sigma^2)$ obtained setting:

$$W_2(x,t)=\begin{cases} w_2(x,t) & \text{in } M(X^0,\sigma)\\ w_2(x,2t^0-t) & \text{in } B(x^0,\sigma)\times(t^0+\rho^2-\sigma^2,t^0).\end{cases} \tag{2.10}$$

$W_2(x,t)$ belongs to $L^2(t^0+\rho^2-\sigma^2,t^0+\sigma^2,H^1(B(x^0,\sigma),\mathbb{R}^N))$ and is a weak solution (in the usual sense) in $Q(X^*,\sigma)$ to the system

$$-\sum_{i,j=1}^n D_i\Big(A^0_{ij}\,D_jW_2\Big)+\frac{\partial w_2}{\partial t}=0\,. \tag{2.11}$$

(7) In the sense indicated in section 1.

(8) Then $w_1=u-v_1\in L^2(t^0,t^0+\sigma^2,H^1_0(B(x^0,\sigma),\mathbb{R}^n))\cap H^{\frac{1}{2}}_{t^0}(t^0,t^0+$ $\sigma^2,L^2(B(x^0,\sigma),\mathbb{R}^N))$.

Then by means of the estimate (2.5) of [2], we get

$$\int_{Q(X^*,\rho)} \|W_2 - (W_2)_{Q(X^*,\rho)}\|^2 \, dX \leq \tag{2.12}$$

$$\leq c(\nu) \left(\frac{\rho}{\sigma}\right)^{n+4} \int_{Q(X^*,\sigma)} \|W_2 - (W_2)_{Q(X^*,\sigma)}\|^2 \, dX$$

from which, in virtue of (2.10) and being $Q(X^*, \rho) = M(X^0, \rho)$, it follows

$$\int_{M(X^0,\rho)} \|W_2 - (W_2)_{M(X^0,\rho)}\|^2 \, dX \leq \tag{2.13}$$

$$\leq c(\nu)\left(\frac{\rho}{\sigma}\right)^{n+4} \int_{Q(X^*,\sigma)} \|W_2 - (W_2)_{Q(X^*,\sigma)}\|^2 \, dX \leq$$

$$\leq c(\nu)\left(\frac{\rho}{\sigma}\right)^{n+4} \int_{Q(X^*,\sigma)} \|W_2 - (W_2)_{M(X^*,\sigma)}\|^2 \, dX \leq$$

$$\leq c(\nu)\left(\frac{\rho}{\sigma}\right)^{n+4} \left\{ \int_{B(x^0,\sigma)\times(t^0+\rho^2-\sigma^2,t^0)} \|W_2 - (W_2)_{M(X^0,\sigma)}\|^2 \, dX + \right.$$

$$\left. + \int_{B(x^0,\sigma)\times(t^0,t^0+\rho^2)} \|W_2 - (W_2)_{M(X^0,\sigma)}\|^2 \, dX \right\} \leq c(\nu) \left(\frac{\rho}{\sigma}\right)^{n+4} \cdot$$

$$\cdot \left\{ \int_{t^0+\rho^2-\sigma^2}^{t^0} dt \int_{B(x^0,\sigma)} \|w_2(x, 2t^0 - t) - (w_2)_{M(X^0,\sigma)}\|^2 \, dx + \right.$$

$$\left. + \int_{t^0}^{t^0+\rho^2} dt \int_{B(x^0,\sigma)} \|w_2(x,t) - (w_2)_{M(X^0,\sigma)}\|^2 \, dx \right\} =$$

$$= c(\nu)\left(\frac{\rho}{\sigma}\right)^{n+4} \left\{ \int_{t^0}^{t^0+\rho^2+\sigma^2} dt \int_{B(x^0,\sigma)} \|w_2(x,t) - (w_2)_{M(X^0,\sigma)}\|^2 \, dx + \right.$$

$$+ \int_{t^0}^{t^0+\rho^2} dt \int_{B(x^0,\sigma)} \|w_2(x,t) - (w_2)_{M(X^0,\sigma)}\|^2 \, dx \leq$$

$$\leq c(\nu)\left(\frac{\rho}{\sigma}\right)^{n+4} \int_{M(X^0,\sigma)} \|w_2 - (w_2)_{M(X^0,\sigma)}\|^2 \, dX.$$

Then it results:

$$(2.14)\qquad \int_{M(X^0,\rho)} \|w_2 - (w_2)_{M(X^0,\rho)}\|^2\, dX \le$$

$$\le c(\nu)\Big(\frac{\rho}{\sigma}\Big)^{n+4} \int_{M(X^0,\sigma)} \|w_2 - (w_2)_{M(X^0,\sigma)}\|^2\, dX\,, \quad \forall \rho \in (0,\sigma).$$

On the other hand, in virtue of Teorema 1.1, v_1 satisfies the estimate

$$\int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i v_1\|^2\, dX \le c\Bigg\{ \|u\|^2_{H^{*\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B(x^0,\sigma),\mathbb{R}^N))} +$$

$$+ \int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2\, dX + \int_{M(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2\, dX \Bigg\}$$

and hence, being $w_1 = u - v_1$

$$\int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w_1\|^2\, dX \le c\Bigg\{ \|u\|^2_{H^{*\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B(x^0,\sigma)\mathbb{R}^N))} +$$

$$+ \int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2\, dX + \int_{M(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2\, dX \Bigg\}.$$

Then being $w_1 \in L^2(t^0, t^0 + \sigma^2, H^1_0(B(x^0,\sigma), \mathbb{R}^n))$, it follows

$$(2.15)\qquad \int_{M(X^0,\sigma)} \|w_1\|^2\, dX \le$$

$$\le c\,\sigma^2\Bigg\{ \|u\|^2_{H^{*\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B(x^0,\sigma)\mathbb{R}^N))} +$$

$$+ \int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2\, dX + \int_{M(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2\, dX \Bigg\}.$$

Since $w = w_1 + w_2$, from (2.14) and (2.15), the assert (2.6) follows.

Using the Lemma 2.1 and the same technique of the lemma 2.3 of [6], we obtain:

Lemma 2.3. *If* $u \in L^2(-T,0,H^1(B^+(x^0,\sigma),\mathbb{R}^N)) \cap H^{*\frac{1}{2}}_{-T}(-T,0,L^2(B^+(x^0,\sigma),\mathbb{R}^N))$, *and* $F^i \in L^2(B^+(x^0,\sigma)\times(-T,0),\mathbb{R}^N)$, $i=0,1,\cdots,n$, *if* $w \in L^2(-T,0,\widetilde{H}^1_0(B^+(x^0,\sigma),\mathbb{R}^N)) \cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(B^+(x^0,\sigma),\mathbb{R}^N))$ (9) *is solution in* $Q^+(X^0,\sigma) \subset B^+(x^0,\sigma)\times(-T,0), X^0=(x^0,t^0)$, *to the system (2.1),* (6) *then there exists a positive constant* $c=c(\nu)$ *such that* $\forall \rho \in (0,\sigma)$

$$\int_{Q^+(X^0,\rho)} \|w - w_{Q^+(X^0,\rho)}\|^2\,dX \leq \tag{2.16}$$
$$\leq c(\nu)\left\{\left(\frac{\rho}{\sigma}\right)^{n+4}\int_{Q^+(X^0,\sigma)}\|w-w_{Q^+(X^0,\sigma)}\|^2\,dX+\right.$$
$$+\int_{Q^+(X^0,\sigma)}\|u-u_{Q^+(X^0,\sigma)}\|^2\,dX+\sigma^2\left[\int_{Q^+(X^0,\sigma)}\sum_{i=1}^n\|D_iu\|^2\,dX+\right.$$
$$\left.\left.+\int_{Q^+(X^0,\sigma)}\sum_{i=0}^n\|F^i\|^2\,dX\right]\right\}.$$

Analgously using the Lemma 2.2 and following the technique of the lemma 2.4 of [6], we obtain

Lemma 2.4. *If* $u \in L^2(-T,0,H^1(B^+(x^0,\sigma),\mathbb{R}^N)) \cap H^{*\frac{1}{2}}_{-T}(-T,0,L^2(B^+(x^0,\sigma),\mathbb{R}^N))$, *and* $F^i \in L^2(B^+(x^0,\sigma)\times(-T,0),\mathbb{R}^N)$, $i=0,1,\cdots,n$, *if* $w \in L^2(-T,0,\widetilde{H}^1_0(B^+(x^0,\sigma),\mathbb{R}^N)) \cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(B^+(x^0,\sigma),\mathbb{R}^N))$ *is solution in* $M^+(X^0,\sigma) \subset B^+(x^0,\sigma)\times(-T,0), X^0=(x^0,t^0)$, $t^0=-T$, *to the system (2.1)* (6), *then there exists a positive constant* $c=c(\nu)$ *such that* $\forall \rho \in (0,\sigma)$

$$\int_{M^+(X^0,\rho)} \|w - w_{M^+(X^0,\rho)}\|^2\,dX \leq \tag{2.17}$$
$$\leq c(\nu)\left\{\left(\frac{\rho}{\sigma}\right)^{n+4}\int_{M^+(X^0,\sigma)}\|w-w_{M^+(X^0,\sigma)}\|^2\,dX+\right.$$
$$+\sigma^2\left[\int_{M^+(X^0,\sigma)}\sum_{i=1}^n\|D_iu\|^2\,dX+\right.$$

(9) $\widetilde{H}^1_0(B^+(x^0,\sigma),\mathbb{R}^N)$ is the class of the functions $u \in H^1(B^+(x^0,\sigma),\mathbb{R}^N)$ whith trace on the iperplane $x_n = x^0_n$ zero.

$$+ \|u\|^2_{H^{*\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B^+(x^0,\sigma),\mathbb{R}^N))} + \int_{M^+(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 \, dX \Big] \Big\}.$$

3. $\mathscr{L}^{2,\lambda}$-Regularity results.

Let $L^{2,\lambda}(Q,\mathbb{R}^N)$, $0 < \lambda < n+2$, be the usual Morrey space related to parabolic metric

$$d(X,Y) = \max\left\{ \|x-y\|, \quad |t-\tau|^{\frac{1}{2}},\ X=(x,t),\ Y=(y-t)\right\}$$

and let $\mathscr{L}^{2,\lambda}(Q,\mathbb{R}^N)$, $\lambda > n+2$, be the Campanato spaces.

Let us denote by $H^{*0,\frac{1}{2},(\lambda)}_{-T}(Q,\mathbb{R}^N)$, (cfr [6]), $0 < \lambda < n+2$, the space of those functions $u(X) \in H^{*\frac{1}{2}}_{-T}(-T,0,L^2(\Omega,\mathbb{R}^N))$ for which

$$[u]^2_{H^{*0,\frac{1}{2},(\lambda)}_{-T}(Q,\mathbb{R}^N)} =$$

$$= \sup_{\substack{x^0\in\Omega\\ \sigma>0}} \frac{1}{\sigma^\lambda}\left(\int_{-T}^{m_\sigma} dt \int_{-T}^{m_\sigma} d\xi \int \frac{\|u(X)-u(x,\xi)\|^2}{|t-\xi|^2}\, dx + \right.$$

$$\left. + \int_{-T}^{m_\sigma} dt \int_{B(x^0,\sigma)\cap\Omega} \frac{\|u(X)-u(x,-T)\|^2}{t+T}\, dx\right) < +\infty$$

where $m_\sigma = \min\,(0, -T+\sigma^2)$.

Let $w \in L^2(-T,0,H^1(\Omega,\mathbb{R}^N)) \cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(\Omega,\mathbb{R}^N))$ a solution in Q to the system

$$(3.1)\qquad -\sum_{i,j=1}^{n} D_i\big(A_{ij}(X)D_jw\big) + \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_iF^i + F^0$$

where, for a certain λ, $0 < \lambda < n+2$

$$(3.2)\qquad F^i \in L^{2,\lambda}(Q,\mathbb{R}^N) \qquad i = 0,1,....,n$$

while u and $A_{ij}(X)$, $i,j = 1,2,....,n$, verify the conditions (1.3), (1.5) and (1.6).

The following theorem holds:

Theorem 3.1. *If* $w \in L^2(-T,0,H^1(\Omega,\mathbb{R}^N)) \cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(\Omega,\mathbb{R}^N))$ *is solution in* Q *to the system (3.1), if the conditions (1.3) (1.5) (1.6) (3.2) are satisfied and if*

$$u \in \mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N) \cap H^{*0,\frac{1}{2},(\lambda)}_{-T}(Q,\mathbb{R}^N) \tag{3.3}$$

$$D_i u \in L^{2,\lambda}(Q,\mathbb{R}^N) \qquad i = 1,2,...,n \tag{3.4}$$

$$D_i w \in L^{2,\lambda}(Q,\mathbb{R}^N) \tag{3.5}$$

with $0 < \lambda < n+2$, *then, for every cilinder* $Q_0 = \Omega_0 \times (-T,0)$ *with* $\Omega_0 \subset\subset \Omega$, *it results*

$$w \in \mathscr{L}^{2,\lambda+2}(Q_0,\mathbb{R}^N) \tag{3.6}$$

and we have

$$\|w\|_{\mathscr{L}^{2,\lambda+2}(Q_0,\mathbb{R}^N)} \le c(\nu,n,\lambda)\left\{\frac{1}{\sigma_0^{\lambda+2}}\|w\|^2_{L^2(Q,\mathbb{R}^N)} + \right. \tag{3.7}$$

$$+\sum_{i=1}^n \|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)} + \sum_{i=1}^n \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} +$$

$$\left. + [u]^2_{H^{*0,\frac{1}{2},(\lambda)}_{-T}(Q,\mathbb{R}^N)} + \sum_{i=0}^n \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}\right\}$$

where σ_0 *is the positive number defined by (3.8).*

Proof. Let us fix $Q_0 = \Omega_0 \times (-T,0)$ with $\Omega_0 \subset\subset \Omega$, let us denoted by R_0 the euclidean distance of Ω_0 from $\partial\Omega$ and let us set

$$\sigma_0 = \frac{1}{2}\min\left(R_0,\sqrt{T}\right). \tag{3.8}$$

Theorem (3.1) will be achieved if for every $X^0 \in \overline{Q}_0$, $\forall \sigma \in (0,\sigma_0]$ and $\forall \rho \in (0,\sigma)$ it results ([10])

$$\int_{I(x^0,\rho)\cap Q} \|w - w_{I(X^0,\rho)\cap Q}\|^2\, dX \le \tag{3.9}$$

([10]) This because

$$\int_{I(x^0,\rho)\cap Q_0} \|w - w_{I(x^0,\rho)\cap Q_0}\|^2\, dX \le$$

$$\le \int_{I(x^0,\rho)\cap Q_0} \|w - w_{I(x^0,\rho)\cap Q}\|^2\, dX \le \int_{I(x^0,\rho)\cap Q} \|w - w_{I(x^0,\rho)\cap Q}\|^2\, dX .$$

$$\leq c(\nu,n,\lambda)\rho^{\lambda+2}\left\{\frac{1}{\sigma_0^{\lambda+2}}\int_{M(x^0,\sigma)}\|w-w_{M(x^0,\sigma)}\|^2\,dX+\right.$$

$$+\sum_{i=1}^{n}\|D_iw\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+\|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)}+\sum_{i=1}^{n}\|D_iu\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+$$

$$\left.+[u]^2_{H_{-T}^{*0,\frac{1}{2}(\lambda)}(Q,\mathbb{R}^N)}+\sum_{i=0}^{n}\|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}\right\}.$$

Let us fix $X^0 = (x^0 = (x^0, t^0) \in \overline{Q}_0$; to achieve (3.9) we must consider two cases (cfr. Theor. 9.1 [1]) :

a) $t^0 = -T$.

For every $\sigma \in (0, \sigma_0]$ we have

$$I(X^0,\sigma)\cap Q = M(X^0,\sigma)\subset B(x^0,\sigma)\times(-T,0)\subset Q\,; \tag{3.10}$$

the function w belongs to

$$L^2(-T,0,H^1(B(x^0,\sigma),\mathbb{R}^N))\cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(B(x^0,\sigma),\mathbb{R}^N))$$

and is solution in $M(X^0,\sigma)$ to the system

$$-\sum_{i,j=1}^{n}D_i\big(A_{ij}(X^0)D_jw\big)+\frac{\partial w}{\partial t}=\frac{\partial u}{\partial t}-\sum_{i=1}^{n}D_if^i+F^0 \tag{3.11}$$

with

$$f^i=F^i+\sum_{j=1}^{n}\left[(A_{ij}(X^0)-A_{ij}(X))D_jw\right]\in L^2\left(B(x^0,\sigma)\times(-T,0,\mathbb{R}^N)\right),$$

$i = 1, 2, ..., n$. The lemma 2.2 assures that $\forall\,\rho \in (0, \sigma)$

$$\int_{M(X^0,\rho)}\|w-w_{M(X^0,\rho)}\|^2\,dX\leq$$

$$\leq c(\nu)\left\{\left(\frac{\rho}{\sigma}\right)^{n+4}\int_{M(X^0,\sigma)}\|w-w_{M(X^0,\sigma)}\|^2\,dX+\right.$$

$$+\sigma^2\left[\int_{M(X^0,\sigma)}\sum_{i=1}^{n}\|D_iu\|^2\,dX+\|u\|^2_{H^{*\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B(x^0,\sigma),\mathbb{R}^N))}+\right.$$

$$+\int_{M(X^0,\sigma)}\sum_{i=1}^{n}\|f^i\|^2\,dX+\int_{M(X^0,\sigma)}|F^0\|^2\,dX\Big]\Big\}$$

and, then, for every $\rho\in(0,\sigma)$

$$\int_{M(X^0,\rho)}\|w-w_{M(X^0,\rho)}\|^2\,dX\le \tag{3.12}$$

$$\le c(\nu)\Big\{\Big(\frac{\rho}{\sigma}\Big)^{n+4}\int_{M(X^0,\sigma)}\|w-w_{M(X^0,\sigma)}\|^2\,dX+$$

$$+\sigma^2\Big[\int_{M(X^0,\sigma)}\sum_{i=1}^{n}\|D_iw\|^2\,dX+\int_{M(X^0,\sigma)}\sum_{i=1}^{n}\|D_iu\|^2\,dX+$$

$$+\|u\|^2_{H^{*\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B(x^0,\sigma)\mathbb{R}^N))}+\int_{M(X^0,\sigma)}\sum_{i=0}^{n}\|F^i\|^2\,dX\Big]\Big\}.$$

On the other hand, taking into account the assumptions on u, F^i and w, it results:

$$\int_{M(X^0,\sigma)}\sum_{i=1}^{n}\|D_iu\|^2\,dX\le\sigma^\lambda\sum_{i=1}^{n}\|D_iu\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)},\ (^{11}) \tag{3.13}$$

$$\|u\|^2_{H^{*\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B(x^0,\sigma),\mathbb{R}^N))}\le \tag{3.14}$$

$$\le\sigma^\lambda\Big\{\|u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+[u]^2_{H^{*0,\frac{1}{2},(\lambda)}_{-T}(Q,\mathbb{R}^N)}\Big\}$$

$$\int_{M(X^0,\sigma)}\sum_{i=0}^{n}\|F^i\|^2\,dX\le\sigma^\lambda\sum_{i=0}^{n}\|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \tag{3.15}$$

and

$$\int_{M(X^0,\sigma)}\sum_{i=1}^{n}\|D_iw\|^2\,dX\le\sigma^\lambda\sum_{i=1}^{n}\|D_iw\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}. \tag{3.16}$$

(11) Using hypothesis (3.3), we get $u\in L^{2,\lambda}(Q,\mathbb{R}^N)$.

From (3.12) (3.13) (3.14) (3.15) (3.16) we reach $\forall\sigma\in(0,\sigma_0]$ and $\forall\rho\in(0,\sigma)$

$$\varphi(\rho)\le c(\nu)\left(\frac{\rho}{\sigma}\right)^{n+4}\varphi(\sigma)+c(\nu)\sigma^{\lambda+2}\mathcal{M}^2,$$

where

$$\varphi(r)=\int_{M(x^0,r)}\|w-w_{M(x^0,r)}\|^2\,dX$$

and

$$\mathcal{M}^2=\sum_{i=1}^{n}\|D_iw\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+\|u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+$$

$$+\sum_{i=1}^{n}\|D_iu\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+[u]^2_{H_{-T}^{*0,\frac{1}{2},(\lambda)}(Q,\mathbb{R}^N)}+\sum_{i=0}^{n}\|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}.$$

Then, applying the Lemma 2.VI of [3], in corrispondence to the number $\epsilon=n+2-\lambda$, $\forall\sigma\in(0,\sigma_0]$ it results:

$$\varphi(\rho)\le c(\nu)\left(\frac{\rho}{\sigma}\right)^{\lambda+2}\varphi(\sigma)+c(\nu,n,\lambda)\mathcal{M}^2\rho^{\lambda+2}$$

from which, $\forall\sigma\in(0,\sigma_0]$ and $\forall\rho\in(0,\sigma)$ we get:

$$\int_{M(X^0,\rho)}\|w-w_{M(X^0,\rho)}\|^2\,dX\le \tag{3.17}$$

$$\le c(\nu,n,\lambda)\,\rho^{\lambda+2}\left\{\frac{1}{\sigma_0^{\lambda+2}}\int_{M(X^0,\sigma)}\|w-w_{M(x^0,\sigma)}\|^2\,dX+\mathcal{M}^2\right\}.$$

Taking into account (3.10), from (3.17) the estimate (3.9) follows for every $\sigma\in(0,\sigma_0]$ and for every $\rho\in(0,\sigma)$.

b) $t^0\in(-T,0]$.

We shall prove that $\forall X^0\in\overline{Q}_0$ with $t^0\in(-T,0]$, $\forall\sigma\in(0,\sigma_0]$ and $\forall\rho\in(0,\sigma)$ it results

$$\int_{Q(X^0,\rho)\cap Q}\|w-w_{Q(X^0,\rho)\cap Q}\|^2\,dX\le \tag{3.18}$$

$$\le c(\nu,n,\lambda)\,\rho^{\lambda+2}\left\{\frac{1}{\sigma_0^{\lambda+2}}\int_{Q(X^0,\sigma)}\|w-w_{Q(X^0,\sigma)}\|^2\,dX+\right.$$

$$+\sum_{i=1}^{n}\|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+\|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)}+\sum_{i=1}^{n}\|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+$$

$$\left.+[u]^2_{H_{-T}^{*0,\frac{1}{2}(\lambda)}(Q,\mathbb{R}^N)}+\sum_{i=0}^{n}\|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}\right\}.$$

Let us suppose just now we obtained this estimate.

Then if $t^0 = 0$, for every $\rho \in (0, \sigma_0)$ it results $I(X^0,\rho)\cap Q = Q(X^0,\rho)\cap Q$ and hence (3.18) coincides with (3.9).

If $t^0 \in (-T, 0)$, for every $\rho \in (0, \sigma_0)$ we have:

$$I(X^0,\rho)\cap Q = \tag{3.19}$$
$$= \left[Q(X^0,\rho)\cap Q\right]\cup\left[M(X^0,\rho)\cap Q\right]\cup B(x^0,\rho)\times\{t^0\} =$$
$$= \left[Q(X^0,\rho)\cap Q\right]\cup\left[Q(X^*,\rho)\cap Q\right]\cup B(x^0,\rho)\times\{t^0\},$$

where $X^* = (x^0, t^*)$, $t^* = \min(0, t^0 + \rho^2)$.

Taking account that $X^* = (x^0, t^*)$ belongs to $\overline{Q}_0$ and that $t^* \in (-T, 0]$, from (3.18) written with X^* instead of X^0, it follows, $\forall \sigma \in (0, \sigma_0]$ and $\forall \rho \in (0, \sigma)$

$$\int_{Q(X^*,\rho)\cap Q}\|w - w_{Q(X^*,\rho)\cap q}\|^2\,dX \le \tag{3.20}$$

$$\le c(\nu,n,\lambda)\rho^{\lambda+2}\left\{\frac{1}{\sigma_0^{\lambda+2}}\int_{Q(X^*,\sigma)}\|w - w_{Q(X^*,\sigma)}\|^2\,dX+\right.$$

$$+\sum_{i=1}^{n}\|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+\|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^n)}+\sum_{i=1}^{n}\|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+$$

$$\left.+[u]^2_{H_{-T}^{*0,\frac{1}{2},(\lambda)}(Q,\mathbb{R}^N)}+\sum_{i=0}^{n}\|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}\right\}.$$

The estimate (3.9) is a consequence of (3.18), (3.19) and (3.20).
Hence it is enough to show the estimate (3.18).

Setting $\sigma^* = \min\left(\sigma^0, \sqrt{T+t^0}\right)$, we have $Q(X^0,\sigma^*) \subset Q$; then, for every $\sigma \in (0, \sigma^*]$, the function

$$w \in L^2(-T,0,H^1(B(x^0,\sigma),\mathbb{R}^N)) \cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(B(x^0,\sigma)\mathbb{R}^N))$$

is solution in $Q(X^0,\sigma) \subset B(x^0,\sigma) \times (-T,0)$ to the system (3.11) with

$$f^i = F^i + \sum_{j=1}^{n} \big(A_{ij}(X^0) - A_{ij}(X)\big) D_j w \in L^2(B(x^0,\sigma) \times (-T,0), \mathbb{R}^N),$$

$i = 1,2,...,n$. Lemma 2.1 assures that $\forall\, \rho \in (0,\sigma)$

$$\int_{Q(X^0,\rho)} \|w - w_{Q(X^0,\rho)}\|^2\, dX \leq$$

$$\leq c(\nu) \Big\{ \Big(\frac{\rho}{\sigma}\Big)^{n+4} \int_{Q(X^0,\sigma)} \|w - w_{Q(X^0,\sigma)}\|^2\, dX +$$

$$+ \int_{Q(X^0,\sigma)} \|u - u_{Q(X^0,\sigma)}\|^2\, dX + \sigma^2 \Big[\int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2\, dX +$$

$$+ \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|f^i\|^2\, dX + \int_{Q(X^0,\sigma)} \|F^0\|^2\, dX \Big] \Big\}$$

and hence, for every $0 < \rho < \sigma \leq \sigma^*$

$$\int_{Q(X^0,\rho)} \|w - w_{Q(X^0,\rho)}\|^2\, dX \leq \tag{3.21}$$

$$\leq c(\nu) \Big\{ \Big(\frac{\rho}{\sigma}\Big)^{n+4} \int_{Q(X^0,\sigma)} \|w - w_{Q(X^0,\sigma)}\|^2\, dX +$$

$$+ \sigma^{\lambda+2} \sum_{i=1}^{n} \|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \sigma^{\lambda+2} \|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)} +$$

$$+ \sigma^{\lambda+2} \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \sigma^{\lambda+2} \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \Big\}.$$

Hence, if we set

$$\varphi(\sigma) = \int_{Q(X^0,\sigma)} \|w - w_{Q(X^0,\sigma)}\|^2\, dX,$$

being $\sigma^* \leq \sigma_0$, if $\sigma \in (0, \sigma^*]$ and $\rho \in (0, \sigma)$, it results

$$\varphi(\rho) \leq c(\nu)\left(\frac{\rho}{\sigma}\right)^{n+4} \varphi(\sigma) +$$

$$+ c(\nu, n, \lambda)\, \sigma^{\lambda+2} \Bigg\{ \sum_{i=1}^{n} \|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} +$$

$$+ \|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)} + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \Bigg\},$$

from which, making use of Lemma 2.VI of [3], we get $\forall \rho \in (0, \sigma^*)$

$$\int_{Q(X^0,\rho)} \|w - w_{Q(X^0,\rho)}\|^2 \, dX \leq \tag{3.22}$$

$$\leq c(\nu, n, \lambda)\, \rho^{\lambda+2} \Bigg\{ \frac{1}{\sigma^{*\lambda+2}} \int_{Q(X^0,\sigma^*)} \|w - w_{Q(X^0,\sigma^*)}\|^2 \, dX +$$

$$+ \sum_{i=1}^{n} \|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)} +$$

$$+ \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \Bigg\}.$$

Now we have two possibilities.

If $\sigma^* = \sigma_0$, from (3.22) it follows (3.18) $\forall \rho \in (0, \sigma_0]$ and $\forall \rho \in (0, \sigma)$. On the contrary if $\sigma^* = \sqrt{T + t^0} < \sigma_0$, it results

$$Q(X^0, \sigma^*) = M(X^*, \sigma^*) \quad , \quad X^* = (x^0, -T),$$

and (3.22) can be written, $\forall \rho \in (0, \sigma^*)$, in the following way

$$\int_{Q(X^0,\rho)} \|w - w_{Q(X^0,\rho)}\|^2 \, dX \leq \tag{3.23}$$

$$\leq c(\nu, n, \lambda) \Bigg\{ \left(\frac{\rho}{\sigma^*}\right)^{\lambda+2} \int_{M(X^*,\sigma^*)} \|w - w_{M(X^*,\sigma^*)}\|^2 \, dX +$$

$$+\rho^{\lambda+2}\left[\sum_{i=1}^{n}\|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+\|u\|^2_{\mathcal{L}^{2,\lambda+2}(Q,\mathbb{R}^N)}+\right.$$

$$\left.+\sum_{i=1}^{n}\|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+\sum_{i=0}^{n}\|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}\right]\Bigg\}.$$

Making use of (3.17), with $\rho=\sigma^*<\sigma_0$ and $X^0=X^*$, we get

$$\frac{1}{\sigma^{*\lambda+2}}\int_{M(X^*,\sigma^*)}\|w-w_{M(X^*,\sigma^*)}\|^2\,dX\le \tag{3.24}$$

$$\le c(\nu,n,\lambda)\left\{\frac{1}{\sigma_0^{\lambda+2}}\|w\|^2_{L^2(Q,\mathbb{R}^N)}+\mathcal{M}^2\right\}.$$

From (3.23) and (3.24) the estimate (3.18) follows for every $\sigma\in\left(0,\sqrt{T+t^0}\right)$ and for every $\rho\in(0,\sigma^*)$. While if $\sqrt{T+t^0}\le\rho<\sigma^0$ we have

$$Q(X^0,\rho)\cap Q\subset M(X^*,\rho)\subset Q,\ X^*=(x^0,-T)$$

and hence from (3.17), with $X^0=X^*$ we obtain

$$\int_{Q(X^0,\rho)\cap Q}\|w-w_{Q(X^0,\rho)\cap Q}\|^2\,dX\le \tag{3.25}$$

$$\le\int_{M(X^*,\rho)}\|w-w_{M(X^*,\rho)}\|^2\,dX\le$$

$$\le c(\nu,n,\lambda)\rho^{\lambda+2}\left\{\frac{1}{\sigma_0^{\lambda+2}}\|w\|^2_{L^2(Q,\mathbb{R}^N)}+\mathcal{M}^2\right\}$$

and hence the assert.

Now let us show the following

Theorem 3.2. *If $w\in L^2(-T,0,\tilde{H}^1_0(B^+(1),\mathbb{R}^N))\cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(B^+(1),\mathbb{R}^N))$ is solution in $Q=Q^+(1)=B^+(1)\times(-T,0)$ to the system (3.1), if the conditions (1.3), (1.5), (1.6), (3.2) are satisfied with $\Omega=B^+(1),Q=Q^+(1)$ and if $u\in\mathcal{L}^{2,\lambda+2}(Q^+(1),\mathbb{R}^N)\cap \overset{*}{H}{}^{0,\frac{1}{2}(\lambda)}_{-T}(Q^+(1),\mathbb{R}^N)$, $D_iu\in L^{2,\lambda}(Q^+(1),\mathbb{R}^N)$, $0<\lambda<n+2$, $i=1,2,...,n$, $D_iw\in L^{2,\lambda}(Q^+(1),\mathbb{R}^N)$, $0<\lambda<n+2$, $i=1,2,...,n$, then $\forall\,R\in(0,1)$ it results*

$$w\in\mathcal{L}^{2,\lambda+2}(Q^+(R),\mathbb{R}^N)$$

and moreover

$$\|w\|^2_{\mathscr{L}^{2,\lambda+2}(Q^+(R),\mathbb{R}^N)} \leq c(\nu,n,\lambda)\left\{\frac{1}{\sigma_0^{\lambda+2}}\|w\|^2_{L^2(Q,\mathbb{R}^N)}+\right. \tag{3.26}$$

$$+\sum_{i=1}^n \|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+\|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)}+\sum_{i=1}^n \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+$$

$$\left.+[u]^2_{H_{-T}^{*0,\frac{1}{2}(\lambda)}(Q,\mathbb{R}^N)}+\sum_{i=0}^n \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}\right\}$$

where $Q^+(R) = B^+(R)\times(-T,0)$, $Q = Q^+(1)$, σ_0 *is the positive number defined by (3.27).*

Proof. Let us fix $R\in(0,1)$ and let us set

$$\sigma_0 = \frac{1}{2}\min\left(1-R, R, \sqrt{T}\right). \tag{3.27}$$

We shall prove the theorem if, $\forall X^0\in\bar{Q}^+(R)$, $\forall\sigma\in(0,\sigma_0]$ and $\forall\rho\in\left(0,\frac{\sigma}{2}\right)$, we have an estimate of the type

$$\int_{I(X^0,\rho)\cap Q}\|w - w_{I(X^0,\rho)\cap Q}\|^2\,dX \leq \tag{3.28}$$

$$\leq c(\nu,n,\lambda)\,\rho^{\lambda+2}\left\{\frac{1}{\sigma_0^{\lambda+2}}\|w\|^2_{L^2(Q,\mathbb{R}^n)}+\sum_{i=1}^n\|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+\right.$$

$$+\|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)}+\sum_{i=1}^n\|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}+$$

$$\left.+[u]^2_{H_{-T}^{*0,\frac{1}{2},(\lambda)}(Q,\mathbb{R}^N)}+\sum_{i=0}^n\|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)}\right\}.$$

Let $X^0=(x^0,t^0)=(x_1^0,x_2^0,\dots,x_n^0,t^0)$ be a point of $\overline{Q^+(R)}$; we must consider several possibilities; let us show (3.28) when:

a) $x_n^0 = 0$, $t^0=-T$.

For every $\sigma\in(0,\sigma_0]$ we have:

$$I(X^0,\sigma)\cap Q = B^+(x^0,\sigma)\times(-T,-T+\sigma^2) = M^*(X^0,\sigma)\subset Q; \tag{3.29}$$

the function

$$w \in L^2(-T,0,\widetilde{H}_0^1(B^+(x^0,\sigma),\mathbb{R}^N)) \cap H^{\frac{1}{2}}_{-T}(-T,0,L^2(B^+(x^0,\sigma),\mathbb{R}^N))$$

and is solution in $M^+(X^0,\sigma) \subset B^+(x^0,\sigma) \times (-T,0)$ to the system (3.11), with

$$f^i = F^i + \sum_{j=1}^{n} (\overset{\circ}{A}_{ij} - A_{ij}) D_j w \in L^2(B^+(x^0,\sigma)\mathbb{R}^N) \quad i = 1,2,...,n.$$

The estimate (2.17) of the lemma 2.4 assures that $\forall \rho \in (0,\sigma)$ we have

$$\text{(3.30)} \qquad \int_{M^+(X^0,\rho)} \|w - w_{M^+(X^0,\rho)}\|^2 \, dX \leq$$

$$\leq c(\nu) \Big\{ \left(\frac{\rho}{\sigma}\right)^{n+4} \int_{M^+(X^0,\sigma)} \|w - w_{M^+(X^0,\sigma)}\|^2 \, dX +$$

$$+ \sigma^2 \Big[\int_{M^+(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 \, dX +$$

$$+ \|u\|^2_{H^{*,\frac{1}{2}}_{t^0}(t^0,t^0+\sigma^2,L^2(B^+(x^0,\sigma),\mathbb{R}^N))} + \int_{M^+(X^0,\sigma)} \sum_{i=1}^{n} \|f^i\|^2 \, dX +$$

$$+ \int_{M^+(X^0,\sigma)} \|F^0\|^2 \, dX \Big] \Big\}.$$

From this estimate, taking into account (3.13), (3.14), (3.15), (3.16) and that

$$f^i = F^i + \sum_{j=1}^{N} (\overset{\circ}{A}_{ij} - A_{ij}) D_j w \quad i = 1,2,...,n,$$

follows $\forall 0 < \rho < \sigma \leq \sigma_0$

$$\varphi(\rho) \leq c(\nu) \left(\frac{\rho}{\sigma}\right)^{n+4} \varphi(\sigma) + c(\nu)\, \sigma^{\lambda+2} \mathcal{M}^2$$

where

$$\varphi(r) = \int_{M^+(X^0,r)} \|w - w_{M^+(X^0,r)}\|^2 \, dX ;$$

$$\mathcal{M}^2 = \sum_{i=1}^{n} \|D_i w\|^2_{L^{2,\lambda}(Q,\mathbb{R}^n)} + \|u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} +$$

$$+ \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + [u]^2_{\overset{*}{H}{}^{0,\frac{1}{2},(\lambda)}_{-T}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} .$$

Hence, in virtue of lemma 2.VI of [3], in corrispondence with the number $\epsilon = n + 2 - \lambda$, $\forall \sigma \in (0, \sigma_0]$ and $\forall \rho \in (0, \sigma)$

$$\varphi(\rho) \le c(\nu) \left(\frac{\rho}{\sigma}\right)^{\lambda+2} \varphi(\sigma) + c(\nu, n, \lambda)\mathcal{M}^2 \rho^{\lambda+2}$$

from which, $\forall \sigma \in (0, \sigma_0]$ and $\forall \rho \in (0, \sigma)$

$$\int_{M^+(X^0,\rho)} \|w - w_{M^+(X^0,\rho)}\|^2 \, dX \le$$

$$\le c(\nu, n, \lambda)\, \rho^{\lambda+2} \left\{ \frac{1}{\sigma_0^{\lambda+2}} \int_{M^+(X^0,\sigma)} \|w - w_{M^+(X^0,\sigma)}\|^2 \, dX + \mathcal{M}^2 \right\}$$

From this estimate, taking into account (3.29), (3.28) follows $\forall \sigma \in (0, \sigma_0]$ and $\forall \rho \in (0, \sigma)$.

With easy variations of calculations and using the same technique of [6] one can achieve the estimate (3.28) also in the following situations:

b) $x_n^0 = 0$, $-T < t^0 \le 0$;

c) $0 < x_n^0 \le \frac{\sigma_0}{2}$ e $t^0 = -T$

d) $\frac{\sigma_0}{2} < x_n^0 \le R$ e $t^0 = -T$

e) $\frac{\sigma_0}{2} < x_n^0 \le R$

f) $0 < x_n^0 \le \frac{\sigma_0}{2}$ e $-T < t^0 \le 0$

and then the assert.

Taking into account theorems 3.1, 3.2 and following the same tecnique used in [6], the final theorem can be easily showed:

Theorem 3.3. *If $v \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap H^{\frac{1}{2}}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ is solution in Q to the problem (1.1), (1.2), if the conditions (1.3), (1.5), (1.6) are fulfilled, if $f^i \in L^{2,\lambda}(Q, \mathbb{R}^N)$, $0 < \lambda < n + 2$, $i = 0, 1, ..., n$, if $u \in \mathcal{L}^{2,\lambda+2}(Q, \mathbb{R}^N) \cap \overset{*}{H}{}^{0,\frac{1}{2},(\lambda)}_{-T}(Q, \mathbb{R}^N)$, $D_i u \in L^{2,\lambda}(Q, \mathbb{R}^N)$ $i = 1, 2, ..., n$, then it results*

$$v \in \mathcal{L}^{2,\lambda+2}(Q, \mathbb{R}^N)$$

and the following estimate holds

$$(3.31) \qquad \|v\|_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)} \leq c \left\{ \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \right.$$

$$\left. + [u]^2_{\overset{*}{H}{}^{0,\frac{1}{2},(\lambda)}_{-T}(Q,\mathbb{R}^N)} + \|u\|^2_{\mathscr{L}^{2,\lambda+2}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n} \|f^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \right\} \text{(12)}.$$

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(12) In virtue of the theorem 4.1 of [6] and taking into account the assumptions of theorem (3.3), it results $D_i v, D_i w \in L^{2,\lambda}(Q, \mathbb{R}^N)$, $i = 1, 2, \ldots, n$, $u = v + w$ and moreover the estimate (4.2) of [6] holds.