ON DERIVED $G$-STRUCTURES

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We study the first order prolongations of derived $G$-structures (in the sense of P. Dazord [9]) on a differentiable manifold. We give necessary and sufficient conditions (in terms of structure functions) for the complete integrability of the differentiable system associated to a derived $G$-structure of reducible structure group.

1. Introduction and statement of main results.

In the present note we build on results in [8], [12] and study mainly prolongations of derived $G$-structures (in the sense of [9]). Let $M$ be a real $n$-dimensional $C^\infty$ differentiable manifold and $T(M) \to M$ its tangent bundle. Then $M$ admits a canonical imbedding in $T(M)$ as the zero cross-section, i.e. let $j : M \to T(M)$ be given by $j(x) = 0_x \in T_x(M)$, for any $x \in M$. Set $V(M) = T(M) \setminus j(M)$ and denote by $\pi : V(M) \to M$ the natural projection. Note that $V(M)$ is an open submanifold of $T(M)$. We shall need the pullback bundle $\pi^{-1}L(TM) \to V(M)$ of $L(TM)$ by $\pi$, where $L(TM) \to M$ is the principal $GL(n, \mathbb{R})$-bundle of linear frames tangent to $M$. Let $G$ be a Lie subgroup of $GL(n, \mathbb{R})$. Then a derived $G$-structure on $M$ is a principal $G$-subbundle $B_G(M) \to V(M)$ of $\pi^{-1}L(TM) \to V(M)$.

In general, if $F \to V$ is a real rank $r$ vector bundle over a $C^\infty$ manifold $V$, we denote by $L(F) \to V$ the principal $GL(r, \mathbb{R})$-bundle of frames in the fibres of $F$. 

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$F$, i.e. $L(F)$ consists of the synthetic objects of the form $z = (u, \{f_1, \ldots, f_r\})$ with $u \in V$ and $f_i \in F_u$, $1 \leq i \leq r$. Let $\pi^{-1}TM \to V(M)$ be the pullback of $T(M)$ by $\pi$. Then a Finslerian $G$-structure on $M$ (cf. [12]) is a principal $G$-subbundle $B_G(M) \to V(M)$ of $L(\pi^{-1}TM) \to V(M)$. Note that $\pi^{-1}L(TM) \cong L(\pi^{-1}TM)$ (a principal $GL(n, \mathbb{R})$-bundle isomorphism) so that the two view points are equivalent (cf. also our §3). Nevertheless, the constructions of the first order structure functions (of a derived $G$-structure) in [9], [12] are distinct (the one in [12] depends upon the choice of a nonlinear connection on $V(M)$, and both approaches leave a number of open problems, as follows.

1) The connection between the developement of the theory of derived $G$-structures in [4], [9], [12] (and more recently [16]-[17]) is not fully understood, as yet.

2) None of the above theories has been applied to an example (other than derived 0 ($n$)-structures, i.e. Finslerian metrics).

3) There is no convenient notion of "flat" derived $G$-structures (cf. the comments in [8], pp. 380-381) and corresponding "adapted" coordinate systems.

4) No theory of "prolongations" of derived $G$-structures has been constructed, as yet (cf. [23] for the theory of prolongations of $G$-structures and their structure functions).

5) There is no integration of the general theory of derived $G$-structures with the (rather large) amount of the work done on the determination of the sets of Finslerian connections adapted to a specific derived $G$-structure (cf. [3], [13], [19], [20], [21], [22]) given in terms of tensor fields, such as Finslerian metrics, Finslerian conformal structures, Finslerian almost complex structures, etc.

The present paper is the first of a series in which the author hopes to address the above unanswered questions. Leaving definitions momentarily aside, we may formulate our main results as follows.

**Theorem 1.**

i) Let $N$ be a nonlinear connection on $V(M)$, tau a direct summand to $\partial \text{Hom}(F, \mathcal{G})$ in $(F^\ast \wedge F^\ast) \otimes F$, and $B_G(M) \to V(M)$ a derived $G$-structure on $M$. Then its first prolongation $B_G(M)^{(1)}_N^{(1), \tau}$ is a $G^{(1)}$-structure on $B_G(M)$. If $\tau$ is another complement the corresponding first prolongations of $B_G(M)$ are conjugate, i.e.

\[
B_G(M)^{(1)}_N^{(1), \tau} = B_G(M)^{(1)}_N^{(1), \tau} \rho(S)
\]

for some $S \in \text{Hom}(F, \mathcal{G})$.

ii) Let $B_G(M_i) \to V(M_i)$, $i = 1, 2$, be two isomorphic derived $G$-structures
and $f : M_1 \to M_2$ a diffeomorphism so that $F(f)(B_G(M_1)) = B_G(M_2)$. Let $N_1$ be a nonlinear connection on $V(M_1)$ and $N_{2,f_*u} = (d_u f_*) N_{1,u}$, for any $u \in V(M_1)$. Then $N_2$ is a nonlinear connection on $V(M_2)$ and:

$$L(F(f))B_G(M_1)^{N_1}_{\tau} = B_G(M_2)^{N_2}_{\tau}$$

i.e. the first prolongations of $B_G(M_i)$, $i = 1, 2$, are isomorphic.

In §2 we recollect the material we need on nonlinear connections, horizontal lifts and the Dombrowski map (cf. [10], [14]). The frame bundle technique we use is presented in §3 together with a comparison between the formalism in [9],[12] (cf. our Proposition 1). The first structure function of a derived $G$-structure is introduced in §4 in a form close to that in [12] (we use an arbitrary nonlinear connection rather then the nonlinear connection of a given regular connection in $\pi^{-1} TM$, and employ properties of the "standard" horizontal vector fields derived in [1]). The sections §5 - §6 are devoted to the proof of our Theorem 1. Especially the proof of the fact that our prolongations give first order information on isomorphism (of derived $G$-structures) is more delicate (than its classical counterpart in [23]) and organized in our Lemmee 1 to 6. Derived substructures are succinctly studied in §7 where we also hint to some open problem.

Let $G$ be a reducible Lie subgroup of $GL(n, \mathbb{R})$, i.e. there is a proper subspace $V \subseteq \mathbb{R}^n$ invariant by $G$. In the presence of a derived $G$-structure, $V$ gives rise to a $\pi$-distribution (in the sense of [11]) $\mathcal{V}$ on $M$. Next $\mathcal{V}$ lifts to a Pfaffian system $\mathcal{D}$ on $V(M)$ (cf. §8) whose integrability in addressed in the following:

**Theorem 2.** Let $B_G(M) \to V(M)$ be a derived $G$-structure on $M$, $N$ a nonlinear connection on $V(M)$, and $\beta : \pi^{-1} TM \to N$ the corresponding horizontal lift. Assume $G$ is reducible and let $\mathcal{V}$ be the associated $\pi$-distribution on $M$. Then $\mathcal{D} = \beta \mathcal{V} \oplus \gamma \mathcal{V}$ is involutive if and only if the first structure function

$$c : B_G(M) \to \frac{((F^* \wedge F^*) \otimes F)}{\partial \text{Hom}(F, \mathcal{D})}$$

of $B_G(M)$ is $\text{Ker}(\overline{\pi})$-valued.

2. Finslerian metrics and nonlinear connections.

The pullback bundle $\pi^{-1} TM \to V(M)$ plays (within Finslerian geometry) a role which is similar to that of the tangent bundle in Riemannian geometry. Precisely, let $E : T(M) \to [0, +\infty)$. Then $E$ is a Finslerian energy function if
i) $E \in C^1(T(M)), E \in C^\infty(V(M))$, ii) $E$ is positive homogeneous of degree 2, i.e. $E(\lambda u) = \lambda^2 E(u)$ for any $\lambda > 0, u \in T(M)$, iii) $E(u) = 0 \iff u = 0$.

To formulate the last axiom, let $(U, x^i)$ be a local coordinate system on $M$ and $(\pi^{-1}(U), x^i, y^i)$ the naturally induced local coordinates on $V(M)$. Define $g_{ij} : \pi^{-1}(U) \to \mathbb{R}$ by setting:

$$g_{ij}(u) = \frac{1}{2} \frac{\partial^2 E}{\partial y^i \partial y^j}(u)$$

for any $u \in \pi^{-1}(U)$. We request that iv) $g_{ij}(u) \xi^i \xi^j \geq 0$ and $= 0 \iff \xi^i = 0$, for any $u \in \pi^{-1}(U)$ and $(\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$, that is the quadratic form $g_{ij}(u)\xi^i \xi^j$ should be positive-definite. A pair $(M, E)$ is a Finslerian manifold. The pullback bundle $\pi^{-1}TM$ of a Finslerian manifold $(M, E)$ is a Riemannian bundle in a natural way. Indeed, let $X : M \to T(M)$ be a tangent vector field on $M$. Its natural lift is the cross-section $\overline{X} : V(M) \to \pi^{-1}TM$ defined by $\overline{X}(u) = (u, X(\pi(u)))$, for any $u \in V(M)$. Cross-sections in $\pi^{-1}TM$ are usually referred to as Finslerian vector fields on $M$. Let $X_i$ be the natural lift of the (local) tangent vector field $\frac{\partial}{\partial x^i}, 1 \leq i \leq n$. Then $\{X_1, \ldots, X_n\}$ is a frame field in $\pi^{-1}TM$ on $\pi^{-1}(U)$. Finally, we define an inner product $g_u$ on $\pi_u^{-1}TM = \{u\} \times T_{\pi(u)}(M)$ by setting $g_u(X, Y) = g_{ij}(u)\xi^i \xi^j$, for any $X, Y \in \pi_u^{-1}TM$, where $X = \xi^i X_i(u), Y = \eta^i X_i(u)$. The definition of $g_u(X, Y)$ does not depend upon the choice of local coordinates $(U, x^i)$ at $x = \pi(u)$ and $u \mapsto g_u$ is a Riemannian bundle metric on $\pi^{-1}TM$.

A $C^\infty$ distribution $N$ on $V(m)$ is a nonlinear connection on $V(M)$ if

$$T_u V(M) = N_u \oplus \text{Ker}(d_u \pi)$$

for any $u \in V(M)$. Cf. also [14].

Define a bundle morphism $L : T(V(M)) \to \pi^{-1}TM$ by $L_u X = (u, (d_u \pi)X)$, for any $u \in V(M), X \in T_u(V(M))$. Given a nonlinear connection $N$ on $V(M)$ the restriction $L : N \to \pi^{-1}TM$ is a vector bundle isomorphism. Set $\beta_u = (L_u|_{N_u})^{-1}$ for any $u \in V(M)$. The bundle isomorphism $\beta : \pi^{-1}TM \to N$ is termed horizontal lift (with respect to $N$).

As to local computations, set $\delta_i = \beta X_i$. Then $\{\delta_i\}$ is a frame field in $N$ on $\pi^{-1}(U)$. One may seek $\delta_i$ as a linear combination $\delta_i = M^i_j \partial_j - N^i_j \hat{\partial}_j$, where $\partial_i = \frac{\partial}{\partial x^i}, \hat{\partial}_i = \frac{\partial}{\partial y^i}$ for the sake of simplicity. Apply $L$ so that to yield $M^i_j = \delta^i_j$ (as $L \partial_i = X_i$ and $L \hat{\partial}_i = 0$). The remaining (uniquely determined) functions $N^i_j : \pi^{-1}(U) \to \mathbb{R}$ are the coefficients of the nonlinear connection.
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$N$ (with respect to $(U, x^i)$). Let $x'^{ti} = x^{ti}(x^1, \ldots, x^n)$, det $\left[ \frac{\partial x'^{ti}}{\partial x^j} \right] \neq 0$ on $U \cap U' \neq \emptyset$, be a transformation of local coordinates on $M$. Taking into account the identities:

$$X_i = \left( \frac{\partial x'^{tj}}{\partial x^i} \circ \pi \right) X'_j$$

$$\partial_i = \frac{\partial x'^{tj}}{\partial x^i} \partial'_j + \frac{\partial^2 x'^{tj}}{\partial x^i \partial x^k} y^k \partial'_j$$

$$\partial'_i = \frac{\partial x'^{tj}}{\partial x^i} \partial'_j$$

$$\delta_i = \partial_i - N^i_j \partial'_j \quad \delta'_i = \partial'_i - N^i'_j \partial'_j$$

one obtains:

$$\delta_i = \frac{\partial x'^{tj}}{\partial x^i} \delta'_j + \left\{ \frac{\partial x'^{tj}}{\partial x^i} N^k_j \partial'_k - \frac{\partial x'^{tk}}{\partial x^j} N^j_i + \frac{\partial^2 x'^{tk}}{\partial x^i \partial x^j} y^i \right\} \partial'_k. \quad (2.2)$$

Finally, as a consequence of (2.2) and of the uniqueness of the direct sum decomposition (cf. (2.1)) it follows that the coefficients of the nonlinear connection $N$ satisfy the transformation law:

$$\frac{\partial x'^{tj}}{\partial x^i} N^k_j \partial'_k = \frac{\partial x'^{tk}}{\partial x^j} N^j_i - \frac{\partial^2 x'^{tk}}{\partial x^i \partial x^j} y^i. \quad (2.3)$$

Vice versa a set of $C^\infty$ functions $N^i_j$ obeying (2.3) under any coordinate transformation $x'^{ti} = x^{ti}(x^1, \ldots, x^n)$ determines a nonlinear connection on $V(M)$ by setting $N_u = \sum_{i=1}^n \Re (\partial_i - N^i_j \partial'_j) u$. The definition of $N_u$ does not depend (by (2.3)) upon the choice of local coordinates $(U, x^i)$ at $\pi(u)$.

**Examples.**

1) Let $\Gamma^i_{jk}$ be a linear connection on $M$. Then $N^i_j(x, y) = \Gamma^i_{jk}(x)y^k$ is a nonlinear connection on $V(M)$.

2) Let $\mathcal{L} : V(M) \rightarrow \pi^{-1}TM$ be the Finslerian vector field given by $\mathcal{L}(u) = (u, u)$, for any $u \in V(M)$. Then $\mathcal{L}$ is referred to as the Liouville vector field. Let $\nabla$ be a connection in $\pi^{-1}TM \rightarrow V(M)$. A tangent vector field $X$ on $V(M)$ is horizontal (with respect to $\nabla$) if $\nabla_X \mathcal{L} = 0$. The horizontal distribution $N(\nabla) : u \rightarrow N(\nabla) u \subset T_u(V(M))$ of $\nabla$ consists of all $Y \in T_u(V(M))$ so that there is a horizontal tangent vector field $X$ on $V(M)$ with $X(u) = Y$. If
$N(\nabla)$ is a nonlinear connection on $V(M)$ then $\nabla$ is termed regular. Cf. also [2]. If $(M, E)$ is a Finslerian manifold, let $\nabla$ be the Cartan-Chern connection in $(\pi^{-1}TM, g)$. Cf. [5], [7]. Then $\nabla$ is regular. Its nonlinear connection $N(\nabla)$ is (locally) given by:

$$N^i_j = \frac{1}{2} \partial_j \left| \begin{array}{c} i \\ 0 \\ 0 \end{array} \right|$$

$$i \quad j \quad k$$

$$\left| i \right| = i | j k | y_i y_j$$

$$\left| i \right| = g^{im} \left| j k, m \right|$$

$$\left| i j, k \right| = \frac{1}{2} \left( \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right)$$

In general, a pair $(\nabla, N)$ consisting of a connection $\nabla$ in the vector bundle $\pi^{-1}TM$ and a nonlinear connection $N$ on $V(M)$ is called a Finslerian connection. Any regular connection gives rise to a Finslerian connection. The converse is false for most of the "canonical" connections of Finslerian geometry (e.g. the Berwald and Rund connections (cf. [18]) are not regular).

3) There is yet another way to look at the nonlinear connection of the Cartan-Chern connection. Let $\gamma : \pi^{-1}TM \to \text{Ker}(d\pi)$ be defined by $\gamma X_i = \partial_i$. Then $\gamma$ is a (globally defined) bundle isomorphism referred to as the vertical lift. The Dombrowski map is the bundle morphism $K : T(V(M)) \to \pi^{-1}TM$ given $K_u = \gamma_u^{-1} \circ Q_u, u \in V(M)$, where $Q_u = T_u(V(M)) \to \text{Ker}(d_u\pi)$ is the natural projection associated with (2.1). Therefore the construction of $K$ depends on a given fixed nonlinear connection $N$ on $V(M)$. Cf. also [10]. The Sasaki metric of a Finslerian manifold $(M, E)$ is the Riemannian metric $G$ on $V(M)$ defined by:

$$G(X, Y) = g(LX, LY) + g(KX, KY)$$

for any $X, Y \in \Gamma^\infty(T(V(M)))$. Here the Dombrowski map $K$ is built with respect to the nonlinear connection $N(\nabla)$ of the Cartan-Chern connection of $(M, E)$. Let $N_u$ be the orthogonal complement of $\text{Ker}(d_u\pi)$ in $T_u(V(M))$ (with respect to $G_u$), $u \in V(M)$. Then $N$ is a nonlinear connection on $V(M)$ and $N = N(\nabla)$. 
3. Finslerian frame bundles and canonical 1-forms.

Let $\Phi : \pi^{-1} L(TM) \to L(\pi^{-1} TM)$ be given by

$$\Phi(z) = (u, \{(u, X_1), \ldots, (u, X_n)\})$$

for any $z = (u, b) \in \pi^{-1} L(TM)$, where $b = (x, \{X_1, \ldots, X_n\}) \in L(TM)$. Then $\Phi$ is a principal $GL(n, \mathbb{R})$-bundle isomorphism.

P. Dazord defines (cf. [9], p. 2730) a 1-form

$$\alpha \in \Gamma^\infty(T^*(\pi^{-1} L(TM)) \otimes \mathbb{R}^n)$$

as follows $\alpha_z = b^{-1} \circ (d_z(\rho))$, $z = (u, b)$, where $\rho : \pi^{-1} L(TM) \to V(M)$ is the natural projection. Note that $\alpha$ is the $h$-basic form of [18], p. 48. On the other hand, together with [12], we may define the 1-form $\theta^h \in \Gamma^\infty(T^*(\pi^{-1} TM) \otimes \mathbb{R}^n)$ by setting $\theta^h_z = z^{-1} \circ L_u \circ (d_z \rho_1)$, for any $z = (u, \{X_i\})$. Where $X_i \in \pi^{-1} TM$ and $\rho_1 : L(\pi^{-1} TM) \to V(M)$ is the natural projection.

**Proposition.** The 1-forms $\alpha, \theta^h$ coincide up to an isomorphism, i.e.

(3.1) $\alpha_z = \theta^h_{\Phi(z)} \circ (d_z \Phi)$

**Proof.** To establish (3.1) we look at the following diagram:

$$
\begin{array}{ccc}
T_u(V(M)) & \xrightarrow{L_u} & \pi^{-1} TM \\
\downarrow{d\Phi(z)\rho_1} & & \downarrow{\Phi(z)^{-1}} \\
T_{\Phi(z)}(L(\pi^{-1} TM)) & \xrightarrow{\theta^h_{\Phi(z)}} & \mathbb{R}^n \\
\downarrow{d_z \Phi} & & \downarrow{1_{\mathbb{R}^n}} \\
T_z(\pi^{-1} L(TM)) & \xrightarrow{\alpha_z} & \mathbb{R}^n \\
\downarrow{b^{-1}} & & \\
T_u(V(M)) & \xrightarrow{d_u \pi} & T_x(M)
\end{array}
$$

where $z = (u, b)$ and $x = \pi(u)$. As the upper and lower rectangles are commutative, it is sufficient to check the commutativity of the big rectangle.
Taking into account \( \rho_1 \circ \Phi = \rho \) and \( \Phi(z)^{-1}(u, X) = b^{-1}(X) \), for any \( X \in T_x(M) \), we may conduct the following calculation:

\[
\Phi(z)^{-1} \circ L_u \circ (d_{\Phi(z)} \rho_1) \circ (d_z \Phi) = \Phi(z)^{-1} \circ L_u \circ (d_z \rho) = \\
= \Phi(z)^{-1}(u, (d_u \pi)(d_z \rho)) = b^{-1} \circ d_z(\pi \rho). \quad \Box
\]

In addition to the \( h \)-basic 1-form we define \( \theta^v \in \Gamma^\infty(T^*(L(\pi^{-1}TM)) \otimes \mathbb{R}^n) \) as follows. Let \( N \) be a fixed nonlinear connection on \( V(M) \) and \( K : T(V(M)) \to \pi^{-1}TM \) the corresponding Dombrowski map.

Set \( \theta^v_z = z^{-1} \circ K_u \circ (d_z \rho_1) \), for any \( z \in L(\pi^{-1}TM), u = \rho_1(z) \).

If \( z = (u, \{X_i\}) \) then \( z : \mathbb{R}^n \to \pi^{-1}TM \) is given by \( z(e_i) = X_i \) where \( \{e_i\} \) is the canonical basis of \( \mathbb{R}^n \). Together with [11] let us define \( \theta \in \Gamma^\infty(T^*(L(\pi^{-1}TM)) \otimes F) \) by \( \theta = \theta^h \oplus \theta^v \) where \( F = \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n \). We may emphasize the importance of considering the 1-form \( \theta \) (rather than \( \theta^h \) or \( \theta^v \) alone) as follows. Let \( H \) be a horizontal distribution in \( L(\pi^{-1}TM) \to V(M) \), that is the following direct sum decomposition holds:

\[
T_z(L(\pi^{-1}TM)) = H_z \oplus \text{Ker}(d_z \rho)
\]

for any \( z \in L(\pi^{-1}TM) \). From now on we do not distinguish between \( \pi^{-1}L(TM) \) and \( L(\pi^{-1}TM) \), (respectively between \( \rho \) and \( \rho_1 \)). Set \( t_z = (d_z \rho)|_{H_z}, z \in L(\pi^{-1}TM) \). Then \( t_z : H_z \to T_u(V(M)) \) is an \( \mathbb{R} \)-linear isomorphism, \( u = \rho(z) \). Note that neither the \( h \)-basic nor the \( v \)-basic 1-forms may play the role of the canonical 1-form in [15], vol. I, p. 118, as their restrictions \( \theta^h_z, \theta^v_z : H_z \to \mathbb{R}^n \) are not isomorphisms.

Indeed \( \text{Ker}(\theta^h_z) = \text{Ker}(d_z \rho) \oplus t_z^{-1}(\text{Ker}(d_u \pi)) \) and \( \text{Ker}(\theta^v_z) = \text{Ker}(d_z \rho) \oplus t_z^{-1}(N_u), u = \rho(z) \). However \( \text{Ker}(\theta_z) = \text{Ker}(d_z \rho) \) so that \( \theta_z : H_z \to F \) is a \( \mathbb{R} \)-linear isomorphism. Then \( \theta \) is referred to as the canonical 1-form of \((M, N)\).

4. Structure functions.

Let \( B_G(M) \to V(M) \) be a derived \( G \)-structure and

\[
\theta \in \Gamma^\infty(T^*(B_G(M)) \otimes F)
\]

the 1-form induced on \( B_G(M) \) by the canonical 1-form of \((M, N)\). Here \( F = \mathbb{R}^{2n} \) and the nonlinear connection \( N \) is fixed (throughout §4). Together with [12] let us define \( C_H : B_G(M) \to (F^* \wedge F^*) \otimes F \), for a given fixed horizontal distribution \( H \) in \( B_G(M) \to V(M) \), as follows. Let \( \xi \in F \) and denote
by \( H(\xi) \in \Gamma^\infty(H) \) the tangent vector field on \( B_G(M) \) defined by \( \theta(H(\xi)) = \xi \).

Note that \( H(\xi) \) is well defined (as \( \theta_z : H_z \to F \) is an isomorphism, for any \( z \in B_G(M) \)) and \( C^\infty \) differentiable. Cf. [1], \( H(\xi) \) possesses properties which are similar to those of the standard horizontal vector fields in [15], vol. I, p. 119. However \( H(\xi) \) depends on the choice of nonlinear connection \( N \) on \( V(M) \), in addition to the data \( (H, \xi) \). Let \( \xi, \eta \in F \) and set:

\[
c_H(z)(\xi \wedge \eta) = (d\theta)_z(H(\xi), H(\eta)).
\]

Let \( \partial : \text{Hom}(F, \mathcal{G}) \to (F^* \wedge F^*) \otimes F \), where \( \mathcal{G} \) is the Lie algebra of \( G \), be defined by \( (\partial T)(\xi \wedge \eta) = T(\xi)\eta - T(\eta)\xi \), for any \( T \in \text{Hom}(F, \mathcal{G}) \) and any \( \xi, \eta \in F \). Here \( \mathcal{G} \) acts canonically on \( F = \mathbb{R}^n \oplus \mathbb{R}^n \), i.e. if \( A \in \mathcal{G} \) and \( \xi = \xi_1 \oplus \xi_2 \in F \) then \( A\xi = A\xi_1 \oplus A\xi_2 \).

Let \( H, H' \) be two horizontal distributions in \( B_G(M) \to V(M) \). Then:

\[
(4.1) \quad c_H(z) - c_{H'}(z) = \frac{1}{2} \partial T
\]

for some \( T \in \text{Hom}(F, \mathcal{G}) \) depending only on \( H, H' \), and for any \( z \in B_G(M) \).

For the sake of completeness, let us prove (4.1). Cf. also Theor.4.1 in [12].

As \( H(\xi)_z - H'(\xi)_z \in \text{Ker}(\theta_z) = \text{Ker}(d\xi \rho) \), there is \( T \in \text{Hom}(F, \mathcal{G}) \) so that \( H'(\xi)_z - H(\xi)_z = T(\xi)^* \). Here, for each \( A \in \mathcal{G} \), we denote by \( A^* \in \Gamma^\infty(\text{Ker}(d\rho)) \) the fundamental vector field associated with \( A \), i.e. \( A^*_z = (d\xi L_z)A_e \) for any \( z \in B_G(M) \). Here \( e \in G \) is the unit \( n \times n \) matrix, while \( L_z : G \to B_G(M) \) is given by \( L_z(g) = zg \), for any \( g \in G \). Then:

\[
c_H(z)(\xi \wedge \eta) - c_{H'}(z)(\xi \wedge \eta) = \frac{1}{2} \{ \theta_z([H(\xi), T(\eta)^*]) - \theta_z([H(\eta), T(\xi)^*]) \} = \frac{1}{2} \{ T(\xi)\eta - T(\eta)\xi \}
\]

Here we made use of a formula in [1], i.e. \( [A^*, H(\xi)] = H(A\xi) \), for any \( A \in \mathcal{G} \), \( \xi \in F \). Finally, let \( c : B_G(M) \to ((F^* \wedge F^*) \otimes F) / \partial \text{Hom}(F, \mathcal{G}) \) be defined by \( c(z) = \Psi(c_H(z)) \) for any \( z \in B_G(M) \) and any horizontal distribution \( H \) in \( B_G(M) \), where

\[
\Psi : (F^* \wedge F^*) \otimes F \to ((F^* \wedge F^*) \otimes F) / \partial \text{Hom}(F, \mathcal{G})
\]

is the natural map. Then \( c \) is well defined as a consequence of (4.1) and is referred to as the first structure function of the derived \( G \)-structure \( B_G(M) \). This appears to be distinct from the structure functions in [4], [9] and the relation between the three is not fully clear.
5. Prolongations of derived $G$-structures.

Let $\mathcal{G}^{(1)} = \text{Ker} (\partial) \subset \text{Hom} (F, \mathcal{G})$ be the first prolongation of $G$. Next consider $\rho : \mathcal{G}^{(1)} \to \text{End}_{\mathbb{R}} (F \oplus \mathcal{G})$ given by $\rho(T) (\xi, A) = (\xi, T(\xi) + A)$, for any $T \in \mathcal{G}^{(1)}$, $\xi \in F$, $A \in \mathcal{G}$. Then $\rho$ is a representation of the additive group $\mathcal{G}^{(1)}$ on $F \oplus \mathcal{G}$.

Let $\{e_1, \ldots, e_{2n}\}$ be the canonical basis of $F$ and $\{A_1, \ldots, A_{n_0}\}$ a fixed basis of $\mathcal{G}$, $n^0 = \text{dim}_\mathbb{R} \mathcal{G}$. Let $h : \text{End}_{\mathbb{R}} (F \oplus \mathcal{G}) \to GL(2n + n^0, \mathbb{R})$ be the isomorphism associated with the linear basis $\{(e_i, 0), (0, A_\alpha)\}$ of $(F \oplus \mathcal{G})$. Then $G^{(1)} = h(\rho(\mathcal{G}^{(1)}))$ is a Lie subgroup of $GL(2n + n^0, \mathbb{R})$, i.e. the first prolongation of $G$.

Let $\tau$ be a direct summand to $\partial \text{Hom} (F, \mathcal{G})$ in $(F^* \wedge F^*) \otimes F$. Let $\mathcal{H}(\tau)$ be the set of all horizontal distributions $H$ in $B_G(M)$ so that $c_H$ is $\tau$-valued. Clearly $\mathcal{H}(\tau)$ depends on a fixed nonlinear connection $N$ on $V(M)$, as well.

Note that given $H \in \mathcal{H}(\tau)$ the rest of the horizontal distributions in $\mathcal{H}(\tau)$ are parametrized by elements of $\mathcal{G}^{(1)}$. Indeed $H'(\xi)_z - H(\xi)_z = T(\xi)_z$ for some $T \in \text{Hom} (F, \mathcal{G})$ depending only on $H$, $H' \in \mathcal{H}(\tau)$. Then (4.1) yields $\partial T \in \tau \cap \partial \text{Hom} (F, \mathcal{G}) = (0)$.

Define $B^{(1)} = B_G(M)^{(1), \tau}_N$ to be the set of all linear frames tangent to $B_G(M)$ of the form $(z, \{H(e_i)_z, A^*_\alpha z\})$ for any $z \in B_G(M)$ and any $H \in \mathcal{H}(\tau)$. Let $\pi^{(1)} : L(TB_G(M)) \to B_G(M)$ be the principal $GL(2n + n^0, \mathbb{R})$-bundle of linear frames tangent to $B_G(M)$. To prove that $B^{(1)} \to B_G(M)$ is a $G^{(1)}$-structure note firstly that $\pi^{(1)}(B^{(1)}) = B_G(M)$. Also, it is clear from the definition that for any $z \in B_G(M)$ there is $U \subset B_G(M)$ open, $z \in U$, and there is a cross-section $\sigma : U \to L(TB_G(M))$ so that $\sigma(U) \subset B^{(1)}$. As $B^{(1)}$ is already a submanifold of $L(TB_G(M))$ it remains to be shown that given $r \in B^{(1)}$ and $a \in G^{(1)}$ we have $ra \in B^{(1)}$ if and only if:

\[
G^{(1)} = \left\{ \begin{bmatrix} \delta^i_j & 0 \\ T(e_i) & \delta^\alpha_\beta \end{bmatrix} : T \in \mathcal{G}^{(1)} \right\}
\]

where $T(e_i) = T(E_i)^\alpha A_\alpha$. Next, if $r = (z, \{H(e_i)_z, A^*_\alpha z\}) \in B^{(1)}$ then $ra = (z, \{H(e_j)_z a^i_j + A^*_\alpha, a^\alpha_i, H(e_i)_z a^i + A^*_\alpha z, a^\alpha_i \})$ where $a = \begin{bmatrix} a^i_j & a^\alpha_i \\ a^\alpha_i & a^\beta_\alpha \end{bmatrix} \in GL(2n + n^0, \mathbb{R})$. Thus $ra \in B^{(1)}$ if and only if:

\[
H(e_j)_z a^i_j + A^*_\alpha, a^\alpha_i = H'(e_i)_z
\]

\[
H(e_i)_z a^i + A^*_\alpha, a^\beta_\alpha = A^*_\alpha, z
\]
for some \( H' \in \mathcal{H}(\tau) \). Apply \( \theta_z \) to both (5.1) - (5.2) so that to get \( a_i^z = \delta_i^1 \) and \( a_i^\alpha = 0 \). Again (5.2) gives \( a_\alpha^\beta = \delta_\alpha^\beta \). Finally (5.1) yields \( T(e_i)_z^* = A_{\alpha z}^* a_i^\alpha \) for some \( T \in \mathcal{G}^{(1)} \). Thus \( a \in \mathcal{G}^{(1)} \). \( \square \)

To justify the second statement in i) of our Theorem 1, let \( \overline{\tau} \) so that \((F^* \wedge F^*) \otimes F = \overline{\tau} \oplus \partial \text{Hom}(F, \mathcal{G})\). Set \( \overline{\mathcal{B}}^{(1)} = B_G(M_N^{(1)}, \overline{\tau}) \) for brevity. Let \( \overline{\tau} = (z, \{\overline{H}(e_i)_z, A_{\alpha z}^*\}) \in \overline{\mathcal{B}}^{(1)}, r = (z, \{H(e_i)_z, A_{\alpha z}^*\}) \) for some \( \overline{H} \in \mathcal{H}(\overline{\tau}) \) and \( H \in \mathcal{H}(\tau) \). Then \( \overline{H}(\xi) - H(\xi) = S(\xi)^* \) for some \( s \in \text{Hom}(F, \mathcal{G}) \) and \( \xi \in F \). It follows that \( \overline{\tau} = r\rho(s) \), where

\[
\rho(s) = \begin{bmatrix}
\delta_i^j \\
T(e_i)^\alpha \\
\delta_\alpha^\beta
\end{bmatrix} \in GL(2n + n^0, \mathbb{R}) \]. \( \square \)

6. Isomorphic derived \( G \)-structures.

Let \( B_G(M_i) \to V(M_i) \), \( i = 1, 2 \), be two derived \( G \)-structures. Then \( B_G(M_i) \), \( i = 1, 2 \), are said to be isomorphic if there is a diffeomorphism \( f : M_1 \to M_2 \) so that \( F(f)(B_G(M_1)) = B_G(M_2) \), where \( F(f) : \pi_1^{-1} L(TM_1) \to \pi_1^{-1} L(TM_2) \) is defined by \( F(f)(u, b) = (f_* u, L(f)(b)) \), for any \( z = (u, b) \in \pi_1^{-1} L(TM_1) \). Here \( f_* : V(M_1) \to V(M_2) \) denotes the differential of \( f \) while \( L(f) : L(TM_1) \to L(TM_2) \) is the naturally induced bundle map, i.e. \( L(f)(b) = (f(x), \{(dx)f(X_i)\}) \) for any \( b = (x, \{X_i\}) \in L(TM_1) \).

If \( A \in \mathcal{G} \) then \( \ell_i(A) \in \Gamma^\infty(\text{Ker}(d\rho_i)), i = 1, 2 \), denotes the fundamental vector field associated with \( A \) (previously denoted by \( A^* \)). At this point we may prove ii) of our Theorem 1. To this end we shall need the following:

**Lemma 1.** Let \( B_G(M_i) \to V(M_i), i = 1, 2 \), be two isomorphic derived \( G \)-structures and \( f : M_1 \to M_2 \) a diffeomorphism so that \( F(f)B_G(M_1) = B_G(M_2) \). Then:

\[
(6.1) \quad F(f)_* \ell_1(A) = \ell_2(A).
\]

**Proof.** Let \( z \in B_G(M_1) \) and \( L_{1,z} : G \to B_G(M_1) \) given by \( L_{1,z}(g) = zg \), for any \( g \in G \). Then:

\[
F(f) \circ L_{1,z}(g) = F(f)(zg) = F(f)(u, bg) = \left( (dx)f(u, L(f)(bg)) \right) = \\
= \left( (dx)f(u, L(f)(b)) \right) = F(f)(u, b)g
\]
for any \( z = (u, b) \in B_G(M_1), g \in G \), where \( x = \pi_1(u) \in M_1 \). We have obtained:

\[
F(f) \circ L_{1,z} = L_{2,F(f)(z)}.
\]

Taking into account (6.2) we may conduct the following calculation:

\[
(d_z F(f)) \ell_1(A)_z = (d_z F(f)) \circ (d_e L_{1,z}) A_e = d_e (F(f) \circ L_{1,z}) A_e =
\]

\[
= d_e (L_{2,F(f)(z)}) A_e = \ell_2(A) F(f)(z). \quad \Box
\]

Set \( P_i = \pi_i^{-1} L(TM_i), i = 1, 2 \), for simplicity. The diffeomorphism \( F(f) : P_1 \to P_2 \) induces the natural bundle map

\[
L(F(f)) : L(TP_1) \to L(TP_2),
\]

\[
L(F(f))(z, \{Z_a\}) = (F(f)(z), \{(d_z F(f) Z_a)\}), z \in P_1, Z_a \in T_z(P_1).
\]

This is the map in (1.2). We shall need:

**Lemma 2.** Let \( H_1 \) be a horizontal distribution in \( B_G(M_1) \to V(M_1) \) and \( H_{2,F(f)(z)} \subset T_{F(f)(z)}(B_G(M_2)) \) defined by:

\[
H_{2,F(f)(z)} = (d_z F(f)) H_{1,z}
\]

for any \( z \in B_G(M_1) \). Then \( H_2 \) is a horizontal distribution in \( B_G(M_2) \to V(M_2) \).

**Proof.** Note that:

\[
(d_z F(f)) \ker (d_z \rho_1) = \ker (d_{F(f)(z)} \rho_2).
\]

This follows from the identity:

\[
\rho_2 \circ F(f) = f_* \circ \rho_1.
\]

Indeed, it is sufficient (since both sides in (6.4) have the same dimension) to check the inclusion \( "\subseteq" \). To this end, let \( X \in \ker (d_z \rho_1) \). Then:

\[
(d_{F(f)(z)} \rho_2) \circ (d_z F(f)) X = d_z (\rho_2 \circ F(f)) X =
\]

\[
= d_z (f_* \circ \rho_1) X = (d_z f_* \circ (d_z \rho_1)) X = 0.
\]

Applying \( d_z F(f) \) to: \( T_z (B_G(M_1)) = H_{1,z} \oplus \ker (d_z \rho_1) \) and using (6.3) - (6.4) shows that \( T_{F(f)(z)}(B_G(M_2)) \) may be written as the sum of \( H_{2,F(f)(z)} \) and \( \ker (d_{F(f)(z)} \rho_2) \). As \( d_z F(f) \) commutes with the intersection the sum is also direct. \( \Box \)
Lemma 3. Let $N_1$ be a nonlinear connection on $V(M_1)$ and $N_2, f_*(u) \subset T_{f_*(u)}(V(M_2))$ defined by:

$$N_{2, f_*(u)} = (d_u f_*) N_{1, u}$$

for any $u \in V(M_1)$. Then $N_2$ is a nonlinear connection on $V(M_2)$.

Proof. Note that:

$$d_u f_* \text{ Ker } (d_u \pi_1) = \text{ Ker } (d_{f_*(u)} \pi_2).$$

As both sides in (6.7) have the same dimension, it is sufficient to check one inclusion. Let $X \in \text{ Ker } (d_u \pi_1)$. Then:

$$(d_{f_*(u)} \pi_2) \circ (d_u f_*) X = d_u (\pi_2 \circ f_*) X = d_u (f \circ \pi_1) X = 0$$

Finally, let us apply $d_u f_*$ to $T_u(V(M_1)) = N_{1, u} \oplus \text{ Ker } (d_u \pi_1)$, etc. □

Lemma 4. Let $N_1$ be a nonlinear connection on $V(M_1)$ and $N_2$ the nonlinear connection given by (6.6). Let $\theta_i \in \Gamma^\infty \left( T^*(B_G(M_i)) \otimes F \right)$ be the canonical 1-form of $B_G(M_i)$, built with respect to $N_i, \ i = 1, 2$. Then:

$$\theta_2, F(f)(z) \circ (d_z F(f)) = \theta_1, z$$

for any $z \in B_G(M_1)$. Here $F = \mathbb{R}^{2n}, \ n = \dim(M_i), \ i = 1, 2$.

Proof. The following diagram is commutative:

$$
\begin{array}{ccc}
T_u(V(M_1)) & \xrightarrow{d_u f_*} & T_{f_*(u)}(V(M_2)) \\
\downarrow \quad L_{1, u} & & \downarrow \quad L_{2, f_*(u)} \\
\pi_{1, u}^{-1} TM_1 & \xrightarrow{(Df)_u} & \pi_{2, f_*(u)}^{-1} TM_2
\end{array}
$$

for any $u \in V(M_1)$, cf. [1]. Here $(Df)_u$ denotes the restriction of $f_\ast \times f_\ast$ to $\pi_{1, u}^{-1} TM_1$. Moreover

$$\begin{align*}
(Df)_u \circ z &= F(f)(z)
\end{align*}$$

for any $z = (u, b) \in B_G(M_1)$, i.e. the following diagram is commutative:

$$
\begin{array}{ccc}
\pi_{1, u}^{-1} TM_1 & \xrightarrow{(Df)_u} & \mathbb{R}^n \\
\downarrow \quad z & & \downarrow \quad F(f) \\
\pi_{2, f_*(u)}^{-1} TM_2 & &
\end{array}
$$
To check (6.9) let $z = (u, b) \cong (u\{(u, X_i)\})$ where $b = (\pi_1(u), \{X_i\})$. Then $(Df)_u \circ z(e_i) = (Df)_u(u, X_i) = (f_*u, f_*X_i) = F(f)(z)(e_i)$ because $F(f)(z) = (f_*u, L(f)(b)) = (f_*u, (f(x), \{f_*X_i\}) \cong (f_*u, \{(f_*u, f_*X_i)\})$. As $\theta_2 = \theta^h_2 \oplus \theta^v_2$ it is sufficient to prove (6.8) for the $h$- and $v$-basic 1-forms. Using the commutative diagrams above we may conduct the following computation:

$$
\theta^h_{2,F(f)(z)} \circ (d_zF(f)) = F(f)(z)^{-1} \circ (d_{2,F_*u} \circ L_{F(f)(z)} \rho_2) \circ (d_zF(f)) =
$$

$$= F(f)(z)^{-1} \circ L_{2,F_*u} \circ d_z(f_* \circ \rho_1) =
$$

$$= F(f)(z)^{-1} \circ (Df)_u \circ L_{1,u} \circ (d_z \rho_1) = z^{-1} \circ L_{1,u} \circ (d_z \rho_1) = \theta^h_{1,z}$$

The proof of (6.8) for the $v$-basic 1-form $\theta^v_2$ is somewhat trickier. Note firstly the commutativity of the following diagram:

$$
\begin{array}{ccc}
T_{f_*u}(V(M_2)) & \xrightarrow{Q_{2,f_*u}} & \text{Ker}(d_{f_*u} \pi_2) \\
\downarrow d_{uf_*} & & \downarrow d_{uf_*} \\
T_u(V(M_1)) & \xrightarrow{Q_{1,u}} & \text{Ker}(d_u \pi_1)
\end{array}
$$

for any $u \in V(M_1)$, as a consequence of Lemma 3. We retain the identity:

$$
(6.10) \quad Q_{2,f_*u} \circ (d_{uf_*}) = (d_{uf_*}) \circ Q_{1,u}.
$$

Next we need to establish the commutativity of the diagram:

$$
\begin{array}{ccc}
\pi^{-1}_{1,u}TM & \xrightarrow{\gamma_{1,u}} & \text{Ker}(d_u \pi_1) \\
\downarrow (Df)_u & & \downarrow d_{uf_*} \\
\pi^{-1}_{2,f_*u}TM_2 & \xrightarrow{\gamma_{2,f_*u}} & \text{Ker}(d_{f_*u} \pi_2)
\end{array}
$$

To this end, note that the definition of the vertical lift (given in terms of local frames in §2) admits the following coordinate-free reformulation. Let $X = (u, v) \in \pi^{-1}_{1,u}TM_1$ and define the $C^\infty$ curve $c_{1,X} : (-\varepsilon, \varepsilon) \to V(M_1)$ by setting $c_{1,X}(t) = u + tv$, for $|t| < \varepsilon$, $\varepsilon > 0$. Then $\gamma_{1,u}X = \frac{dc_{1,X}}{dt}(0)$. Note that

$$
(6.11) \quad f_* \circ c_{1,X} = c_{2,D(f)_u x}.
$$
Using (6.11) we may perform the following calculation:

\[(d_u f^*) \circ \gamma_{1,u}(X) = d_0(f^* \circ c_{1,x}) \frac{d}{dt} \bigg|_{t=0} = (d_0 c_{2,D(f)_u x}) \frac{d}{dt} \bigg|_{t=0} = \]

\[= \gamma_{2,f_u} \circ (Df)_u X \]

Let us compose with \(\gamma_{2,f_u}^{-1}\) (at the left) in (6.10). We obtain:

\[(6.12) \quad K_{2,f_u} \circ (d_u f^*) = (Df)_u \circ K_{1,u} \]

i.e. the following diagram is commutative:

\[
\begin{array}{ccc}
T_{f_u}(V(M_2)) & \xrightarrow{K_{2,f_u}} & \pi_{2,f_u}^{-1} TM_2 \\
\downarrow d_u f^* & & \downarrow (Df)_u \\
T_u(V(M_1)) & \xrightarrow{K_{1,u}} & \pi_{1,u}^{-1} TM_1
\end{array}
\]

for any \(u \in V(M_1)\). Using (6.12) we have:

\[\theta_{2,F(f)(z)} \circ (d_z F(f)) = F(f)(z)^{-1} \circ K_{2,f_u} \circ (d_{F(f)(z)} \rho_2) \circ (d_z F(f)) = \]

\[= F(f)(z)^{-1} \circ K_{2,f_u} \circ (d_u f^*) \circ (d_z \rho_1) = \]

\[= F(f)(z)^{-1} \circ (Df)_u \circ K_{1,u} \circ (d_z \rho_1) = z^{-1} \circ K_{1,u} \circ (d_z \rho_1) = \theta_{1,z}^v \]

and the proof of Lemma 4 is complete. □

**Lemma 5.** Let \(H_1\) be a horizontal distribution in \(B_G(M_1) \to V(M_2)\) and \(H_2\) defined by (6.3). Then:

\[(6.13) \quad (d_z F(f)) H_1(\xi)_z = H_2(\xi) F(f)(z) \]

for any \(z \in B_G(M_1), \xi \in F\).

**Proof.** As a consequence of (6.8) we have \(\theta_{2,F(f)(z)} H_2(\xi) F(f)(z) - (d_z F(f)) H_1(\xi)_z = 0\) so that \(H_{2,F(f)(z)} - (d_z F(f)) H_1(\xi)_z \in H_{2,,F(f)(z)} \cap \text{Ker}(d_{F(f)(z)} \rho_2) = (0)\). □
Lemma 6. Let $\tau \subset (F^* \wedge F^*) \otimes (F)$ be a direct sumand to $\partial \operatorname{Hom}(F, \mathcal{G})$ and $H_1$ a horizontal distribution in $B_G(M_1)$. Let $H_2$ be given by (6.3). Then:

$$H_1 \in \mathcal{H}(\tau) \Rightarrow H_2 \in \mathcal{H}(\tau)$$

i.e. $c_{H_2}$ is $\tau$-valued.

Proof. Here $c_{H_2}$ is built from the data $(H_2, N_2)$, where $N_2$ is given by (6.6). Using the Lemmata 5 and 4 we have:

$$c_{H_2}(F(f)(z))(\xi \wedge \eta) = (d\theta_2)(H_2(\xi), H_2(\eta))_{F(f)(z)} =$$

$$= d(F(f)^*\theta_2)(H_1(\xi), H_1(\eta))_z = (d\theta_1)(H_1(\xi), H_1(\eta))_z = c_{H_1}(z)(\xi \wedge \eta)$$

so that the following diagram:

$$\begin{array}{ccc}
B_G(M_1) & \xrightarrow{c_{H_1}} & (F^* \wedge F^*) \otimes F \\
\downarrow & & \downarrow \circlearrowright \\
B_G(M_2) & \xrightarrow{c_{H_2}} & F(f)
\end{array}$$

is commutative. Our Lemma 6 is proved. \(\square\)

Finally, let $\tau = (z, \{H_1(e_i)_z, e_1(A_{\alpha})_z\})$ be a linear frame tangent to $B_G(M_1)$, adapted to the $G^{(1)}$-structure $B_G(M_1)^{(1),\tau}_{N_1}$, where $H_1 \in \mathcal{H}(\mathcal{G})$. Then:

$$L(F(f))(\tau) = (F(f)(z), \{(d_zF(f))H_1(e_i)_z, (d_zF(f))e_1(A_{\alpha})_z\}) =$$

$$= (F(f)(z), \{H_2(e_i)_{F(f)(z)}, e_2(A_{\alpha})_{F(f)(z)}\}) \in B_G(M_2)^{(1),\tau}_{N_2}$$

as a consequence of our Lemmata 1, 5 and 6. The inclusion $L(F(f))(B_G(M_1)^{(1),\tau}_{N_1}) \subseteq B_G(M_2)^{(1),\tau}_{N_2}$, yields (1.2) since $\dim_{\mathbb{R}} B_G(M_i)^{(1),\tau}_{N_i} = \dim_{\mathbb{R}} G^{(1)} + \dim_{\mathbb{R}} B_G(M_i) = \dim_{\mathbb{R}} G^{(1)} + \dim_{\mathbb{R}} G + 2n$, for $i = 1, 2$. Our Theorem 1 is completely proved.
7. Derived substructures.

Let \( B_G(M) \to V(M) \) be a derived \( G \)-structure on \( M \). If \( G' \) is a Lie subgroup of \( G \) then a \textit{derived substructure} is a principal \( G' \)-subbundle \( B_{G'}(M) \to V(M) \) of \( B_G(M) \to V(M) \). As far as the base manifold is the same, the theory of derived substructures is a direct extension of the classical theory of substructures of a given \( G \)-structure (cf. e.g. [23]) so that we allow ourselves to be somewhat sketchy. If \( H' \) is a horizontal distribution in \( B_{G'}(M) \) and \( H \) an extension of \( H' \) to \( B_G(M) \), then \( c_H \circ j = c_{H'} \), where \( j : B_{G'}(M) \to B_G(M) \) is the given imbedding. Both \( c_H, c_{H'} \) are built with respect to the same given nonlinear connection \( N \) on \( V(M) \) (fixed throughout §7). This follows from \( \theta' = j^* \theta \), where \( \theta, \theta' \) are the canonical 1-forms of \( B_G(M) \), and \( B_{G'}(M) \), respectively. The Lie algebra \( \mathcal{G}' \) of \( G' \) is a subalgebra of \( \mathcal{G} \). Then:

\[
\frac{(F^* \wedge F^*) \otimes F}{\partial \text{Hom}(F, \mathcal{G})} \cong \frac{(F^* \wedge F^*) \otimes F}{\partial \text{Hom}(F, \mathcal{G}')},
\]

(isomorphisms of linear spaces). Thus, for any \( z \in B_{G'}(M) \), \( c(j(z)) \) is the class of \( c'(z) \) modulo \( \partial \text{Hom}(F, \mathcal{G}') \).

More interesting seems to be the case where the base manifolds are distinct. Precisely, let \( B_H(A) \to V(A) \) be a derived \( H \)-structure on a \( C^\infty \) manifold \( A \) and \( N \) a nonlinear connection on \( V(A) \). Let \( G \subset H \) be a Lie subgroup, \( B_G(M) \to V(M) \) a derived \( G \)-structure on \( M \) and \( f : M \to A \) a \( C^\infty \) imbedding. Assume that there is an imbedding \( j : B_G(M) \to B_H(A) \) so that \( (j, f_*) \) is a principal bundle monomorphism. It is an open problem to relate the first structure functions of \( B_G(M) \), \( B_H(A) \). One should seek for the natural candidate for the nonlinear connection "induced" on \( V(M) \) by \( N \). If, for instance, \( A \) is a Finslerian manifold and \( N \) the nonlinear connexion of its unique regular Cartan-Chern connection then one may endow \( M \) with the nonlinear connection of the induced connection (and apply the theory developed in [1]).

8. Derived \( G \)-structures with reducible structure group.

Assume \( G \) is \textit{reducible}, i.e. there is a \( G \)-invariant proper subspace \( V \subset \mathbb{R}^n \). Let \( B_G(M) \to V(M) \) be a derived \( G \)-structure. For any \( u \in V(M) \) define \( \mathcal{Y}_u \subset \pi_u^{-1}TM \) as follows. Let \( z \in \rho^{-1}(u) \subset B_G(M) \) and set \( \mathcal{Y}_u = z(V) \). Then \( \mathcal{Y}_u \) is well defined (i.e. the definition does not depend upon the choice of
\[ z \in \rho^{-1}(u) \] due to the \( G \)-invariance of \( V \). Then \( \mathcal{Y} : u \mapsto \mathcal{Y}_u \) is a \( \pi \)-distribution on \( M \) (cf. [11], [6]). For any \( u \in V(M) \) define \( \mathcal{D}_u \subset T_u(V(M)) \) by setting \( \mathcal{D}_u = \beta_u \mathcal{Y}_u \oplus \gamma_u \mathcal{Y}_u \). Here \( \beta \) denotes the horizontal lift associated with the nonlinear connection \( N \) on \( V(M) \) (fixed throughout §8). Then \( \mathcal{D} \) is a \( 2p \)-dimensional distribution on \( V(M) \), \( p = \dim_{\mathbb{R}} V \). It is our purpose to formulate necessary and sufficient conditions for the integrability of \( \mathcal{D} \) in terms of the structure function \( c \) of \( B_G(M) \).

**Lemma 7.** Let \( Z \in \Gamma^\infty(T(V(M))) \) and \( \hat{Z} \in \Gamma^\infty(T(B_G(M))) \) so that \( \hat{Z} \) is \( \rho \)-related to \( Z \), i.e. \( (d\rho)_z \hat{Z}_z = Z_{\rho(z)} \), for any \( z \in B_G(M) \). Then \( Z \in \mathcal{D} \) if and only if \( \theta(\hat{Z})_z \in V \oplus V \) for any \( z \in B_G(M) \).

**Proof.** Let \( z \in B_G(M) \), \( u = \rho(z) \). Then \( Z_u \in \mathcal{D}_u \) iff \( (d\rho)_u \hat{Z}_z \in \mathcal{D}_u = (\beta \mathcal{Y})_u \oplus (\gamma \mathcal{Y})_u \) i.e. iff:

\[
(8.1) \quad (d\rho)_u \hat{Z}_z = \beta_u z(\xi) + \gamma_u z(\xi_2)
\]

for some \( \xi_1, \xi_2 \in V \). Let us apply \( L_u \), respectively \( K_u \), to the identity (8.1). Thus \( z^{-1} \circ L_u \circ (d\rho) \hat{Z}_z = \xi_1 \) and \( z^{-1} \circ K_u \circ (d\rho) \hat{Z}_z = \xi_2 \) which is equivalent to \( \theta^h(\hat{Z})_z \in V \) and \( \theta^v(\hat{Z})_z \in V \).

**Lemma 8.** Let \( H \) be a horizontal distribution in \( B_G(M) \to V(M) \). Then the following statements are equivalent:

i) \( \mathcal{D} \) is involutive.

ii) For any \( z \in B_G(M) \):

\[
(8.2) \quad c_H(z)((V \oplus V) \wedge (V \oplus V)) \subset V \oplus V
\]

**Proof.** Assume i) holds. We wish to compute \( c_H(z)(\xi \wedge \eta) \) for \( z \in B_G(M) \) and \( \xi, \eta \in V \oplus V \). Set \( \xi = \xi_1 \oplus \xi_2, \eta = \eta_1 \oplus \eta_2, \xi_i, \eta_i \in V, \ i = 1, 2 \). Then \( \beta_u z(\xi_1) + \gamma_u z(\xi_2) \) and \( \beta_u z(\eta_1) + \gamma_u z(\eta_2) \) are elements of \( \mathcal{D}_u \), where \( u = \rho(z) \). Next consider \( Y, Z \in \mathcal{D} \) so that \( Y_u = \beta_u z(\xi_1) + \gamma_u z(\xi_2) \) and \( Z_u = \beta_u z(\eta_1) + \gamma_u z(\eta_2) \). This choice is always possible (not unique) by standard theorems on the \( C^\infty \) extension of sections of a vector bundle (here \( \mathcal{D} \)) defined on some closed subset (here \( \{u\} \) of the base space. Let \( \hat{Y}, \hat{Z} \in \Gamma^\infty(H) \) be \( \rho \)-related to \( Y, Z \), respectively. Then

\[
(8.3) \quad H(\xi)_z = \hat{Y}_z.
\]

for \( z \in B_G(M) \) fixed above. Indeed, as both sides of (8.3) are horizontal (with respect to \( H \)) it is sufficient to show that \( \theta(\hat{Y}_z) = \xi \). This follows from the calculation below:

\[
\theta_z(\hat{Y}_z) = (\theta^h \hat{Y})_z \oplus (\theta^v \hat{Y})_z = (z^{-1} \circ L_u \circ (d\rho) \hat{Y}_z) \oplus (z^{-1} \circ K_u \circ (d\rho) \hat{Y}_z) =
\]
\[(z^{-1} L_u Y_u) \oplus (z^{-1} K_u Y_u) = \xi_1 \oplus \xi_2 = \xi.\]

Analogously \(H(\eta)_z = \widehat{Z}_z\). As \(\mathcal{D}\) is involutive \([Y, Z] \in \mathcal{D}\). On the other hand \([\widehat{Y}, \widehat{Z}]\) is \(\rho\)-related to \([Y, Z]\) (cf. Prop.1.3 in [15], vol I, p. 65) so that, by Lemma 7, \(\theta([\widehat{Y}, \widehat{Z}])_z \in V \oplus V\). Therefore:

\[c_H(z)(\xi \wedge \eta) = (d\theta)_z(H(\xi)_z, H(\eta)_z) = (d\theta)_z(\widehat{Y}_z, \widehat{Z}_z) =\]

\[= \frac{1}{2} \{\widehat{Y}_z(\theta(\widehat{Z})) - \widehat{Z}_z(\theta(\widehat{Y})) - \theta([\widehat{Y}, \widehat{Z}])_z\} \in V \oplus V\]

and (8.2) is proved. The proof of ii) \(\Rightarrow\) i) is similar and therefore left as an exercise to the reader.

Let:

\[\tau : (F^* \wedge F^*) \otimes F \to [(V \oplus V)^* \wedge (V \oplus V)^*] \otimes \left[ F \over (V \oplus V) \right]\]

be defined by \((\tau L)(\xi \wedge \eta) = \Phi(L(\xi \wedge \eta))\) for any \(L \in (F^* \wedge F^*) \otimes F\) and \(\xi, \eta \in V \oplus V\). Here \(\Phi : F \to {F \over (V \oplus V)}\) is the canonical map. As \(V\) is \(G\)-invariant, it is \(\mathcal{G}\)-invariant, as well. Thus:

\[\partial \text{Hom}(F, \mathcal{G}) \subset \text{Ker}(\tau).\]

and \(\tau\) induces a linear map:

\[\overline{\tau} : \left( {F^* \wedge F^*} \otimes F \over \partial \text{Hom}(F, \mathcal{G}^d') \right) \to [(V \oplus V)^* \wedge (V \oplus V)^*] \otimes \left[ F \over (V \oplus V) \right]\]

so that \(\tau = \overline{\tau} \circ \Psi\). At this point we may complete the proof of Theorem 2. Assume \(\mathcal{D}\) is involutive.

Then by Lemma 8, for any \(\xi, \eta \in V \oplus V\), \(c_H(z)(\xi \wedge \eta) \in V \oplus V\). Consequently \(\tau(c_H(z)(\xi \wedge \eta)) = \Phi(c_H(z)(\xi \wedge \eta))\) and then \(\overline{\tau}(z) = \overline{\tau}(\Psi(c_H(z))) = \tau(c_H(z)) = 0\) for any \(z \in B_G(M)\). The proof of the converse is similar (and thus omitted).
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