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ON DERIVED G-STRUCTURES

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We study the first order prolongations of derived G-structures (in the sense of P. Dazord [9]) on a differentiable manifold. We give necessary and sufficient conditions (in terms of structure functions) for the complete integrability of the differentiable system associated to a derived G-structure of reducible structure group.

1. Introduction and statement of main results.

In the present note we build on results in [8], [12] and study mainly prolongations of derived G-structures (in the sense of [9]).
Let M be a real n-dimensional C^∞ differentiable manifold and $T(M) \to M$ its tangent bundle. Then M admits a canonical imbedding in $T(M)$ as the zero cross-section, i.e. let $j : M \to T(M)$ be given by $j(x) = 0_x \in T_x(M)$, for any $x \in M$. Set $V(M) = T(M) \setminus j(M)$ and denote by $\pi : V(M) \to M$ the natural projection. Note that $V(M)$ is an open submanifold of $T(M)$. We shall need the pullback bundle $\pi^{-1}L(TM) \to V(M)$ of $L(TM)$ by π, where $L(TM) \to M$ is the principal $GL(n, \mathbb{R})$-bundle of linear frames tangent to M. Let G be a Lie subgroup of $GL(n, \mathbb{R})$. Then a *derived G-structure on M* is a principal G-subbundle $B_G(M) \to V(M)$ of $\pi^{-1}L(TM) \to V(M)$.

In general, if $F \to V$ is a real rank r vector bundle over a C^∞ manifold V, we denote by $L(F) \to V$ the principal $GL(r, \mathbb{R})$-bundle of frames in the fibres of

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F, i.e. $L(F)$ consists of the synthetic objects of the form $z = (u, \{f_1, \dots, f_r\})$ with $u \in V$ and $f_i \in F_u$, $1 \leq i \leq r$. Let $\pi^{-1}TM \to V(M)$ be the pullback of $T(M)$ by π. Then a *Finslerian G-structure* on M (cf. [12]) is a principal G-subbundle $B_G(M) \to V(M)$ of $L(\pi^{-1}TM) \to V(M)$. Note that $\pi^{-1}L(TM) \cong L(\pi^{-1}TM)$ (a principal $GL(n, \mathbb{R})$-bundle isomorphism) so that the two view points are equivalent (cf. also our §3). Nevertheless, the constructions of the first order structure functions (of a derived G-structure) in [9], [12] are distinct (the one in [12] depends upon the choice of a nonlinear connection on $V(M)$, and both approaches leave a number of open problems, as follows.

1) The connection between the developement of the theory of derived G-structures in [4], [9], [12] (and more recently [16]-[17]) is not fully understood, as yet.

2) None of the above theories has been applied to an example (other than derived $0(n)$-structures, i.e. Finslerian metrics).

3) There is no convenient notion of "flat" derived G-structures (cf. the comments in [8], pp. 380-381) and corresponding "adapted" coordinate systems.

4) No theory of "prolongations" of derived G-structures has been constructed, as yet (cf. [23] for the theory of prolongations of G-structures and their structure functions).

5) There is no integration of the general theory of derived G-structures with the (rather large) amount of the work done on the determination of the sets of Finslerian connections adapted to a specific derived G-structure (cf. [3], [13], [19], [20], [21], [22]) given in terms of tensor fields, such as Finslerian metrics, Finslerian conformal structures, Finslerian almost complex structures, etc.

The present paper is the first of a series in which the author hopes to address the above unanswered questions. Leaving definitions momentarily aside, we may formulate our main results as follows.

Theorem 1.

i) Let N be a nonlinear connection on $V(M)$, tau a direct sumand to $\partial\mathrm{Hom}\,(F, \mathcal{G})$ in $(F^* \wedge F^*) \otimes F$, and $B_G(M) \to V(M)$ a derived G-structure on M. Then its first prolongation $B_G(M)_N^{(1),\tau}$ is a $G^{(1)}$-structure on $B_G(M)$. If $\overline{\tau}$ is another complement the corresponding first prolongations of $B_G(M)$ are conjugate, i.e.

$$B_G(M)_N^{(1),\overline{\tau}} = B_G(M)_N^{(1),\tau} \rho(S) \tag{1.1}$$

for some $S \in \mathrm{Hom}\,(F, \mathcal{G})$.

ii) Let $B_G(M_i) \to V(M_i)$, $i = 1, 2$, be two isomorphic derived G-structures *and $f : M_1 \to M_2$ a diffeomorphism so that $F(f)(B_G(M_1)) = B_G(M_2)$. Let N_1 be a nonlinear connection on $V(M_1)$ and $N_{2,f_*(u)} = (d_u f_*)N_{1,u}$, for any $u \in V(M_1)$. Then N_2 is a nonlinear connection on $V(M_2)$ and:*

$$L(F(f))B_G(M_1)^{(1),\tau}_{N_1} = B_G(M_2)^{(1),\tau}_{N_2} \tag{1.2}$$

i.e. the first prolongations of $B_G(M_i)$, $i = 1, 2$, are isomorphic.

In §2 we recollect the material we need on nonlinear connections, horizontal lifts and the Dombrowski map (cf. [10], [14]). The frame bundle technique we use is presented in §3 together with a comparison between the formalism in [9],[12] (cf. our Proposition 1). The first structure function of a derived G-structure is introduced in § 4 in a form close to that in [12] (we use an arbitrary nonlinear connection rather then the nonlinear connection of a given regular connection in $\pi^{-1}TM$, and employ properties of the "standard" horizontal vector fields derived in [1]). The sections §5 - §6 are devoted to the proof of our Theorem 1. Especially the proof of the fact that our prolongations give first order information on isomorphism (of derived G-structures) is more delicate (than its classical counterpart in [23]) and organized in our Lemmae 1 to 6. Derived substructures are succintly studied in §7 where we also hint to some open problem.

Let G be a reducible Lie subgroup of $GL(n, \mathbb{R})$, i.e. there is a proper subspace $V \subseteq \mathbb{R}^n$ invariant by G. In the presence of a derived G-structure, V gives rise to a π-distribution (in the sense of [11]) $\mathscr{V}$ on M. Next $\mathscr{V}$ lifts to a Pfaffian system $\mathscr{D}$ on $V(M)$ (cf. §8) whose integrability in adressed in the following:

Theorem 2. *Let $B_G(M) \to V(M)$ be a derived G-structure on M, N a nonlinear connection on $V(M)$, and $\beta : \pi^{-1}TM \to N$ the corresponding horizontal lift. Assume G is reducible and let $\mathscr{V}$ be the associated π-distribution on M. Then $\mathscr{D} = \beta\mathscr{V} \oplus \gamma\mathscr{V}$ is involutive if and only if the first structure function*

$$c : B_G(M) \to \frac{((F^* \wedge F^*) \otimes F)}{\partial \mathrm{Hom}\,(F, \mathscr{G})}$$

of $B_G(M)$ is $\mathrm{Ker}\,(\overline{\tau})$*-valued.*

2. Finslerian metrics and nonlinear connections.

The pullback bundle $\pi^{-1}TM \to V(M)$ plays (within Finslerian geometry) a role which is similar to that of the tangent bundle in Riemannian geometry. Precisely, let $E : T(M) \to [0, +\infty)$. Then E is a Finslerian energy function if

i) $E \in C^1(T(M))$, $E \in C^\infty(V(M))$, ii) E is positive homogeneous of degree 2, i.e. $E(\lambda u) = \lambda^2 E(u)$ for any $\lambda > 0$, $u \in T(M)$, iii) $E(u) = 0 \Leftrightarrow u = 0$. To formulate the last axiom, let (U, x^i) be a local coordinate system on M and $(\pi^{-1}(U), x^i, y^i)$ the naturally induced local coordinates on $V(M)$. Define $g_{ij} : \pi^{-1}(U) \to \mathbb{R}$ by setting:

$$g_{ij}(u) = \frac{1}{2} \frac{\partial^2 E}{\partial y^i \partial y^j}(u)$$

for any $u \in \pi^{-1}(U)$. We request that iv) $g_{ij}(u)\,\xi^i \xi^j \geq 0$ and $= 0 \Leftrightarrow \xi^i = 0$, for any $u \in \pi^{-1}(U)$ and $(\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$, that is the quadratic form $g_{ij}(u)\xi^i\xi^j$ should be positive-definite. A pair (M, E) is a *Finslerian manifold*. The pullback bundle $\pi^{-1}TM$ of a Finslerian manifold (M, E) is a Riemannian bundle in a natural way. Indeed, let $X : M \to T(M)$ be a tangent vector field on M. Its *natural lift* is the cross-section $\overline{X} : V(M) \to \pi^{-1}TM$ defined by $\overline{X}(u) = (u, X(\pi(u)))$, for any $u \in V(M)$. Cross-sections in $\pi^{-1}TM$ are usually referred to as *Finslerian vector fields* on M. Let X_i be the natural lift of the (local) tangent vector field $\dfrac{\partial}{\partial x^i}$, $1 \leq i \leq n$. Then $\{X_1, \ldots, X_n\}$ is a frame field in $\pi^{-1}TM$ on $\pi^{-1}(U)$. Finally, we define an inner product g_u on $\pi_u^{-1}TM = \{u\} \times T_{\pi(u)}(M)$ by setting $g_u(X, Y) = g_{i,j}(u)\xi^i\xi^j$, for any $X, Y \in \pi_u^{-1}TM$, where $X = \xi^i X_i(u)$, $Y = \eta^i X_i(u)$. The definition of $g_u(X, Y)$ does not depend upon the choice of local coordinates (U, x^i) at $x = \pi(u)$ and $u \mapsto g_u$ is a Riemannian bundle metric on $\pi^{-1}TM$.

A C^∞ distribution N on $V(m)$ is a *nonlinear connection* on $V(M)$ if

$$T_u V(M) = N_u \oplus \text{Ker}\,(d_u \pi) \tag{2.1}$$

for any $u \in V(M)$. Cf. also [14].

Define a bundle morphism $L : T(V(M)) \to \pi^{-1}TM$ by $L_u X = (u, (d_u\pi)X)$, for any $u \in V(M), X \in T_u(V(M))$. Given a nonlinear connection N on $V(M)$ the restriction $L : N \to \pi^{-1}TM$ is a vector bundle isomorphism. Set $\beta_u = (L_u|_{N_u})^{-1}$ for any $u \in V(M)$. The bundle isomorphism $\beta : \pi^{-1}TM \to N$ is termed *horizontal lift* (with respect to N).

As to local computations, set $\delta_i = \beta X_i$. Then $\{\delta_i\}$ is a frame field in N on $\pi^{-1}(U)$. One may seek δ_i as a linear combination $\delta_i = M_i^j\, \partial_j - N_i^j\, \dot{\partial}_j$, where $\partial_i = \dfrac{\partial}{\partial x^i}$, $\dot{\partial}_i = \dfrac{\partial}{\partial y^i}$ for the sake of simplicity. Apply L so that to yield $M_j^i = \delta_j^i$ (as $L\,\partial_i = X_i$ and $L\,\dot{\partial}_i = 0$). The remaining (uniquely determined) functions $N_j^i : \pi^{-1}(U) \to \mathbb{R}$ are the *coefficients of the nonlinear connection*

N (with respect to (U, x^i)). Let $x'^i = x'^i(x^1, \ldots, x^n)$, $\det\left[\frac{\partial x'^i}{\partial x^j}\right] \neq 0$ on $U \cap U' \neq \phi$, be a transformation of local coordinates on M. Taking into account the identities:

$$X_i = \left(\frac{\partial x'^j}{\partial x^i} \circ \pi\right) X'_j$$

$$\partial_i = \frac{\partial x'^j}{\partial x^i} \partial'_j + \frac{\partial^2 x'^j}{\partial x^i \partial x^k} y^k \dot{\partial}'_j$$

$$\dot{\partial}_i = \frac{\partial x'^j}{\partial x^i} \dot{\partial}'_j$$

$$\delta_i = \partial_i - N_i^j \dot{\partial}_J \qquad \delta'_i = \partial'_i - N_i'^j \dot{\partial}'_j$$

one obtains:

$$\delta_i = \frac{\partial x'^j}{\partial x^i} \delta'_j + \left\{ \frac{\partial x'^j}{\partial x^i} N_j'^k - \frac{\partial x'^k}{\partial x^j} N_i^j + \frac{\partial^2 x'^k}{\partial x^i \partial x^j} y^i \right\} \dot{\partial}'_k . \tag{2.2}$$

Finally, as a conseguence of (2.2) and of the uniqueness of the direct sum decomposition (cf. (2.1)) it follows that the coefficients of the nonlinear connection N satisfy the transformation law:

$$\frac{\partial x'^j}{\partial x^i} N_j'^k = \frac{\partial x'^k}{\partial x^j} N_i^j - \frac{\partial^2 x'^k}{\partial x^i \partial x^j} y^j . \tag{2.3}$$

Viceversa a set of C^∞ functions N_j^i obeying (2.3) under any coordinate transformation $x'^i = x'^i(x^1, \ldots, x^n)$ determines a nonlinear connection on $V(M)$ by setting $N_u = \sum_{i=1}^{n} \mathbb{R}(\partial_i - N_i^j \dot{\partial}_j)_u$. The definition of N_u does not depend (by (2.3)) upon the choice of local coordinates (U, x^i) at $\pi(u)$.

Examples.

1) Let Γ_{jk}^i be a linear connection on M. Then $N_j^i(x, y) = \Gamma_{jk}^i(x) y^k$ is a nonlinear connection on $V(M)$.

2) Let $\mathscr{L} : V(M) \to \pi^{-1}TM$ be the Finslerian vector field given by $\mathscr{L}(u) = (u, u)$, for any $u \in V(M)$. Then $\mathscr{L}$ is referred to as the *Liouville vector field.* Let ∇ be a connection in $\pi^{-1}TM \to V(M)$. A tangent vector field X on $V(M)$ is *horizontal* (with respect to ∇) if $\nabla_X \mathscr{L} = 0$. The *horizontal distribution* $N(\nabla) : u \to N(\nabla)_u \subset T_u(V(M))$ of ∇ consists of all $Y \in T_u(V(M))$ so that there is a horizontal tangent vector field X on $V(M)$ with $X(u) = Y$. If

$N(\nabla)$ is a nonlinear connection on $V(M)$ then ∇ is termed *regular*. Cf. also [2]. If (M, E) is a Finslerian manifold, let ∇ be the Cartan-Chern connection in $(\pi^{-1}TM, g)$. Cf. [5], [7]. Then ∇ is regular. Its nonlinear connection $N(\nabla)$ is (locally) given by:

$$N^i_j = \frac{1}{2}\overset{\bullet}{\partial}_j \begin{vmatrix} i \\ 0\,0 \end{vmatrix}$$

$$\begin{vmatrix} i \\ 0\,0 \end{vmatrix} = \begin{vmatrix} i \\ j\,k \end{vmatrix} y^i y^j$$

$$\begin{vmatrix} i \\ j\,k \end{vmatrix} = g^{im} \,|\, jk,\, m|$$

$$|\, ij,\, k| = \frac{1}{2}\,(\partial_i\, g_{jk} + \partial_j\, g_{ik} - \partial_k\, g_{ij})$$

In general, a pair (∇, N) consisting of a connection ∇ in the vector bundle $\pi^{-1}TM$ and a nonlinear connection N on $V(M)$ is called a *Finslerian connection.* Any regular connection gives rise to a Finslerian connection. The converse is false for most of the "canonical" connections of Finslerian geometry (e.g. the Berwald and Rund connections (cf. [18]) are not regular).

3) There is yet another way to look at the nonlinear connection of the Cartan-Chern connection. Let $\gamma : \pi^{-1}TM \to \text{Ker}\,(d\pi)$ be defined by $\gamma X_i = \overset{\bullet}{\partial}_i$. Then γ is a (globally defined) bundle isomorphism referred to as the *vertical lift.* The *Dombrowski map* is the bundle morphism $K : T(V(M)) \to \pi^{-1}TM$ given $K_u = \gamma_u^{-1} \circ Q_u$, $u \in V(M)$, where $Q_u = T_u(V(M)) \to \text{Ker}(d_u\pi)$ is the natural projection associated with (2.1). Therefore the construction of K depends on a given fixed nonlinear connection N on $V(M)$. Cf. also [10]. The *Sasaki metric* of a Finslerian manifold (M, E) is the Riemannian metric G on $V(M)$ defined by:

$$G(X, Y) = g(LX, LY) + g(KX, KY)$$

for any $X, Y \in \Gamma^\infty(T(V(M)))$. Here the Dombrowski map K is built with respect to the nonlinear connection $N(\nabla)$ of the Cartan-Chern connection of (M, E). Let N_u be the orthogonal complement of $\text{Ker}(d_u\pi)$ in $T_u(V(M))$ (with respect to G_u), $u \in V(M)$. Then N is a nonlinear connection on $V(M)$ and $N = N(\nabla)$.

3. Finslerian frame bundles and canonical 1-forms.

Let $\Phi : \pi^{-1}L(TM) \to L(\pi^{-1}TM)$ be given by

$$\Phi(z) = (u, \{(u, X_1), \ldots, (u, X_n)\})$$

for any $z = (u, b) \in \pi^{-1}L(TM)$, where $b = (x, \{X_1, \ldots, X_n\}) \in L(TM)$. Then Φ is a principal $GL(n, \mathbb{R})$-bundle isomorphism.
P. Dazord defines (cf. [9], p. 2730) a 1-form

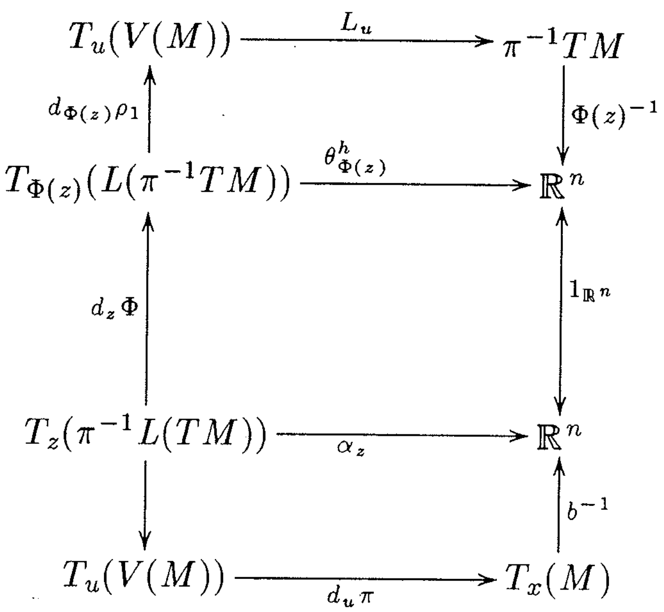
$$\alpha \in \Gamma^\infty(T^*(\pi^{-1}L(TM)) \otimes \mathbb{R}^n)$$

as follows $\alpha_z = b^{-1} \circ (d_z(\pi\rho))$, $z = (u, b)$, where $\rho : \pi^{-1}L(TM) \to V(M)$ is the natural projection. Note that α is the h-basic form of [18], p. 48. On the other hand, together with [12], we may define the 1-form $\theta^h \in \Gamma^\infty(T^*(L(\pi^{-1}TM)) \otimes \mathbb{R}^n)$ by setting $\theta^h_z = z^{-1} \circ L_u \circ (d_z \rho_1)$, for any $z = (u, \{X_i\})$. Where $X_i \in \pi_u^{-1}TM$ and $\rho_1 : L(\pi^{-1}TM) \to V(M)$ is the natural projection.

Proposition. *The* 1*-forms* α, θ^h *coincide up to an isomorphism, i.e.*

$$\alpha_z = \theta^h_{\Phi(z)} \circ (d_z \Phi) \tag{3.1}$$

Proof. To establish (3.1) we look at the following diagram:



where $z = (u, b)$ and $x = \pi(u)$. As the upper and lower rectangles are commutative, it is sufficient to check the commutativity of the big rectangle.

Taking into account $\rho_1 \circ \Phi = \rho$ and $\Phi(z)^{-1}(u,X) = b^{-1}(X)$, for any $X \in T_x(M)$, we may conduct the following calculation:

$$\Phi(z)^{-1} \circ L_u \circ (d_{\Phi(Z)}\rho_1) \circ (d_z\Phi) = \Phi(z)^{-1} \circ L_u \circ (d_z\rho) =$$

$$= \Phi(z)^{-1}(u,(d_u\pi)(d_z\rho)) = b^{-1} \circ d_z(\pi\rho). \qquad \square$$

In addition to the h-basic 1-form we define $\theta^v \in \Gamma^\infty(T^*(L(\pi^{-1}TM)) \otimes \mathbb{R}^n)$ as follows. Let N be a fixed nonlinear connection on $V(M)$ and $K : T(V(M)) \to \pi^{-1}TM$ the corresponding Dombrowski map.
Set $\theta^v_z = z^{-1} \circ K_u \circ (d_z\rho_1)$, for any $z \in L(\pi^{-1}TM), u = \rho_1(z)$.
If $z = (u,\{X_i\})$ then $z : \mathbb{R}^n \to \pi_u^{-1}TM$ is given by $z(e_i) = X_i$ where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$. Together with [11] let us define $\theta \in \Gamma^\infty(T^*(L(\pi^{-1}TM))\otimes F)$ by $\theta = \theta^h \oplus \theta^v$ where $F = \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$. We may emphasize the importance of considering the 1-form θ (rather than θ^h or θ^v alone) as follows. Let H be a *horizontal distribution* in $L(\pi^{-1}TM) \to V(M)$, that is the following direct sum decomposition holds:

$$T_z(L(\pi^{-1}TM)) = H_z \oplus \text{Ker}\,(d_z\rho)$$

for any $z \in L(\pi^{-1}TM)$. From now on we do not distinguish between $\pi^{-1}L(TM)$ and $L(\pi^{-1}TM)$, (respectively between ρ and ρ_1). Set $t_z = (d_z\rho)_{|H_z}$, $z \in L(\pi^{-1}TM)$. Then $t_z : H_z \to T_u(V(M))$ is an $\mathbb{R}$-linear isomorphism, $u = \rho(z)$. Note that neither the h-basic nor the v-basic 1-forms may play the role of the canonical 1-form in [15], vol. I, p. 118, as their restrictions $\theta^h_z, \theta^v_z : H_z \to \mathbb{R}^n$ are not isomorphisms.
Indeed Ker $(\theta^h_z) = \text{Ker}(d_z\rho) \oplus t_z^{-1}(\text{Ker}(d_u\pi))$ and Ker $(\theta^v_z) = \text{Ker}(d_z\rho) \oplus t_z^{-1}(N_u), u = \rho(z)$. However $\text{Ker}(\theta_z) = \text{Ker}(d_z\rho)$ so that $\theta_z : H_z \to F$ is a $\mathbb{R}$-linear isomorphism. Then θ is referred to as the *canonical* 1-*form* of (M,N).

4. Structure functions.

Let $B_G(M) \to V(M)$ be a derived G-structure and

$$\theta \in \Gamma^\infty(T^*(B_G(M)) \otimes F)$$

the 1-form induced on $B_G(M)$ by the canonical 1-form of (M,N). Here $F = \mathbb{R}^{2n}$ and the nonlinear connection N is fixed (throughout §4). Together with [12] let us define $C_H : B_G(M) \to (F^* \wedge F^*) \otimes F$, for a given fixed horizontal distribution H in $B_G(M) \to V(M)$, as follows. Let $\xi \in F$ and denote by $H(\xi) \in \Gamma^\infty(H)$ the tangent vector field on $B_G(M)$ defined by $\theta(H(\xi)) = \xi$. Note that $H(\xi)$ is well defined (as $\theta_z : H_z \to F$ is an isomorphism, for any $z \in B_G(M)$) and C^∞ differentiable. Cf. [1], $H(\xi)$ possesses properties which are similar to those of the standard horizontal vector fields in [15], vol. I, p. 119. However $H(\xi)$ depends on the choice of nonlinear connection N on $V(M)$, in addition to the data (H, ξ). Let $\xi, \eta \in F$ and set:

$$c_H(z)(\xi \wedge \eta) = (d\theta)_z(H(\xi)_z, H(\eta)_z).$$

Let $\partial : \mathrm{Hom}(F, \mathcal{G}) \to (F^* \wedge F^*) \otimes F$, where $\mathcal{G}$ is the Lie algebra of G, be defined by $(\partial T)(\xi \wedge \eta) = T(\xi)\eta - T(\eta)\xi$, for any $T \in \mathrm{Hom}(F, \mathcal{G})$ and any $\xi, \eta \in F$. Here $\mathcal{G}$ acts canonically on $F = \mathbb{R}^n \oplus \mathbb{R}^n$, i.e. if $A \in \mathcal{G}$ and $\xi = \xi_1 \oplus \xi_2 \in F$ then $A\xi = A\xi_1 \oplus A\xi_2$.

Let H, H' be two horizontal distributions in $B_G(M) \to V(M)$. Then:

$$c_H(z) - c_{H'}(z) = \frac{1}{2}\, \partial T \tag{4.1}$$

for some $T \in \mathrm{Hom}(F, \mathcal{G})$ depending only on H, H', and for any $z \in B_G(M)$. For the sake of completeness, let us prove (4.1). Cf. also Theor.4.1 in [12]. As $H(\xi)_z - H'(\xi)_z \in \mathrm{Ker}(\theta_z) = \mathrm{Ker}(d_z\rho)$, there is $T \in \mathrm{Hom}(F, \mathcal{G})$ so that $H'(\xi)_z - H(\xi)_z = T(\xi)^*_z$. Here, for each $A \in \mathcal{G}$, we denote by $A^* \in \Gamma^\infty(\mathrm{Ker}(d\rho))$ the fundamental vector field associated with A, i.e. $A^*_z = (d_e L_z)A_e$ for any $z \in B_G(M)$. Here $e \in G$ is the unit $n \times n$ matrix, while $L_z : G \to B_G(M)$ is given by $L_z(g) = zg$, for any $g \in G$. Then:

$$c_H(z)(\xi \wedge \eta) - c_{H'}(z)(\xi \wedge \eta) =$$

$$= \frac{1}{2}\{\theta_z([H(\xi), T(\eta)^*]) - \theta_z([H(\eta), T(\xi)^*])\} = \frac{1}{2}\{T(\xi)\eta - T(\eta)\xi\}$$

Here we made use of a formula in [1], i.e. $[A^*, H(\xi)] = H(A\xi)$, for any $A \in \mathcal{G}$, $\xi \in F$. Finally, let $c : B_G(M) \to ((F^* \wedge F^*) \otimes F) / \partial\mathrm{Hom}(F, \mathcal{G})$ be defined by $c(z) = \Psi(c_H(z))$ for any $z \in B_G(M)$ and any horizontal distribution H in $B_G(M)$, where

$$\Psi : (F^* \wedge F^*) \otimes F) \to ((F^* \wedge F^*) \otimes F) / \partial\,\mathrm{Hom}(F, \mathcal{G})$$

is the natural map. Then c is well defined as a consequence of (4.1) and is referred to as *the first structure function* of the derived G-structure $B_G(M)$. This appears to be distinct from the structure functions in [4], [9] and the relation between the three is not fully clear.

5. Prolongations of derived G-structures.

Let $\mathscr{G}^{(1)} = \operatorname{Ker}(\partial) \subset \operatorname{Hom}(F, \mathscr{G})$ be the *first prolongation* of G. Next consider $\rho : \mathscr{G}^{(1)} \to \operatorname{End}_{\mathbb{R}}(F \oplus \mathscr{G})$ given by $\rho(T)(\xi, A) = (\xi, T(\xi) + A)$, for any $T \in \mathscr{G}^{(1)}$, $\xi \in F$, $A \in \mathscr{G}$. Then ρ is a representation of the additive group $\mathscr{G}^{(1)}$ on $F \oplus \mathscr{G}$.
Let $\{e_1, \ldots, e_{2n}\}$ be the canonical basis of F and $\{A_1, \cdots, A_{n^0}\}$ a fixed basis of $\mathscr{G}$, $n^0 = \dim_{\mathbb{R}} \mathscr{G}$. Let $h : \operatorname{End}_{\mathbb{R}}(F \oplus \mathscr{G}) \to GL(2n + n^0, \mathbb{R})$ be the isomorphism associated with the linear basis $\{(e_i, 0), (0, A_\alpha)\}$ of $(F \oplus \mathscr{G})$. Then $G^{(1)} = h(\rho(\mathscr{G}^{(1)}))$ is a Lie subgroup of $GL(2n + n^0, \mathbb{R})$, i.e. the *first prolongation* of G.

Let τ be a direct sumand to $\partial\operatorname{Hom}(F, \mathscr{G})$ in $(F^* \wedge F^*) \otimes F$. Let $\mathscr{H}(\tau)$ be the set of all horizontal distributions H in $B_G(M)$ so that c_H is τ-valued. Clearly $\mathscr{H}(\tau)$ depends on a fixed nonlinear connection N on $V(M)$, as well. Note that given $H \in \mathscr{H}(\tau)$ the rest of the horizontal distributions in $\mathscr{H}(\tau)$ are parametrized by elements of $\mathscr{G}^{(1)}$. Indeed $H'(\xi)_z - H(\xi)_z = T(\xi)^*_z$ for some $T \in \operatorname{Hom}(F, \mathscr{G})$ depending only on $H, H' \in \mathscr{H}(\tau)$. Then (4.1) yields $\partial T \in \tau \cap \partial\operatorname{Hom}(F, \mathscr{G}) = (0)$.

Define $B^{(1)} = B_G(M)^{(1),\tau}_N$ to be the set of all linear frames tangent to $B_G(M)$ of the form $(z, \{H(e_i)_z, A^*_{\alpha,z}\})$ for any $z \in B_G(M)$ and any $H \in \mathscr{H}(\tau)$. Let $\pi^{(1)} : L(TB_G(M)) \to B_G(M)$ be the principal $GL(2n + n^0, \mathbb{R})$-bundle of linear frames tangent to $B_G(M)$. To prove that $B^{(1)} \to B_G(M)$ is a $G^{(1)}$-structure note firstly that $\pi^{(1)}(B^{(1)}) = B_G(M)$. Also, it is clear from the definition that for any $z \in B_G(M)$ there is $U \subseteq B_G(M))$ open, $z \in U$, and there is a cross-section $\sigma : U \to L(TB_G(M))$ so that $\sigma(U) \subseteq B^{(1)}$. As $B^{(1)}$ is already a submanifold of $L(TB_G(M))$ it remains to be shown that given $r \in B^{(1)}$ and $a \in G^{(1)}$ we have $ra \in B^{(1)} \Leftrightarrow a \in G^{(1)}$. Note that

$$G^{(1)} = \left\{ \begin{bmatrix} \delta_i^j & 0 \\ T(e_i)^\alpha & \delta_\beta^\alpha \end{bmatrix} : T \in \mathscr{G}^{(1)} \right\}$$

where $T(e_i) = T(E_i)^\alpha A_\alpha$. Next, if $r = (z, \{H(e_i)_z, A^*_{\alpha,z}\}) \in B^{(1)}$ then $ra = (z, \{H(e_j)_z\, a_i^j + A^*_{\alpha,z}\, a_i^\alpha, H(e_i)_z\, a_\alpha^i + A^*_{\beta,z}\, a_\alpha^\beta\})$ where $a = \begin{bmatrix} a_i^j & a_\beta^j \\ a_i^\alpha & a_\beta^\alpha \end{bmatrix} \in GL(2n + n^0, \mathbb{R})$. Thus $ra \in B^{(1)}$ if and only if:

$$H(e_j)_z\, a_i^j + A^*_{\alpha,z}\, a_i^\alpha = H'(e_i)_z \tag{5.1}$$

$$H(e_i)_z\, a_\alpha^i + A^*_{\beta,z}\, a_\alpha^\beta = A^*_{\alpha,z} \tag{5.2}$$

for some $H' \in \mathscr{H}(\tau)$. Apply θ_z to both (5.1) - (5.2) so that to get $a_i^j = \delta_i^j$ and $a_\alpha^i = 0$. Again (5.2) gives $a_\alpha^\beta = \delta_\alpha^\beta$. Finally (5.1) yields $T(e_i)_z^* = A_{\alpha,z}^* \, a_i^\alpha$ for some $T \in \mathscr{G}^{(1)}$. Thus $a \in G^{(1)}$. □

To justify the second statement in i) of our Theorem 1, let $\overline{\tau}$ so that $(F^* \wedge F^*) \otimes F = \overline{\tau} \oplus \partial\mathrm{Hom}(F, \mathscr{G})$. Set $\overline{B}^{(1)} = B_G(M)_N^{(1),\overline{\tau}}$ for brevity. Let $\overline{r} = (z, \{\overline{H}(e_i)_z, A_{\alpha,z}^*\}) \in \overline{B}^{(1)}$, $r = (z, \{H(e_i)_z, A_{\alpha,z}^*\})$ for some $\overline{H} \in \mathscr{H}(\overline{\tau})$ and $H \in \mathscr{H}(\tau)$. Then $\overline{H}(\xi) - H(\xi) = S(\xi)^*$ for some $s \in \mathrm{Hom}(F, \mathscr{G})$ and $\xi \in F$. It follows thet $\overline{r} = r\rho(s)$, where

$$\rho(s) = \begin{bmatrix} \delta_i^j & 0 \\ T(e_i)^\alpha & \delta_\beta^\alpha \end{bmatrix} \in GL(2n + n^0, \mathbb{R}). \qquad \square$$

6. Isomorphic derived G-structures.

Let $B_G(M_i) \to V(M_i)$, $i = 1, 2$, be two derived G-structures. Then $B_G(M_i)$, $i = 1, 2$, are said to be *isomorphic* if there is a diffeomorphism $f : M_1 \to M_2$ so that $F(f)(B_G(M_1)) = B_G(M_2)$, where $F(f) : \pi_1^{-1}L(TM_1) \to \pi_2^{-1}L(TM_2)$ is defined by $F(f)(u, b) = \big(f_* u, L(f)(b)\big)$, for any $z = (u, b) \in \pi_1^{-1}L(TM_1)$. Here $f_* : V(M_1) \to V(M_2)$ denotes the differential of f while $L(f) : L(TM_1) \to L(TM_2)$ is the naturally induced bundle map, i.e. $L(f)(b) = \big(f(x), \{(d_x f)X_i\}\big)$ for any $b = \big(x, \{X_i\}\big) \in L(TM_1)$.

If $A \in \mathscr{G}$ then $\ell_i(A) \in \Gamma^\infty(\mathrm{Ker}\,(d\rho_i))$, $i = 1, 2$, denotes the fundamental vector field associated with A (previously denoted by A^*). At this point we may prove ii) of our Theorem 1. To this end we shall need the following:

Lemma 1. *Let $B_G(M_i) \to V(M_i)$, $i = 1, 2$, be two isomorphic derived G-structures and $f : M_1 \to M_2$ a diffeomorphism so that $F(f)B_G(M_1) = B_G(M_2)$. Then:*

$$F(f)_* \ell_1(A) = \ell_2(A). \tag{6.1}$$

Proof. Let $z \in B_G(M_1)$ and $L_{1,z} : G \to B_G(M_1)$ given by $L_{1,z}(g) = zg$, for any $g \in G$. Then:

$$F(f) \circ L_{1,z}(g) = F(f)(zg) = F(f)(u, bg) = \Big((d_x f)u,\ L(f)(bg)\Big) =$$

$$= \Big((d_x f)u,\ L(f)(b)g\Big) = F(f)(u, b)g$$

for any $z = (u, b) \in B_G(M_1)$, $g \in G$, where $x = \pi_1(u) \in M_1$. We have obtained:

$$F(f) \circ L_{1,z} = L_{2,F(f)(z)}. \tag{6.2}$$

Taking into account (6.2) we may conduct the following calculation:

$$\Big(d_z F(f)\Big)\ell_1(A)_z = \Big(d_z F(f)\Big) \circ \Big(d_e L_{1,z}\Big)A_e = d_e\Big(F(f) \circ L_{1,z}\Big)A_e =$$

$$= d_e(L_{2,F(f)(z)})A_e = \ell_2(A)_{F(f)(z)}. \qquad \square$$

Set $P_i = \pi_i^{-1}L(TM_i)$, $i = 1, 2$, for simplicity. The diffeomorphism $F(f) : P_1 \to P_2$ induces the natural bundle map

$$L(F(f)) : L(TP_1) \to L(TP_2),$$

$$L(F(f))\,(z, \{Z_a\}) = \big(F(f)(z), \{(d_z F(f) Z_a\}\big),\ z \in P_1, Z_a \in T_z(P_1).$$

This is the map in (1.2). We shall need:

Lemma 2. *Let H_1 be a horizontal distribution in $B_G(M_1) \to V(M_1)$ and $H_{2,F(f)(z)} \subset T_{F(f)(z)}(B_G(M_2))$ defined by:*

$$H_{2,F(f)(z)} = (d_z F(f))H_{1,z} \tag{6.3}$$

for any $z \in B_G(M_1)$. Then H_2 is a horizontal distribution in

$$B_G(M_2) \to V(M_2).$$

Proof. Note that:

$$(d_z F(f))\,\mathrm{Ker}\,(d_z\rho_1) = \mathrm{Ker}\,(d_{F(f)(z)}\rho_2). \tag{6.4}$$

This follows from the identity:

$$\rho_2 \circ F(f) = f_* \circ \rho_1. \tag{6.5}$$

Indeed, it is sufficient (since both sides in (6.4) have the same dimension) to check the inclusion "$\subseteq$". To this end, let $X \in \mathrm{Ker}\,(d_z\rho_1)$. Then:

$$\big(d_{F(f)(z)}\rho_2\big) \circ \big(d_z F(f)\big)\,X = d_z(\rho_2 \circ F(f))X =$$

$$= d_z(f_* \circ \rho_1)X = (d_u f_*) \circ (d_z\rho_1)X = 0\,.$$

Applying $d_z F(f)$ to: $T_z\big(B_G(M_1)\big) = H_{1,z} \oplus \mathrm{Ker}\,(d_z\rho_1)$ and using (6.3) - (6.4) shows that $T_{F(f)(z)}\big(B_G(M_2)\big)$ may be written as the sum of $H_{2,F(f)(z)}$ and $\mathrm{Ker}\,(d_{F(f)(z)}\rho_2)$. As $d_z F(f)$ commutes with the intersection the sum is also direct. $\square$

Lemma 3. *Let N_1 be a nonlinear connection on $V(M_1)$ and $N_{2,f_*(u)} \subset T_{f_*(u)}\big(V(M_2)\big)$ defined by:*

$$N_{2,f_*(u)} = (d_u f_*) N_{1,u} \tag{6.6}$$

for any $u \in V(M_1)$. Then N_2 is a nonlinear connection on $V(M_2)$.

Proof. Note that:

$$(d_u f_*)\,\mathrm{Ker}\,(d_u \pi_1) = \mathrm{Ker}\,(d_{f_*(u)} \pi_2). \tag{6.7}$$

As both sides in (6.7) have the same dimension, it is sufficient to check one inclusion. Let $X \in \mathrm{Ker}\,(d_u\pi_1)$. Then:

$$(d_{f_*(u)}\pi_2) \circ (d_u f_*)X = d_u(\pi_2 \circ f_*)\,X = d_u(f \circ \pi_1)\,X = 0$$

Finally, let us apply $d_u f_*$ to $T_u(V(M_1)) = N_{1,u} \oplus \mathrm{Ker}\,(d_u\pi_1)$, etc. □

Lemma 4. *Let N_1 be a nonlinear connection on $V(M_1)$ and N_2 the nonlinear connection given by (6.6). Let $\theta_i \in \Gamma^\infty\big(T^*(B_G(M_i)) \otimes F\big)$ be the canonical 1-form of $B_G(M_i)$, built with respect to N_i, $i = 1, 2$. Then:*

$$\theta_{2,F(f)(z)} \circ \Big(d_z F(f)\Big) = \theta_{1,z} \tag{6.8}$$

for any $z \in B_G(M_1)$. Here $F = \mathbb{R}^{2n}$, $n = \dim\,(M_i)$, $i = 1, 2$.

Proof. The following diagram is commutative:

$$\begin{array}{ccc} T_u(V(M_1)) & \xrightarrow{\;d_u f_*\;} & T_{f_*(u)}(V(M_2)) \\ {\scriptstyle L_{1,u}}\big\downarrow & & \big\downarrow{\scriptstyle L_{2,f_*(u)}} \\ \pi^{-1}_{1,u}TM_1 & \xrightarrow[\;(Df)_u\;]{} & \pi^{-1}_{2,f_*(u)}TM_2 \end{array}$$

for any $u \in V(M_1)$, cf. [1]. Here $(Df)_u$ denotes the restriction of $f_* \times f_*$ to $\pi^{-1}_{1,u}TM_1$. Moreover

$$(Df)_u \circ z = F(f)\,(z) \tag{6.9}$$

for any $z = (u, b) \in B_G(M_1)$, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}_{1,u}TM_1 & & \\ & \nwarrow^{z} & \\ {\scriptstyle (Df)_u}\big\downarrow & & \mathbb{R}^n \\ & \swarrow_{F(f)} & \\ \pi^{-1}_{2,f_*(u)}TM_2 & & \end{array}$$

To check (6.9) let $z = (u,b) \cong (u\{(u,X_i)\})$ where $b = (\pi_1(u), \{X_i\})$. Then $(Df)_u \circ z(e_i) = (D_f)_u(u,X_i) = (f_*u, f_*X_i) = F(f)(z)(e_i)$ because $F(f)(z) = (f_*u, L(f)(b)) = (f_*u, (f(x), \{f_*X_i\}) \cong (f_*u, \{(f_*u, f_*X_i)\})$. As $\theta_2 = \theta_2^h \oplus \theta_2^v$ it is sufficient to prove (6.8) for the h- and v-basic 1-forms. Using the commutative diagrams above we may conduct the following computation:

$$\theta^h_{2,F(f)(z)} \circ (d_z F(f)) = F(f)(z)^{-1} \circ (d_{2,f_*(u)} \circ L_{F(f)(z)}\rho_2) \circ (d_z F(f)) =$$

$$= F(f)(z)^{-1} \circ L_{2,f_*(u)} \circ d_z(f_* \circ \rho_1) =$$

$$= F(f)(z)^{-1} \circ (Df)_u \circ L_{1,u} \circ (d_z\rho_1) = z^{-1} \circ L_{1,u} \circ (d_z\rho_1) = \theta^h_{1,z}$$

The proof of (6.8) for the v-basic 1-form θ_2^v is somewthat trickier. Note firstly the commutativity of the following diagram:

$$\begin{array}{ccc} T_{f_*(u)}(V(M_2)) & \xrightarrow{Q_{2,F_*(u)}} & \operatorname{Ker}(d_{f_*(u)}\pi_2) \\ \downarrow{\scriptstyle d_u f_*} & & \downarrow{\scriptstyle d_u f_*} \\ T_u(V(M_1)) & \xrightarrow[Q_{1,u}]{} & \operatorname{Ker}(d_u\pi_1) \end{array}$$

for any $u \in V(M_1)$, as a consequence of Lemma 3. We retain the identity:

$$Q_{2,f_*(u)} \circ (d_u f_*) = (d_u f_*) \circ Q_{1,u}. \tag{6.10}$$

Next we need to estabilish the commutativity of the diagram:

$$\begin{array}{ccc} \pi^{-1}_{1,u}TM & \xrightarrow{\gamma_{1,u}} & \operatorname{Ker}(d_u\pi_1) \\ \downarrow{\scriptstyle (Df)_u} & & \downarrow{\scriptstyle d_u f_*} \\ \pi^{-1}_{2,f*(u)}TM_2 & \xrightarrow[\gamma_{2,f_*}(u)]{} & \operatorname{Ker}(d_{f_*(u)}\pi_2) \end{array}$$

To this end, note that the definition of the vertical lift (given in terms of local frames in §2) admits the following coordinate-free reformulation. Let $X = (u,v) \in \pi^{-1}_{1,u}TM_1$ and define the C^∞ curve $c_{1;X} : (-\varepsilon, \varepsilon) \to V(M_1)$ by setting $c_{1,x}(t) = u + tv$, for $|t| < \varepsilon$, $\varepsilon > 0$. Then $\gamma_{1,u}X = \dfrac{dc_{1,x}}{dt}(0)$. Note that

$$f_* \circ c_{1,x} = c_{2,D(f)_u x}. \tag{6.11}$$

Using (6.11) we may perform the following calculation:

$$(d_u f*) \circ \gamma_{1,u}(X) = d_0(f_* \circ c_{1,x}) \frac{d}{dt}\Big|_{t=0} = (d_0\, c_{2,D(f)_u x}) \frac{d}{dt}\Big|_{t=0} =$$

$$= \gamma_{2,f_*(u)} \circ (Df)_u X$$

Let us compose with $\gamma^{-1}_{2,f_*(u)}$ (at the left) in (6.10). We obtain:

$$K_{2,f_*(u)} \circ (d_u f_*) = (Df)_u \circ K_{1,u} \tag{6.12}$$

i.e. the following diagram is commutative:

$$\begin{array}{ccc} T_{f_*(u)}(V(M_2)) & \xrightarrow{K_{2,f_*(u)}} & \pi^{-1}_{2,f_*(u)} TM_2 \\ {\scriptstyle d_u f_*}\downarrow & & \downarrow{\scriptstyle (Df)_u} \\ T_u(V(M_1)) & \xrightarrow[K_{1,u}]{} & \pi^{-1}_{1,u} TM_1 \end{array}$$

for any $u \in V(M_1)$. Using (6.12) we have:

$$\theta^v_{2,F(f)(z)} \circ (d_z F(f)) = F(f)(z)^{-1} \circ K_{2,f_*(u)} \circ (d_{F(f)(z)}\rho_2) \circ (d_z F(f)) =$$

$$= F(f)(z)^{-1} \circ K_{2,f_*(u)} \circ (d_u f_*) \circ (d_z \rho_1) =$$

$$= F(f)(z)^{-1} \circ (Df)_u \circ K_{1,u} \circ (d_z\rho_1) = z^{-1} \circ K_{1,u} \circ (d_z \rho_1) = \theta^v_{1,z}$$

and the proof of Lemma 4 is complete. □

Lemma 5. *Let H_1 be a horizontal distribution in $B_G(M_1) \to V(M_2)$ and H_2 defined by (6.3). Then:*

$$(d_z F(f)) H_1(\xi)_z = H_2(\xi)_{F(f)(z)} \tag{6.13}$$

for any $z \in B_G(M_1)$, $\xi \in F$.

Proof. As a consequence of (6.8) we have $\theta_{2,F(f)(z)}\ (H_2(\xi)_{F(f)(z)} - (d_z\ F(f))\ H_1(\xi)_z) = 0$ so that $H_{2,F(f)(z)} - (d_z F(f))\ H_1(\xi)_z \in H_{2,,F(f)(z)} \cap \mathrm{Ker}\,(d_{F(f)(z)}\rho_2) = (0)$. □

Lemma 6. *Let $\tau \subset (F^* \wedge F^*) \otimes (F)$ be a direct sumand to $\partial\mathrm{Hom}\,(F, \mathscr{G})$ and H_1 a horizontal distribution in $B_G(M_1)$. Let H_2 be given by (6.3). Then:*

$$H_1 \in \mathscr{H}(\tau) \Rightarrow H_2 \in \mathscr{H}(\tau)$$

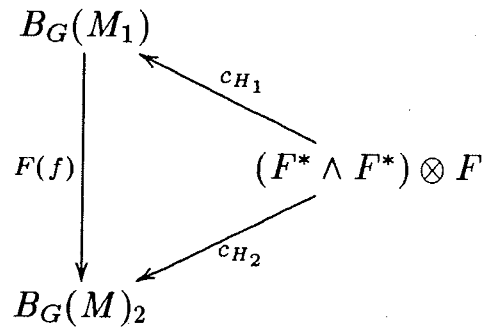
i.e. c_{H_2} is τ-valued.

Proof. Here c_{H_2} is built from the data (H_2, N_2), where N_2 is given by (6.6). Using the Lemmae 5 and 4 we have:

$$c_{H_2}(F(f)(z))(\xi \wedge \eta) = (d\,\theta_2)(H_2(\xi), H_2(\eta))_{F(f)(z)} =$$

$$= d(F(f)^*\theta_2)\,(H_1(\xi), H_1(\eta))_z = (d\,\theta_1)\,(H_1(\xi), H_1(\eta))_z = c_{H_1}(z)(\xi \wedge \eta)$$

so that the following diagram:



is commutative. Our Lemma 6 is proved. □

Finally, let $r = (z, \{H_1(e_i)_z, e_1(A_\alpha)_z\})$ be a linear frame tangent to $B_G(M_1)$, adapted to the $G^{(1)}$-structure $B_G(M_1)^{(1),\tau}_{N_1}$, where $H_1 \in \mathscr{H}(\mathscr{G})$. Then:

$$L(F(f))(r) = (F(f)(z), \{(d_zF(f))H_1(e_i)_z, (d_zF(f))e_1(A_\alpha)_z\}) =$$

$$= (F(f)(z), \{H_2(e_i)_{F(f)(z)}, e_2(A_\alpha)_{F(f)(z)}\}) \in B_G(M_2)^{(1),\tau}_{N_2}$$

as a consequence of our Lemmae 1, 5 and 6.
The inclusion $L(F(f))(B_G(M_1)^{(1),\tau}_{N_1}) \subseteq B_G(M_2)^{(1),\tau}_{N_2}$, yields (1.2) since $\dim_{\mathbb{R}} B_G(M_i)^{(1),\tau}_{N_i} = \dim_{\mathbb{R}} G^{(1)} + \dim_{\mathbb{R}} B_G(M_i) = \dim_{\mathbb{R}} G^{(1)} + \dim_{\mathbb{R}} G + 2n$, for $i = 1, 2$. Our Theorem 1 is completely proved.

7. Derived substructures.

Let $B_G(M) \to V(M)$ be a derived G-structure on M. If G' is a Lie subgroup of G then a *derived substructure* is a principal G'-subbundle $B_{G'}(M) \to V(M)$ of $B_G(M) \to V(M)$. As far as the base manifold is the same, the theory of derived substructures is a direct extension of the classical theory of substructures of a given G-structure (cf. e.g. [23]) so that we allow ourselves to be somewhat sketchy. If H' is a horizontal distribution in $B_{G'}(M)$ and H an extension of H' to $B_G(M)$, then $c_H \circ j = c_{H'}$, where $j : B_{G'}(M) \to B_G(M)$ is the given imbedding. Both $c_H, c_{H'}$ are built with respect to the same given nonlinear connection N on $V(M)$ (fixed throughout §7). This follows from $\theta' = j^*\theta$, where θ, θ' are the canonical 1-forms of $B_G(M)$, and $B_{G'}(M)$, respectively. The Lie algebra $\mathcal{G}'$ of G' is a subalgebra of $\mathcal{G}$. Then:

$$\frac{((F^* \wedge F^*) \otimes F)}{\partial \mathrm{Hom}\,(F, \mathcal{G})} \cong \frac{\dfrac{(F^* \wedge F^*) \otimes F}{\partial \mathrm{Hom}\,(F, \mathcal{G}')}}{\dfrac{\partial \mathrm{Hom}\,(F, \mathcal{G})}{\partial \mathrm{Hom}\,(F, \mathcal{G}')}} \cong \frac{\dfrac{(F^* \wedge F^*) \otimes F}{\partial \mathrm{Hom}\,(F, \mathcal{G}')}}{\partial \mathrm{Hom}\,(F, \frac{\mathcal{G}}{\mathcal{G}'})}$$

(isomorphisms of linear spaces). Thus, for any $z \in B_{G'}(M)$, $c(j(z))$ is the class of $c'(z)$ modulo $\partial \mathrm{Hom}\,(F, \dfrac{\mathcal{G}}{\mathcal{G}'})$.

More interesting seems to be the case where the base manifolds are distinct. Precisely, let $B_H(A) \to V(A)$ be a derived H-structure on a C^∞ manifold A and N a nonlinear connection on $V(A)$. Let $G \subset H$ be a Lie subgroup, $B_G(M) \to V(M)$ a derived G-structure on M and $f : M \to A$ a C^∞ imbedding. Assume that there is an imbedding $j : B_G(M) \to B_H(A)$ so that (j, f_*) is a principal bundle monomorphism. It is an open problem to relate the first structure functions of $B_G(M)$, $B_H(A)$. One should seek for the natural candidate for the nonlinear connection "induced" on $V(M)$ by N. If, for instance, A is a Finslerian manifold and N the nonlinear connexion of its unique regular Cartan-Chern connection then one may endow M with the nonlinear connection of the induced connection (and apply the theory developed in [1]).

8. Derived G-structures with reducible structure group.

Assume G is *reducible*, i.e. there is a G-invariant proper subspace $V \subset \mathbb{R}^n$. Let $B_G(M) \to V(M)$ be a derived G-structure. For any $u \in V(M)$ define $\mathcal{V}_u \subset \pi_u^{-1}TM$ as follows. Let $z \in \rho^{-1}(u) \subset B_G(M)$ and set $\mathcal{V}_u = z(V)$. Then $\mathcal{V}_u$ is well defined (i.e. the definition does not depend upon the choice of

$z \in \rho^{-1}(u)$) due to the G-invariance of V. Then $\mathscr{V} : u \mapsto \mathscr{V}_u$ is a π-distribution on M (cf. [11], [6]). For any $u \in V(M)$ define $\mathscr{D}_u \subset T_u(V(M))$ by setting $\mathscr{D}_u = \beta_u \mathscr{V}_u \oplus \gamma_u \mathscr{V}_u$. Here β denotes the horizontal lift associated with the nonlinear connection N on $V(M)$ (fixed throughout §8). Then $\mathscr{D}$ is a $2p$-dimensional distribution on $V(M)$, $p = \dim_{\mathbb{R}} V$. It is our purpose to formulate necessary and sufficient conditions for the integrability of $\mathscr{D}$ in terms of the structure function c of $B_G(M)$.

Lemma 7. *Let $Z \in \Gamma^\infty(T(V(M)))$ and $\widehat{Z} \in \Gamma^\infty(T(B_G(M)))$ so that $\widehat{Z}$ is ρ-related to Z, i.e. $(d_z\rho)\widehat{Z}_z = Z_{\rho(z)}$, for any $z \in B_G(M)$. Then $Z \in \mathscr{D}$ if and only if $\theta(\widehat{Z})_z \in V \oplus V$ for any $z \in B_G(M)$.*

Proof. Let $z \in B_G(M)$, $u = \rho(z)$. Then $Z_u \in \mathscr{D}_u$ iff $(d_z\rho)\widehat{Z}_z \in \mathscr{D}_u = (\beta\mathscr{V})_u \oplus (\gamma\mathscr{V})_u$ i.e. iff:

$$(8.1) \qquad (d_z\rho)\widehat{Z}_z = \beta_u z(\xi_1) + \gamma_u z(\xi_2)$$

for some $\xi_1, \xi_2 \in V$. Let us apply L_u, respectively K_u, to the identity (8.1). Thus $z^{-1} \circ L_u \circ (d_z\rho)\widehat{Z}_z = \xi_1$ and $z^{-1} \circ K_u \circ (d_z\rho)\widehat{Z}_z = \xi_2$ which is equivalent to $\theta^h_z(\widehat{Z}_z) \in V$ and $\theta^v_z(\widehat{Z}_z) \in V$. □

Lemma 8. *Let H be a horizontal distribution in $B_G(M) \to V(M)$. Then the following statements are equivalent:*

i) $\mathscr{D}$ is involutive.

ii) For any $z \in B_G(M)$:

$$(8.2) \qquad c_H(z)((V \oplus V) \wedge (V \oplus V)) \subset V \oplus V$$

Proof. Assume i) holds. We wish to compute $c_H(z)(\xi \wedge \eta)$ for $z \in B_G(M)$ and $\xi, \eta \in V \oplus V$. Set $\xi = \xi_1 \oplus \xi_2$, $\eta = \eta_1 \oplus \eta_2, \xi_i, \eta_i \in V$, $i = 1, 2$. Then $\beta_u z(\xi_1) + \gamma_u z(\xi_2)$ and $\beta_u z(\eta_1) + \gamma_u z(\eta_2)$ are elements of $\mathscr{D}_u$, where $u = \rho(z)$. Next consider $Y, Z \in \mathscr{D}$ so that $Y_u = \beta_u z(\xi_1) + \gamma_u z(\xi_2)$ and $Z_u = \beta_u z(\eta_1) + \gamma_u z(\eta_2)$. This choice is always possible (not unique) by standard theorems on the C^∞ extension of sections of a vector bundle (here $\mathscr{D}$) defined on some closed subset (here $\{u\}$) of the base space. Let $\widehat{Y}, \widehat{Z} \in \Gamma^\infty(H)$ be ρ-related to Y, Z, respectively. Then

$$(8.3) \qquad H(\xi)_z = \widehat{Y}_z .$$

for $z \in B_G(M)$ fixed above. Indeed, as both sides of (8.3) are horizontal (with respect to H) it is sufficient to show that $\emptyset(\widehat{Y}_z) = \xi$. This follows from the calculation below:

$$\theta_z(\widehat{Y}_z) = (\theta^h\widehat{Y})_z \oplus (\theta^v\widehat{Y})_z = (z^{-1} \circ L_u \circ (d_z\rho)\widehat{Y}_z) \oplus (z^{-1} \circ K_u \circ (d_z\rho)\widehat{Y}_z) =$$

$$= (z^{-1}L_uY_u) \oplus (z^{-1}K_uY_u) = \xi_1 \oplus \xi_2 = \xi\,.$$

Analogously $H(\eta)_z = \widehat{Z}_z$. As $\mathcal{D}$ is involutive $[Y, Z] \in \mathcal{D}$. On the other hand $[\widehat{Y}, \widehat{Z}]$ is ρ-related to $[Y, Z]$ (cf. Prop.1.3 in [15], vol I, p. 65) so that, by Lemma 7, $\theta([\widehat{Y}, \widehat{Z}])_z \in V \oplus V$. Therefore:

$$c_H(z)(\xi \wedge \eta) = (d\theta)_z(H(\xi)_z,\ H(\eta)_z) = (d\theta)_z(\widehat{Y}_z, \widehat{Z}_z) =$$

$$= \frac{1}{2}\{\widehat{Y}_z(\theta(\widehat{Z})) - \widehat{Z}_z(\theta(\widehat{Y})) - \theta([\widehat{Y}, \widehat{Z}])_z\} \in V \oplus V$$

and (8.2) is proved. The proof of ii) $\Rightarrow$ i) is similar and therefore left as an exercise to the reader.

Let:

$$\tau : (F^* \wedge F^*) \otimes F \to [(V \oplus V)^* \wedge (V \oplus V)^*] \otimes \left[\frac{F}{(V \oplus V)}\right]$$

be defined by $(\tau L)(\xi \wedge \eta) = \Phi(L(\xi \wedge \eta))$ for any $L \in (F^* \wedge F^*) \otimes F$ and $\xi, \eta \in V \oplus V$. Here $\Phi : F \to \dfrac{F}{(V \oplus V)}$ is the canonical map. As V is G-invariant, it is $\mathcal{G}$-invariant, as well. Thus:

$$\partial \mathrm{Hom}\,(F, \mathcal{G}) \subset \mathrm{Ker}\,(\tau)\,.$$

and τ induces a linear map:

$$\overline{\tau} : \frac{(F^* \wedge F^*) \otimes F}{\partial\, \mathrm{Hom}\,(F, \mathcal{G}')} \to [(V \oplus V)^* \wedge (V \oplus V)^*] \otimes \left[\frac{F}{(V \oplus V)}\right]$$

so that $\tau = \overline{\tau} \circ \Psi$. At this point we may complete the proof of Theorem 2. Assume $\mathcal{D}$ is involutive.
Then by Lemma 8, for any $\xi, \eta \in V \oplus V$, $c_H(z)(\xi \wedge \eta) \in V \oplus V$. Consequently $\tau(c_H(z)(\xi \wedge \eta)) = \Phi(c_H(z)(\xi \wedge \eta))$ and then $\overline{\tau}(c(z)) = \overline{\tau}(\Psi(c_H(z)) = \tau(c_H(z)) = 0$ for any $z \in\ \in B_G(M)$. The proof of the converse is similar (and thus omitted).

REFERENCES

[1] L.M. Abatangelo, *A study of submanifolds in Finslerian manifolds by principal bundle technics*, Riv. Mat. Univ. Parma, (5) 1 (1992), pp. 1-13.

[2] H. Akbar-Zadeh, *Les espaces de Finsler et de certains leurs generalisations*, Ann. Sci. Ecole Norm. Sup., Paris, (3) 80 (1963), pp. 1-79.

[3] G. Atanasiu, *Structures et connexions Finsler presque hermitienes*, Brasov, 1 (1980), pp. 41-55.

[4] F. Brickell - R.S. Clark - M.S. Al-Borney, *(G, E)-structures*, Topics in Differential Geometry, Ed. H.Rund & W.F. Forbes, Academic Press, New York, 1976, pp. 29-43.

[5] E. Cartan, *Les espaces de Finsler,* Actualites 79, Paris, 2nd edit. 1971.

[6] B. Casciaro - S. Dragomir, *On reducible Finsler spaces with a vanishing* $R^i_{j\,k}$ *torsion tensor field,* Acta Math. Hung., (1-2) 54 (1989), pp. 29-37.

[7] S.S. Chern, *On the Euclidean connections in a Finsler space*, Proc. Nat. Acad. Sci. U.S.A., 29 (1943), pp. 33-37.

[8] L. Di Terlizzi - S. Dragomir, *On lifts of Finslerian G-structures associated with a nonlinear connection*, Rend. Mat. Appl., Roma (3) 6 (1986), pp. 365-381.

[9] P. Dazord, *Tenseur de structure d'une G-structure dérivée*, C. R. Acad. Sci. Paris, 258 (1964), pp. 2730-2733.

[10] P. Dombrowski, *On the geometry of the tangent bundles*, J. Reine Angew. Math., 210, (1962), pp. 73-88.

[11] S. Dragomir, *π-Distributions on differentiable manifolds*, An. St. Univ. "Al.I. Cuza", Iasi, 28 (1982), pp. 55-58.

[12] S. Dragomir, *On the geometry of Finslerian G-structures on differentiable manifolds*, Boll. U.M.I., (1) 5-D (1986), pp. 71-86.

[13] I. Ghinea, *Connexions Finsleriennes presque complexes*, Math. Rev. Anal. Numer. Theo. Appro. Math., 19 (1977), pp. 55-61.

[14] A. Kawaguchi, *On the theory of nonlinear connections I - II*, Tensor N. S., 2 (1952), 123-142, 6 (1956), pp. 165-199.

[15] S. Kobayashi - K. Nomizu, *Foundations of differential geometry,* Intersci. Publishers, New York, Vol I, 1963, Vol II, 1969.

[16] Y. Ichijyo, *On almost Finsler structures*, An. St. Univ. "Al. I. Cuza", Iasi, 30 (1984), pp. 27-30.

[17] Y. Ichijyo, *On some G-structures defined on tangent bundles*, preprint, Tokushima University, 1984.

[18] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kasheisa Press, Kyoto, 1986.

[19] R. Miron - M. Hashiguchi, *Metrical Finsler connections*, Rep. Fac. Sci. Kagoshima Univ., 12 (1979), pp. 21-35.

[20] R. Miron - M. Hashiguchi, *Conformal Finsler connections,* Rev. Roum. Math. Pures Appl., (6) 24 (1981), pp. 861-878.

[21] R. Miron - M. Hashiguchi, *Almost symplectic Finsler structures,* Rep. Fac. Sci. Kagoshima Univ., 14 (1981), pp. 9-19.

[22] P. Stavre - F.C. Klepp, *General Finsler connection transformations with invariant $I^i_{j\,k}$-tensors,* Proc. Romanian-Japanesse Coll. on Finsler Geometry, Brasov, 1984, pp. 165-170.

[23] S. Sternberg, *Lectures on differential geometry,* Prentice Hall, Englewood Cliffs, New Jersey, 1964.

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