NILPOTENT GROUPS OF SEMILINEAR TRANSFORMATIONS WHICH ARE MONOMIAL

ANDREA LUCCHINI - M. CHIARA TAMBURINI

Let $H$ be a nilpotent subgroup of $\Gamma L_n(q) = \langle \varphi \rangle \text{GL}_n(q)$, where $\varphi$ denotes the field automorphism induced by the Frobenius map. We give a condition on the primes dividing $|H \cap \text{GL}_n(q)|$ under which $H$ is conjugate to a subgroup of the generalized monomial group $\langle \varphi \rangle \text{Diag}_n(\mathbb{F}_q^*) \text{Sym}(n)$. We show an application of this result to the determination of Carter subgroups of finite groups.

1. Introduction

Let $\mathbb{F}$ be a field. For a subgroup $T$ of $\mathbb{F}^*$, we denote by $\text{Diag}_n(T)$ the subgroup of $\text{GL}_n(\mathbb{F})$ consisting of diagonal matrices with entries in $T$. The product of $\text{Diag}_n(\mathbb{F}^*)$ with the group $\text{Sym}(n)$ of permutation matrices is called the monomial subgroup of $\text{GL}_n(\mathbb{F})$. It is well known that a finite nilpotent group $H$ is an IM group, i.e., every representation of $H$ over an algebraically closed field of characteristic 0 or prime to $|H|$, is monomial [2, Theorem 52.1, page 356]. In particular, if $\mathbb{F}$ is algebraically closed, a finite nilpotent subgroup of $\text{GL}_n(\mathbb{F})$ of order prime to the characteristic (when positive), is conjugate to a subgroup of the monomial group. Clearly this property no longer holds over a finite field. For example a Sylow 2-subgroup of $\text{GL}_2(3)$ has order $2^4$, whereas the monomial subgroup has order $2^3$. On the other hand, if $q$ is any power of a prime

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ANDREA LUCCHINI - M. CHIARA TAMBURINI

$r_0$, and $p$ is an odd prime such that $q \equiv 1 \pmod{p}$, the monomial subgroup of $\text{GL}_n(q)$ contains a Sylow $p$-subgroup of $\text{GL}_n(q)$. And the same holds for $p = 2$, provided that $q \equiv 1 \pmod{2}$. A similar property is still valid in the group $\Gamma L_n(q)$ of semilinear transformations, with respect to a natural generalization of the monomial group. Let $\varphi$ denote the field automorphism of $\text{GL}_n(q)$ induced by the Frobenius map $\alpha \mapsto \alpha^{r_0}$. Thus $\Gamma L_n(q) = \langle \varphi \rangle \text{GL}_n(q)$. As $\varphi$ normalizes $\text{Diag}_n(F_q^*) \text{Sym}(n)$, we may consider the product

$$M := \langle \varphi \rangle \text{Diag}_n(F_q^*) \text{Sym}(n)$$

and call $M$ the generalized monomial subgroup of $\Gamma L_n(q)$. Under the above hypothesis on $p$ it is still true that $M$ contains a Sylow $p$-subgroup of $\Gamma L_n(q)$. The main aim of this paper is to prove the following generalization of this fact.

**Theorem 1.1.** Let $H$ be a nilpotent subgroup of $\Gamma L_n(q)$ and assume

$$H \text{GL}_n(q) = \langle \psi \rangle \text{GL}_n(q)$$

where $\psi \in \langle \varphi \rangle$ and $r = |C_{F_q} (\psi)|$. Let $p_1, \ldots, p_t$ be the primes which divide $|H \cap \text{GL}_n(q)|$ and for $j \leq t$ denote by $R_{p_j} = (F_q^*)_{p_j}$ the Sylow $p_j$-subgroup of $F_q^*$. Suppose that $r \equiv 1 \pmod{p_j}$ for all $j \leq t$, and if $|H \cap \text{GL}_n(q)|$ is even, suppose further that $q \equiv 1 \pmod{4}$. Then, for some $g \in \text{GL}_n(q)$:

$$H^g \leq \langle \psi \rangle \text{Diag}_n(R_{p_1} \cdots R_{p_t}) \text{Sym}(n) \leq M.$$
Corollary 1.2. Let $\text{SL}_n(q) \leq A \leq \Gamma L_n(q)$, with $q \equiv 1 \pmod{4}$. Then the projective image of $A$ cannot be a minimal counterexample to the conjugacy conjecture of Carter subgroups.

If $A$ is as in the statement of the previous Corollary, with $q$ odd, and has a Carter subgroup $H$ of order coprime to $q$, then $H$ contains a Sylow 2-subgroup of $A$ (see [14]). When $q \equiv 1 \pmod{4}$, an inductive argument on $n$ allows to deduce this fact from Theorem 1.1 and our concluding result.

Theorem 1.3. Let $H$ be a Carter subgroup of $M_0 = D \langle \psi \rangle \text{Sym}(n)$ where $D \leq \text{Diag}_n(F_q^*)$ is normal in $M_0$ and $\langle \psi \rangle \leq \langle \phi \rangle$. Then $H$ contains a Sylow 2-subgroup of $M_0$.

2. Notations and basic facts

Let $p$ be a prime. For an integer $z > 1$ we write $z = z_p z_{p'}$ where $z_p$ is a $p$-power and $p$ does not divide $z_{p'}$. Similarly, for an element $g$ of a group $G$, we write $g = g_p g_{p'}$ where $g_p \in \langle g \rangle$ has order a $p$-power and $g_{p'}$ has order prime to $p$. Finally $G_p$ denotes a Sylow $p$-subgroup of $G$. For the reader’s convenience we recall some well known facts. In particular, a proof of the following Lemma for $p$ odd is given in [8, Lemma 8.1, page 503].

Lemma 2.1. Let $x \in \mathbb{N}$, with $x \equiv 1 \pmod{p}$. Then, for each $y \in \mathbb{N}$:

i) $(x^y - 1)_p = (x^{yp} - 1)_p$;

ii) $(x^{yp} - 1)_p = (x - 1)_p y_p$ provided that $x \equiv 1 \pmod{4}$ if $p = 2$.

Proof. i) $x^y - 1 = (x^{yp} - 1) \left(x^{yp(y_p - 1)} + \cdots + x^{yp} + 1\right)$. As $x \equiv 1 \pmod{p}$, the second factor is congruent to $y_p \pmod{p}$. Thus it is not divisible by $p$.

ii) We set $y_p = p^a$. Our claim is clear when $a = 0$. So let us assume $a > 0$ and put $z = x^{p^{a-1}}$. It follows:

$$x^{p^a} - 1 = z^p - 1 = (z - 1) \left(z^{p-1} + \cdots + 1\right).$$

By induction $(z - 1)_p = (x - 1)_p p^{a-1}$. From $z \equiv 1 \pmod{p}$:

$$(z^{p-1} + \cdots + 1) = p + \frac{p(p - 1)}{2} p + kp^2, \quad k \in \mathbb{Z}.$$ 

Thus, if $p > 2$, we have $(z^{p-1} + \cdots + 1) \equiv p \pmod{p^2}$. On the other hand, if $p = 2$, we are assuming $x \equiv 1 \pmod{4}$. It follows that $z \equiv 1 \pmod{4}$, hence $z + 1 \equiv 2 \pmod{4}$. In both cases we conclude that $(z^{p-1} + \cdots + 1)_p = p$. \qed
As in the Introduction we assume that \( q \) is a power of the prime \( r_0 \) and that \( \varphi \) is the field automorphism of \( \text{GL}_n(q) \) induced by the map \( \alpha \mapsto \alpha^{r_0} \).

For a partition \( n = n_1 + \cdots + n_t \), we set \( \Gamma L_{n_j}(q) = \langle \varphi \rangle \text{GL}_{n_j}(q) \), \( j \leq \ell \), and identify \( (\Gamma L_{n_1}(q) \times \cdots \times \Gamma L_{n_t}(q)) \cap \Gamma L_n(q) \) with

\[
\langle \varphi \rangle (\text{GL}_{n_1}(q) \times \cdots \times \text{GL}_{n_t}(q))
\]

where

\[
\text{GL}_{n_1}(q) \times \cdots \times \text{GL}_{n_t}(q) := \left\{ \begin{pmatrix} A_1 & \cdots & A_{\ell} \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_{\ell} \end{pmatrix} \mid A_j \in \text{GL}_{n_j}(q) \right\}.
\]

**Definition 2.2.** We say that a subgroup of \( \Gamma L_n(q) \) is indecomposable if it is not conjugate, under \( \text{GL}_n(q) \), to a subgroup of (1), for any partition of \( n \) with \( \ell > 1 \).

For each \( j \leq \ell \), let us denote by \( \pi_j \) the projection from (1) onto \( \Gamma L_{n_j}(q) \).

**Lemma 2.3.** Suppose that \( K \) is a subgroup of \( \Gamma L_n(q) \) contained in (1) and set \( K \text{GL}_n(q) = \langle \varphi^k \rangle \text{GL}_n(q) \). Then, for each \( j \leq \ell \):

i) \( \pi_j(K) \text{GL}_{n_j}(q) = \langle \varphi^k \rangle \text{GL}_{n_j}(q) \);

ii) \( \pi_j(K) \cap \text{GL}_{n_j}(q) \leq \pi_j(K \cap \text{GL}_n(q)) \).

In particular the primes which divide the order of \( \pi_j(K) \cap \text{GL}_{n_j}(q) \) are a subset of those which divide the order of \( K \cap \text{GL}_n(q) \).

**Proof.** i) \( K = \langle \varphi^k g \rangle (K \cap \text{GL}_n(q)) \) for some \( g \in \text{GL}_n(q) \). As \( \langle \varphi \rangle \cap \text{GL}_n(q) = 1 \), the assumption that \( K \) is contained in (1) implies that \( g \in (2) \). Thus \( \pi_j(\varphi^k g) = \varphi^k g_j \) for some \( g_j \in \text{GL}_{n_j}(q) \).

ii) Take \( j = 1 \), say, and let \( x_1 \in \pi_1(K) \cap \text{GL}_{n_1}(q) \). Choose \( y \in K \) such that \( y = (x_1, \cdots, x_\ell) \) with \( x_j \in \Gamma L_{n_j}(q) \). From \( x_1 \in \Gamma L_{n_1}(q) \) it follows easily that \( x_j \in \text{GL}_{n_j}(q) \) for all \( j \geq 2 \). Thus \( y \in K \cap \text{GL}_n(q) \). We conclude that \( x_1 \in \pi_1(K \cap \text{GL}_{n_1}(q)) \).

From now on we fix a factorization \( |\varphi| = im \) and set

\[
\psi = \varphi^i, \quad r = |C_{F_q}(\psi)|.
\]

Thus

\[
r = r_0^i, \quad m = |\psi|, \quad q = r^m, \quad \mathbb{F}_r^* = (\mathbb{F}_q^*)^{1+r+\cdots+r^{m-1}}.
\]

**Lemma 2.4.** Let \( p \) be a prime such that \( r \equiv 1 \pmod{p} \) and, if \( p = 2 \) and \( |\varphi| \) is odd, assume further that \( r_0 \equiv 1 \pmod{4} \). Denote by \( R_p \) a Sylow \( p \)-subgroup of \( \mathbb{F}_r^* \) and by \( \Sigma_p \) a Sylow \( p \)-subgroup of \( \text{Sym}(n) \). Then:
\begin{align*}
1) \quad & R_p \leq \mathbb{F}^*_{r^p} ; \\
2) \quad & \Gamma_p := \langle \psi_p \rangle \text{ Diag}_n(R_p) \Sigma_p \text{ is a Sylow } p\text{-subgroup of } \langle \psi \rangle \text{ GL}_n(q) ; \\
3) \quad & \Gamma_p \leq \langle \psi \rangle \text{ GL}_n(p^m) ; \\
4) \quad & \Gamma_p \text{ is (absolutely) irreducible if and only if } n \text{ is a power of } p .
\end{align*}

**Proof.**

1) \((q - 1)_p = (r^m - 1)_p = (r^m - 1)_p \text{ by point i) of Lemma 2.1.} \)

2) We must show that \(|GL_n(q)|_p = ((q - 1)_p)^n |\Sigma_p|\). In fact
\[ |GL_n(q)| = q^{\frac{n(n+1)}{2}} \prod_{\ell=1}^{n} (q^\ell - 1) \]

and, by Lemma 2.1, \((q^\ell - 1)_p = (q^\ell - 1)_p = (q - 1)_p \ell_p \) for each \( \ell \).

3) Is an immediate consequence of 1) and 2).

4) \( \Sigma_p \) is transitive on the canonical basis \( \{e_1, \ldots, e_n\} \) of \( \mathbb{F}_q^n \) only if \( n \) is a power of \( p \). So this condition is necessary for the irreducibility of \( \Gamma_p \). On the other hand, assume that \( n \) is a power of \( p \) and let \( 0 \neq W \) be a \( \Gamma_p \)-invariant subspace. Denote by \( w = \alpha_1 e_1 + \cdots + \alpha_n e_n \) a non-zero vector in \( W \) and assume \( \alpha_i \neq 0 \). Then there exists a diagonal matrix \( d = (\lambda_1, \ldots, \lambda_n) \in \Gamma_p \) with \( \lambda_i \neq 1 \) and \( \lambda_j = 1 \) for all \( j \neq i \). From \( w - dw \in W \), it follows that \( e_i \in W \). By the transitivity of \( \Sigma_p \) the canonical basis is contained in \( W \), hence \( W = \mathbb{F}_q^n \).

**Definition 2.5.**

Considering the factorization into distinct primes
\[ r - 1 = p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \quad \alpha_j > 0 \tag{5} \]

set \( R = R_{p_1} \cdots R_{p_k} \) where \( R_{p_j} \) denotes the Sylow \( p_j \)-subgroup of \( \mathbb{F}_q^* \).

Note that \( \lambda \in \mathbb{F}_q^* \) and \( \lambda^{r-1} \in R \) implies \( \lambda \in R \). In fact the primes which divide the order of \( \lambda \) must belong to \( \{p_1, \ldots, p_k\} \).

**Lemma 2.6.**

For every \( d \in \mathbb{F}_q^* \), the group \( \langle \psi d \rangle R \) is a Carter subgroup of \( \langle \psi \rangle \mathbb{F}_q^* \). In particular \( \langle \psi d \rangle R \) is conjugate to \( \langle \psi \rangle \mathbb{F}_q^* \) under \( \mathbb{F}_q^* \). Moreover every nilpotent subgroup \( M \) of \( \langle \psi \rangle \mathbb{F}_q^* \) such that \( \langle \psi \rangle \mathbb{F}_q^* = MF_q^* \), is contained in a Carter subgroup of \( \langle \psi \rangle \mathbb{F}_q^* \).

**Proof.**

We fix \( p \in \{p_1, \ldots, p_k\} \). For every \( x \in R_p \) we have: \( x^{\psi d} = x^\psi = x^r \). But \( x^r \equiv x \pmod{x^p} \), by the assumption \( r \equiv 1 \pmod{p} \). Thus \( \psi d \) centralizes each composition factor of \( R_p \). It follows that \( \langle \psi d \rangle_{p^r} \) centralizes \( R_p \). By the same argument, \( \langle \psi d \rangle_{p} \) centralizes \( R_p \). We conclude that \( \langle \psi d \rangle R \) is nilpotent.
Now let $N = N_{(ψ)} F_q(⟨ψd⟩R) = ⟨ψd⟩(N ∩ F_q^*)$ and choose $λ ∈ N ∩ F_q^*$. By the definition of $N$, there exist $μ ∈ R$ and $ℓ ∈ N$ such that:

$$λ^{-1}(ψd)λ = (ψd)^ℓμ.$$ 

On the other hand $λ^{-1}(ψd)λ = (ψd)λ^{1−r}$. Thus $μλ^{-1+r} ∈ ⟨ψd⟩$. In particular $μλ^{-1+r} ∈ F_r^*$ as it is centralized by $ψd$. Noting that $F_r^* ⊆ R$, we have $λ^{-1+r} ∈ R$. Hence $λ ∈ R$, by the observation after Definition 2.5. We conclude that $⟨ψd⟩R$ is a Carter subgroup of $⟨ψ⟩F_q^*$.

Finally let $M$ be as in the statement. Thus $M = ⟨ψd⟩ (M ∩ F_q^*)$ for some $d ∈ F_q^*$. If $p$ is a prime which divides $|M ∩ F_q^*|$, then $M_p ∩ F_q^*$ is a non-trivial normal subgroup of $M_p$ which gives $Z(M_p) ∩ F_q^* ≠ 1$. From $Z(M_p) ≤ Z(M)$ centralized by $ψd$, we deduce that $Z(M_p) ∩ F_q^*$ is centralized by $ψ$. Thus $Z(M_p) ∩ F_q^* ≤ F_r^*$, whence $p$ divides $r − 1$. We conclude that $M ∩ F_q^* ≤ R$, which gives $M ≤ ⟨ψd⟩R$.

**Lemma 2.7.** Let $z ∈ GL_a(q)$ be a Singer cycle of order $q^a − 1$. Then there exists $ν ∈ GL_a(q)$ such that $ψν$ has order $|ψ|a = ma$ and

$$N_{(ψ)} GL_a(q)(⟨z⟩) = ⟨ψν⟩⟨z⟩ \text{ with } ⟨ψν⟩ ⟨z⟩ = 1.$$ 

**Proof.** The subalgebra $⟨z⟩ \cup \{0\}$ of $Mat_a(q)$ can be identified with $F_q^a$. The normalizer in $ΓL_n(q)$ of this subalgebra induces a group of automorphisms of $F_q^a$ and the kernel of this action is the centralizer of $z$. Considering $z$ as a permutation of $F_q^a$, it generates an abelian regular group. Thus $⟨z⟩$ is selfcentralizing in $Sym(q^a − 1)$ and, a fortiori, in $ΓL_a(q)$. In particular $F_q^a I_a ≤ ⟨z⟩$. By definition, $ψ$ acts as the identity on $F_r I_a$. Clearly conjugation by elements of $GL_a(q)$ induces the identity on $F_r I_a$. Thus:

$$\frac{N_{(ψ)} GL_a(q)(⟨z⟩)}{⟨z⟩} ≤ Gal(F_q^a : F_r). \quad (6)$$

min($z$) is irreducible over $F_q$ of degree $a$. It follows that min($z$) = char($z$). From min($z^ν$) = min($z^r$) we deduce that $z^ν$ is conjugate to $z^r$. So there exists $μ ∈ GL_a(q)$ such that $z^νμ = z^r$, i.e. $ψμ$ normalizes $F_q^a$ inducing the automorphism $z → z^r$. This automorphism generates $Gal(F_q^a : F_r)$, which has order $ma$: thus $ma$ divides $|ψμ|$ and in (6) we have an equality. It follows:

$$N_{(ψ)} GL_a(q)(⟨z⟩) = ⟨ψμ⟩⟨z⟩, \quad |N_{(ψ)} GL_a(q)(⟨z⟩)| = |z|ma. \quad (7)$$

Set $(ψμ)^{ma} = z^k$ and note that $z^k ∈ F_r I_a$ since it is centralized by $ψμ$. By (4), there exists $ℓ ∈ N$ such that $(z^ℓ)^{1+r+⋯+r^{ma−1}} = z^{−k}$. Thus

$$(ψμ z^ℓ)^{ma} = (ψμ)^{ma}(z^ℓ)^{1+r+⋯+r^{ma−1}} = z^k z^{−k} = 1.$$
Setting $v = \mu z^t$, we have $(\psi v)^{ma} = 1$ and $\langle \psi \mu \rangle \langle z \rangle = \langle \psi v \rangle \langle z \rangle$. We conclude that $\psi v$ has order $ma$ and that $\langle \psi v \rangle \cap \langle z \rangle = 1$ from (7).

In particular this Lemma gives $N_{\Gamma L_n(q)}(\langle z \rangle) = \Gamma_1(q^a)$.

3. The main result

The aim of this Section is to prove Theorem 1.1. To this purpose, we fix $h \in H$ such that $\psi^{-1}h \in \text{GL}_n(q)$. Thus:

$$H = \langle h \rangle (H \cap \text{GL}_n(q)).$$

(8)

Lemma 3.1. If $|h| = |\psi|$, there exists $x \in \text{GL}_n(q)$ such that $h^x = \psi$.

Proof. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_q$ and $\Psi : \text{GL}_n(\mathbb{F}) \to \text{GL}_n(\mathbb{F})$ the Frobenius map $\alpha \mapsto \alpha^q$. Thus $\psi$ is the restriction of $\Psi$ to $\text{GL}_n(q)$. Consider the epimorphism $\pi : \langle \Psi \rangle \text{GL}_n(q) \to \langle \psi \rangle \text{GL}_n(q)$ defined by

$$\Psi^j y \mapsto \psi^j y, \quad y \in \text{GL}_n(q), \ j \in \mathbb{Z}.$$ 

By a Theorem of Lang–Steinberg [10], there exists $x \in \text{GL}_n(\mathbb{F})$ such that

$$\psi^{-1}h = g = \Psi^{-1}x \Psi x^{-1}. \quad (9)$$

Thus $x \Psi x^{-1} = \Psi g \mapsto \psi g = h$. Now $h^m = 1$ implies that $(x \Psi x^{-1})^m$ lies in $\ker \pi = \langle \Psi^m \rangle$. It follows that $x \Psi^m x^{-1} \Psi^{-m} = \text{GL}_n(\mathbb{F}) \cap \langle \Psi \rangle = 1$. We conclude $x \in C_{\text{GL}_n(\mathbb{F})} \Psi^m = \text{GL}_n(q)$. Thus (9) becomes $g = \psi^{-1}x \psi x^{-1}$, whence $h^x = \psi$.

Lemma 3.2. In the proof of Theorem 1.1 we may assume that $H \cap \text{GL}_n(q)$ is non-scalar.

Proof. If $H \cap \text{GL}_n(q)$ is scalar, we have:

$$H \cap \text{GL}_n(q) \leq \langle R_{p_1} \cdots R_{p_t} \rangle I_n.$$ 

From $h^m \in H \cap \text{GL}_n(q)$ we deduce that $h^m = \lambda I_n$, for some $\lambda \in R_{p_1} \cdots R_{p_t}$. It follows that $h^m$ is centralized by $\psi^{-1}h \in \text{GL}_n(q)$, hence by $\psi$. This gives $\lambda \in \mathbb{F}_r^*$ and, by (4), there exists $\rho \in \mathbb{F}_q^*$ such that $\rho^{1+r+\cdots+m-1} = \lambda^{-1}$. Write $o(\rho) = p_1^{\gamma_1} \cdots p_t^{\gamma_t} c$ where $(p_1 \cdots p_t, c) = 1$. Setting $1 = cy_1 + o(\lambda)y_2$ we have

$$(\rho^{1+r+\cdots+m-1})^{cy_1} = (\lambda^{-1})^{cy_1} = \lambda^{-1}.$$
From \(o(ρ^e) = p_1^{n_1} \cdots p_t^{n_t}\) we deduce that \(μ := ρ^{c_1} ∈ R_{p_1} \cdots R_{p_t}\). Moreover:

\[(hμ)^m = h^mμ^1 + r\cdots + r^{m-1} = λλ^{-1}I_n = I_n.\]

By Lemma 3.1, there exists \(g ∈ GL_n(q)\) such that \((hμ)^s = ψ\). From \(H ≤ \langle h, (R_{p_1} \cdots R_{p_t})I_n \rangle = \langle hμ, (R_{p_1} \cdots R_{p_t})I_n \rangle\) we get \(H^s ≤ \langle ψ, (R_{p_1} \cdots R_{p_t})I_n \rangle\).

**Lemma 3.3.** Assume that Theorem 1.1 is false and let \(n\) be the smallest degree for which there exists a counterexample \(H\). Then:

1) \(H\) is indecomposable;

2) if \(H\) is chosen so that the number \(t\) of prime divisors of its order is minimum with respect to all counterexamples of degree \(n\), every prime \(p\) which divides \(|H|\) also divides \(|H ∩ GL_n(q)|\).

**Proof.** 1) If \(H\) is decomposable, we can apply Lemma 2.3 with \(ℓ > 1\) and \(K = H^{x_i}\), for an appropriate \(x ∈ GL_n(q)\). By the minimality of \(n\), for each \(j ≤ ℓ\) there exists \(g_j ∈ GL_{n_j}(q)\) such that

\[(π_j(H^{x_i}))^{s_j} ≤ \langle ψ\rangle Diag_{n_j}(R_{p_1} \cdots R_{p_t}) Sym(n_j).\]

Taking \(g = (g_1, \cdots, g_ℓ)\) we get \(H^{x_g} ≤ \langle ψ\rangle Diag_n(R_{p_1} \cdots R_{p_t}) Sym(n)\).

2) If \(p\) does not divide \(|H ∩ GL_n(q)|\), we have \(H_p ∩ GL_n(q) = 1\), hence

\[|H_p| = |h_p| = |ψ_p|.\]

By Lemma 3.1, there exists \(y ∈ GL_n(q)\) such that \(h_p^γ = ψ_p\). Substituting \(H\) with \(H^y\) we have that \(H_p = \langle ψ_p\rangle\). It follows:

\[H_p^{γ} ≤ C_{(ψ) GL_n(q)}(ψ_p) = \langle ψ_p\rangle × \langle ψ_p^r\rangle GL_n(C_{ψ_p^r}(ψ_p)).\]

Hence \(H_p^{γ} ≤ \langle ψ_p^r\rangle GL_n(C_{ψ_p^r}(ψ_p))\). By the minimality of \(t\), there exists \(g ∈ GL_n(C_{ψ_p^r}(ψ_p))\) such that \(H_p^{γ} ≤ \langle ψ_p^r\rangle Diag_n(R_{p_1} \cdots R_{p_t}) Sym(n)\). Noting that \(g\) centralizes \(ψ_p\), we have that \(H\) satisfies Theorem 1.1. As this fact contradicts our assumptions, we conclude that \(p\) divides \(|H ∩ GL_n(q)|\). □

**Proof. (Theorem 1.1).**

Assume that Theorem 1.1 is false and let \(n, H\) and \(t\) be such that points 1) and 2) of Lemma 3.3 hold. By Lemma 2.4, \(t > 1\), and by Lemma 3.2, there exists a non-scalar Sylow \(p\)-subgroup \(P\) of \(H ∩ GL_n(q)\). Say \(p = p_1\), and set

\[C := C_{Mat_n(q)}(P), \quad Z = Z(C).\]
Thus
\[ H \leq N_{(\gamma)GL_n(q)}(C), \quad H \leq N_{(\gamma)GL_n(q)}(Z). \] (11)

**Case 1** \( P \) has a unique homogeneous component \( W \), of dimension \( m \), say. As \( P_W \) is an irreducible subgroup of \( GL_m(q) \), a Sylow \( p \)-subgroup of \( GL_m(q) \) must be irreducible. From \( q \equiv 1 \mod p \), we have that \( m \) is a power of \( p \). \( Z \) is a field extension of \( \mathbb{F}_q \) and we claim that it has order \( q^{\beta^\alpha} \), for some \( \alpha \geq 0 \). Indeed, up to conjugation, we may assume
\[ Z^* = \left\langle \begin{pmatrix} z & \cdots & z \end{pmatrix} \right\rangle \text{ is irreducible.} \]

The characteristic polynomial \( c(t) \) of each block \( z \) has degree which divides \( m \). As \( c(t) \) is also the minimum polynomial of \( z \), our claim follows.

1.1 \( \alpha > 0 \). Let \( \psi \langle z \rangle = N_{(\gamma)GL_p^\alpha(q)}(\langle z \rangle) \), with \( \psi \) defined as in Lemma 2.7, with \( a = p^\alpha \). The kernel of the homomorphism
\[ f : N_{GL_p^\alpha(q)}(z) \to \text{Gal}(\mathbb{F}_{q^p^\alpha} : \mathbb{F}_q) \]
induced by the conjugation action, coincides with \( \langle z \rangle \). From \( \langle \psi \rangle \cap \langle z \rangle = 1 \), we deduce that the restriction of \( f \) to \( \langle \psi \rangle \cap GL_p^\alpha(q) \), is injective. Hence \( \langle \psi \rangle \cap GL_p^\alpha(q) \) is a \( p \)-group. It follows that \( |(\psi \langle z \rangle)_p| = |\psi_p| \) and, by Lemma 3.1, up to conjugation under \( GL_p^\alpha(q) \), we may suppose that \( (\psi \langle z \rangle)_p = \psi_p \), i.e.
\[ \langle \psi \rangle = \langle (\psi \langle z \rangle)_p \rangle \times \langle \psi_p \rangle. \]

From \( |z| = q^{p^\alpha} - 1 \) with \( p = p_1 \) and the assumption \( r \equiv 1 \mod p_j \), \( j \leq k \), it follows that for \( j \geq 2 \) the Sylow \( p_j \)-subgroup of \( \langle z \rangle \) coincides with \( R_{p_j} \). Hence it is scalar. Let \( \overline{R}_p \) be the Sylow \( p \)-subgroup of \( \langle z \rangle \). By Lemma 2.6, with \( \psi \) and \( q \) replaced respectively by \( \psi \) and \( q^{p^\alpha} \), the group
\[ \langle \psi \rangle \overline{R}_p \text{ Diag}_p(\overline{R}_p \cdots \overline{R}_p) = \langle (\psi \langle z \rangle)_p \rangle \overline{R}_p \langle \psi_p \rangle \text{ Diag}_p(\overline{R}_p \cdots \overline{R}_p) \]
is a Carter subgroup of \( \langle \psi \rangle \langle z \rangle \). In particular \( \langle (\psi \langle z \rangle)_p \rangle \overline{R}_p \) is centralized by \( \psi_p \), hence:
\[ \langle (\psi \langle z \rangle)_p \rangle \overline{R}_p \leq \langle \psi \rangle GL_p^\alpha(r_{m_p}). \]

Note that, if \( p = 2 \) and \( im_p \) is odd, then \( |\phi| = im \) is odd and, in this case, we are assuming \( r_0 \equiv 1 \mod 4 \). So, by Lemma (2.4), there exists \( x \in GL_p^\alpha(r_{mp}) \) such that \( \langle (\psi \langle z \rangle)_p \rangle \overline{R}_p \rangle^x \) lies in \( \langle \psi \rangle \text{ Diag}_p^\alpha(\overline{R}_p) \text{ Sym}(p^\alpha) \). Substituting \( z \) with \( z^x \) we may suppose:
\[ \langle \psi \rangle \overline{R}_p \leq \langle \psi \rangle \text{ Diag}_p^\alpha(\overline{R}_p) \text{ Sym}(p^\alpha). \] (12)
1.1.1 \( n = p^\alpha \). In this case \( Z^* = \langle z \rangle \), hence \( H \leq \langle \psi \nu \rangle \langle z \rangle \). By Lemma 2.6, up to conjugation under \( \langle z \rangle \), we may suppose \( H \leq \langle \psi \nu \rangle R_p (R_{p^2} \cdots R_{p_t}) I_n \). Hence \( H \) satisfies Theorem 1.1 in virtue of (12).

1.1.2 \( n > p^\alpha \). From \( \mathbb{F}_{q^n}^* \leq \langle Z \rangle \) and \( C_{\langle \psi \rangle \text{GL}_n(q)}(\mathbb{F}_{q^n} I_n) = \text{GL}_n(q) \):}

\[
C_{\langle \psi \rangle \text{GL}_n(q)}(Z) = C_{\text{GL}_n(q)}(Z) = \text{GL}_{\frac{n}{14}}(q^{p^\alpha}).
\]

Thus, by (11), setting \( \Psi = \psi(\nu, \cdots, \nu) \) with \( \nu \in \text{GL}_{p^\alpha}(q) \) as above:

\[
H \leq N_{\langle \psi \rangle \text{GL}_n(q)}(Z) = \langle \Psi \rangle \text{GL}_{\frac{n}{14}}(q^{p^\alpha}).
\]

Note that \( \langle \Psi \rangle \) intersects trivially \( \text{GL}_{\frac{n}{14}}(q^{p^\alpha}) \) as the automorphism induced by \( \Psi \) on the center of \( \text{GL}_{\frac{n}{14}}(q^{p^\alpha}) \) has order \( mp^\alpha = |\Psi| \). From

\[
H \cap \text{GL}_{\frac{n}{14}}(q^{p^\alpha}) \leq H \cap \text{GL}_n(q)
\]

and our assumptions on \( n \), there exists \( g \in \text{GL}_{\frac{n}{14}}(q^{p^\alpha}) \) such that

\[
H^g \leq \langle \Psi \rangle \text{Diag}_{\frac{n}{14}}(\overline{R}_p (R_{p^2} \cdots R_{p_t})) \text{Sym} \left( \frac{n}{p^\alpha} \right).
\]

We have that \( H \) satisfies Theorem 1.1 in virtue of (12), recalling the definition of \( \Psi \) given just before (14).

1.2 \( \alpha = 0 \). In the notation of (2), up to conjugation under \( \text{GL}_n(q) \) we may suppose that \( P \leq \text{GL}_u(q) \times \cdots \times \text{GL}_u(q) = (\text{GL}_u(q))^\ell \) and, moreover, that:

\[
P = \left\{ \begin{pmatrix} A & \cdots & A \\ \vdots & \ddots & \vdots \\ A & \cdots & A \end{pmatrix} \mid A \in \pi_1(P) \right\} = \pi_1(P) \otimes I_\ell
\]

where \( \pi_1(P) \) is an absolutely irreducible subgroup of \( \text{GL}_u(q) \). It follows:

\[
C = C_{\text{Mat}_u(q)}(P) = I_u \otimes \text{Mat}_\ell(q)
\]

\[
C_{\text{Mat}_u(q)}(C) = \text{Mat}_u(q) \otimes I_\ell.
\]

By a result of Skolem-Noether [3, Theorem 3.62, page 69], every automorphism of \( \text{Mat}_\ell(q) \) which induces the identity on the center is inner. Hence:

\[
N = N_{\langle \psi \rangle \text{GL}_u(q)}(C) = \langle \psi \rangle \text{GL}_u(q) \otimes \text{GL}_\ell(q).
\]

Let let \( h \) be as in (8). Recalling that \( H \leq N \) we have \( H_p = \langle h_p \rangle P \) with:

\[
h_p = \psi_p a \otimes b, \quad a \in \text{GL}_u(q), \; b \in \text{GL}_\ell(q).
\]
We claim that, up to conjugation under \( \text{GL}_\ell(q) \), we may suppose \( b \) scalar. Indeed, call \( e \) the order of \( \psi_p \). Then:

\[
h_p^e = a^{1+\psi+\cdots+\psi^{e-1}} \otimes b^{1+\psi+\cdots+\psi^{e-1}} \in P.
\]

This forces \( b^{1+\psi+\cdots+\psi^{e-1}} \) to be scalar, i.e. \( (\psi_p b)^e = \rho I_\ell \). Clearly \( \psi_p b \) centralizes \( \rho I_\ell \), hence \( \rho \in C_{F_q}(\psi_p) \). It follows from (4) that \( \rho = \lambda^{1+\psi+\cdots+\psi^{e-1}} \) for some \( \lambda \in F_q \). So \( \psi_p b \lambda^{-1} \) has the same order \( e \) of \( \psi_p \) and our claim follows from Lemma 3.1. So in (18) we may assume \( b \) scalar, i.e.

\[
H_p \leq \langle \psi_p \rangle \text{GL}_u(q) \otimes I_\ell
\]

and it makes sense to consider the projection \( \pi_1 : H_p \to \langle \psi_p \rangle \text{GL}_u(q) \).

In particular \( C_{\text{Mat}_u(q)}(\pi_1(P)) = F_q I_n \). So, if we set \( C_{F_q}(\psi_p) = F_{q_0} \), then:

\[
C_{\text{Mat}_u(q)}(\pi_1(H_p)) = F_{q_0} I_u.
\]

From (15) and (19) it follows:

\[
C_{\text{Mat}_u(q)}(H_p) = I_u \otimes \text{Mat}_\ell(q_0).
\]

Under our assumptions, there exists a Sylow \( p \)-subgroup \( \Gamma_{u,p} \) of \( \langle \psi \rangle \text{GL}_u(q) \) which is monomial. Noting that \( \psi \) normalizes \( \Gamma_{u,p} \), up to conjugation under \( \text{GL}_u(q) \) we may assume \( \pi_1(H_p) \leq \Gamma_{u,p} \). So we have \( H_p \leq \Gamma_p \) where \( \Gamma_p \) is defined as in Lemma 2.4. Clearly \( H_{p'} \leq \langle \psi_{p'} \rangle \text{GL}_n(q) \). By Lemma 2.6 \( \psi_{p'} \) centralizes \( \Gamma_p \) and, a fortiori, \( H_p \). Hence

\[
H_{p'} \leq C_{\langle \psi_{p'} \rangle \text{GL}_n(q)}(H_p) = \langle \psi_{p'} \rangle (I_u \otimes \text{GL}_\ell(q_0)).
\]

By the minimality of \( n \), there exists \( x \in I_u \otimes \text{GL}_\ell(q_0) \) such that \( H_{p'}^x \) is monomial. Since \( x \) centralizes \( H_p \), this completes this case.

**Case 2** The homogeneous components of \( P \) are more than one. They are permuted transitively by \( h \), defined in (8), since \( H \) is indecomposable. Let \( M \) be a maximal subgroup of \( H \) containing the stabilizer of a component \( V_0 \) and call \( W \) the subspace generated by \( \{m(V_0) \mid m \in M\} \). Then \( M \) is a normal subgroup of prime index \( s \) and

\[
F_q^m = W \oplus h(W) \oplus \cdots \oplus h^{s-1}(W)
\]

where each direct summand is stabilized by \( M \). Thus, in the notation of (2), we may assume that \( H \) is a subgroup of

\[
\langle \psi \rangle \left( \text{GL}_2(q) \times \cdots \times \text{GL}_2(q) \right) \langle \sigma \rangle = \langle \psi \rangle \text{GL}_2(q)^s \langle \sigma \rangle
\]

(21)
where \( \sigma \) is a permutation matrix of order \( s \), which permutes the direct factors of \( \text{GL}_\mathbb{F}_q^s(q) \) and

\[
H_s' \leq M \leq \langle \psi \rangle \text{GL}_\mathbb{F}_q^s(q)^s. \tag{22}
\]

Let \( \eta \) be a preimage of \( \sigma \) in \( H_s \). Thus:

\[
\eta = \psi^\ell \sigma(y_1, \cdots, y_s), \tag{23}
\]

where \( \psi^\ell \) can be chosen to be an \( s \)-element and each \( y_j \in \text{GL}_\mathbb{F}_q^s(q) \).

We will use the fact that, for each \( g = (g_1, \cdots, g_s) \in \text{GL}_\mathbb{F}_q^s(q)^s \):

\[
\eta(g_1, \cdots, g_s) = \psi^\ell \sigma(\bar{y}_1, \cdots, \bar{y}_s), \quad \bar{y}_j \in \text{GL}_\mathbb{F}_q^s(q). \tag{24}
\]

Since we are assuming that point 2) of Lemma 3.3 holds, we can say \( s = p_i \). Recalling (22) and the minimality of \( n \), after a first conjugation of \( H \) by some \( g \in \text{GL}_\mathbb{F}_q^s(q)^s \), we can suppose that

\[
H_{s'} \leq \langle \psi \rangle \left( \text{Diag}_{\mathbb{F}_q^s}(R_{p_1} \cdots R_{p_{i-1}}) \text{Sym}(\frac{n}{s}) \right)^s.
\]

Recalling (23) this conjugation takes \( \eta \) into an element of the same shape. Note that

\[
R_{p_1} \cdots R_{p_{i-1}} \leq \prod_{j=m_{p_1}, \cdots, m_{p_{i-1}}} \text{Sym}(p_j)
\]
as a consequence of point 2) of Lemma 2.4. It follows that \( \psi^\ell \), whose order divides \( m_s = m_{p_1} \), centralizes \( H_{s'} \). Imposing that \( \eta \) centralizes \( H_{s'} \) we see that, up to a new conjugation by some element in \( \text{GL}_\mathbb{F}_q^s(q)^s \), we may assume that \( H_{s'} \) consists of elements of shape:

\[
\psi^\ell \left( \begin{array}{c}
x \\
\vdots \\
x
\end{array} \right), \quad x \in \text{GL}_\mathbb{F}_q^s(q).
\]

Set \( G := \langle \psi \rangle \text{GL}_\mathbb{F}_q^s(q)^s \langle \sigma \rangle \) and let \( S \) be a Sylow \( s \)-subgroup of \( C_G(H_{s'}) \) which contains \( H_s \). Then \( \hat{H} := S \times H_{s'} \) is nilpotent and \( H \leq \hat{H} \leq G \). As \( \sigma \) centralizes \( H_{s'} \), there exists \( c \in C_G(H_{s'}) \) such that \( \sigma \in S^c \). Then

\[
\hat{H}^c = \langle \sigma \rangle \left( \hat{H}^c \cap \langle \psi \rangle \text{GL}_\mathbb{F}_q^s(q)^s \right).
\]

The conjugation action of \( \sigma \) on the second factor implies that, for each \( j \leq s \),

\[
\pi_j \left( \hat{H}^c \cap \langle \psi \rangle \text{GL}_\mathbb{F}_q^s(q)^s \right) = \pi_1 \left( \hat{H}^c \cap \langle \psi \rangle \text{GL}_\mathbb{F}_q^s(q)^s \right)
\]

Moreover, by the minimality of \( n \), there exists \( g_0 \in \text{GL}_\mathbb{F}_q^s(q) \) such that

\[
\left( \pi_1 \left( \hat{H}^c \cap \langle \psi \rangle \text{GL}_\mathbb{F}_q^s(q)^s \right) \right)^{g_0} \leq \langle \psi \rangle \text{Diag}_{\mathbb{F}_q^s}(R_{p_1} \cdots R_{p_{i-1}}) \left( \frac{n}{s} \right).
\]

Taking \( g := (g_0, \ldots, g_0) \) we have that \( \sigma^g = \sigma \), hence

\[
H^g \leq \hat{H}^g \leq \langle \psi \rangle \text{Diag}(R_{p_1} \cdots R_{p_{i-1}}) \text{Sym}(n).
\]

\[\square\]
4. An application to Carter Subgroups

Denote by $F_G$ the set of conjugacy classes of Carter subgroups of a finite group $G$. As an easy consequence of [1], it is shown in [11] the following.

**Lemma 4.1.** Let $N$ be a finite normal solvable subgroup of $G$. The canonical epimorphism $\pi : G \rightarrow G/N$ induces a bijection $\hat{\pi} : F_G \rightarrow F_{G/N}$.

So the conjugacy conjecture holds for $G$ if and only if it holds for $G/N$.

**Lemma 4.2.** For each $n \geq 2$:

1) Two semisimple elements of $GL_n(q)$ are conjugate under $GL_n(q)$ if and only if they are conjugate under $SL_n(q)$;

2) the Jordan unipotent block $J_n$ is conjugate to its inverse under $GL_n(\mathbb{Z})$;

3) if $q \equiv 1 \pmod{4}$, then every element $z$ of order $r_0$ is conjugate to its inverse under $SL_n(q)$.

**Proof.** 1) A consequence of the fact that the centralizer of a semisimple element contains matrices of all possible determinants. Enough to see this fact for the companion matrix $m$ of an irreducible polynomial. The algebra generated by $m$ over $F_q$ is a field, whose multiplicative group is generated by a Singer cycle $c$ of order $q^n - 1$. As $\langle c \rangle \cap SL_n(q)$ has index $q - 1$ in $\langle c \rangle$ [7, Satz 7.3, page 187], the determinant of $c$ generates $F_q^\times$.

2) Set $A_1 := \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and, for $s \geq 1$, $A_{s+1} := \begin{pmatrix} J_s A_s \\ (-1)^s A_s \end{pmatrix}$.

Then $(A_{s+1})^2 = I$ and $A_{s+1} J_{s+1} A_{s+1} = J_{s+1}^{-1}$. In fact, by induction:

$$A_{s+1} J_{s+1} A_{s+1} = A_{s+1} \begin{pmatrix} J_s \\ e_s \\ 1 \end{pmatrix} A_{s+1} = \begin{pmatrix} J_{s+1}^{-1} \\ (-1)^{s+1} e_s J_s A_s \\ 1 \end{pmatrix}.$$

The conclusion follows noting that $(-1)^s e_s = e_s A_s$, hence

$$(-1)^{s+1} e_s J_s A_s = -(-1)^s e_s J_s A_s = -e_s A_s J_s A_s = -e_s J_s^{-1}.$$

3) By the previous result, modulo $r_0$, each block $J_m$ of the Jordan form of $z$ is conjugate to $J_m^{-1}$, via $A_m$. Our claim follows from the fact that $A_m$ has determinant 1 if $m \equiv 0, 3 \pmod{4}$ and has determinant $-1$ if $m \equiv 1, 2 \pmod{4}$. But in the second case, the assumption $q \equiv 1 \pmod{4}$ ensures that in $SL_n(q)$ there are scalar matrices of determinant $-1$. □

A more general statement than point 3) of the previous Lemma can be found in [13, Theorem 1.4 (i), (ii)].
Proof. (Corollary 1.2)

Suppose, by contradiction, that the projective image of $A$ is a minimal counterexample to the conjecture. We show that every Carter subgroup $H$ of $A$ is conjugate to a subgroup of $A \cap M$, where $M$ is the generalised monomial subgroup, and this will easily lead to a contradiction. For a fixed $H$, we may choose our notation so that $H \cap GL_n(q) = \langle \psi \rangle \cap GL_n(q)$ for an appropriate power $\psi$ of $\varphi$, as in Theorem 1.1, and set $r = |C_{\mathbb{F}_q}(\psi)|$. Now let $p$ be a prime which divides $|H \cap GL_n(q)|$. As $H \cap GL_n(q)$ is normal in $H$, there exists $z \in Z(H) \cap GL_n(q)$ of order $p$. If $p = r_0$, we have that $z$ is conjugate to $z^{-1}$ under $A$ by point 3) of the previous Lemma. But, as explained in the Introduction, the projective image of $z$ cannot be conjugate to its inverse [12, Lemma 3.1 (b)]. This forces $z = z^{-1}$. Hence $r_0 = 2$, which contradicts the assumption $q \equiv 1 \pmod{4}$. Thus $p \neq r_0$.

In the notation of (8), we have $z = z^h$ where $h \in H$ is such that $h \varphi z^{-1}$ is conjugate to $A$. Therefore, $z$ is conjugate to $z^\varphi$ under $GL_n(q)$. This gives that the Jordan canonical form $J$ of $z$ is conjugate to the Jordan canonical form $J^\varphi$ of $z^\varphi$ under $GL_n(\mathbb{F})$, where $\mathbb{F}$ denotes the algebraic closure of $\mathbb{F}_q$. But $J^\varphi = J^r$ gives that $z$ and $z^r$ are conjugate under $GL_n(q)$ [9, Corollary 2, page 397]. We conclude that $z$ is conjugate to $z^r$ under $A$, by point 1) of the previous Lemma. In particular, if $z$ is scalar, we have $z = z^r$ whence $r \equiv 1 \pmod{p}$. So assume that $z$ is non-scalar. Again by [12, Lemma 3.1 (b)] the projective images of $z$ and $z^r$ must be the same, i.e. $z^r = \rho z$ for some $\rho \in \mathbb{F}_q^*$. This gives that $z^{-1}$ is scalar and we conclude that $r \equiv 1 \pmod{p}$, otherwise $z$ would belong to $\langle z^{-1} \rangle$. By Theorem 1.1, there exists $g \in GL_n(q)$ such that $H^g \leq M = \langle \varphi \rangle \cap GL_n(\mathbb{F}_q^*) \cap Sym(n)$. For an appropriate $d \in \text{Diag}_n(\mathbb{F}_q^*)$ we have that $a = gd \in SL_n(q) \leq A$. Thus $H$ is conjugate, under $A$, to a subgroup of $A \cap M$. To reach the desired contradiction note that, by the assumption $SL_n(q) \leq A$:

$$
\frac{A \cap M}{A \cap \langle \varphi \rangle \cap \text{Diag}_n(\mathbb{F}_q^*)} \sim Sym(n).
$$

The Carter subgroups of $Sym(n)$ are its Sylow 2-subgroups, by [5]. Thus, by Lemma 4.1, all Carter subgroups of $A \cap M$ are conjugate. We conclude that all Carter subgroups of $A$ are conjugate, a contradiction.

Proof. of Theorem 1.3.

$HD/D$ is a Carter subgroup of $M_0/D = \langle \psi \rangle \times Sym(n)$. Thus, by [5], $HD/D$ coincides with $\langle \psi \rangle \Sigma_2$, where $\Sigma_2$ is a Sylow 2-subgroup of $Sym(n)$. It follows that $H$ is a Carter subgroup of

$$
HD = \langle \psi \rangle \Sigma_2 D = \langle \psi \rangle \Sigma_2 D_2.
$$
If \( d \in D \), we have \( d^\psi = d^r \). This means that \( \psi \) normalizes every subgroup of \( D \). In particular \( \langle \psi \rangle_{2'} \) stabilizes any composition series of \( D_2 \), inducing the identity on each composition factor. It follows that \( \langle \psi \rangle_{2'} \) centralizes \( D_2 \) [6, Theorem 3.2, page 178]. Thus \( \langle \psi \rangle \Sigma_2 D_2 \) is a nilpotent group. Note that \( D_{2'} \) is characteristic in \( D \), hence normal in \( HD \). Again \( HD_{2'} / D_{2'} \) is a Carter subgroup of \( HD / D_2 = \langle \psi \rangle \Sigma_2 D_2 \). It follows that \( HD_{2'} = HD \). As \( HD \) contains a Sylow 2-subgroup of \( M_0 \), the same is true for \( H \).

\[ \square \]

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ANDREA LUCCHINI

*Dipartimento di Matematica Pura ed Applicata*

*Università degli Studi di Padova*

*Via Trieste 63*

*35121 Padova, Italy*

*e-mail: lucchini@math.unipd.it*

M. CHIARA TAMBURINI

*Dipartimento di Matematica e Fisica*

*Università Cattolica del Sacro Cuore*

*Via Musei 41*

*25121 Brescia, Italy*

*e-mail: c.tamburini@dmf.unicatt.it*