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$W^{2,p}$-REGULARITY FOR A CLASS OF ELLIPTIC SECOND ORDER EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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We prove a well-posedness result in the class $W^{2,p} \cap W_0^{1,p}$ for the Dirichlet problem $(*)$ below. We assume L to be an elliptic second order operator with discontinuous coefficients and lower order terms. The paper extends a recent result (see [1], [2]) for operators restricted to leading terms.

Introduction.

In this paper we consider an elliptic second order operator with discontinuous coefficients and lower order terms proving the well-posedness of the Dirichlet problem

$$(*) \qquad \begin{cases} Lu = f \in L^p \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \end{cases} \qquad 1 < p < +\infty .$$

The result was previously known under the assumption that the leading terms' coefficients a_{ij} are in the space $C^0(\overline{\Omega})$ and that the lower order terms have coefficients b_i, c belonging to suitable L^p spaces (see [3], [5], [6], [7]). Recently F. Chiarenza, M. Frasca and P. Longo in the papers [1] [2] were able to give L^p

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estimates and to establish the well-posedness of $(*)$ when Lu is an elliptic second order operator without lower order terms and the coefficients a_{ij} are assumed to be in VMO. VMO is the subspace of JOHN-NIRENBERG's space BMO whose elements have norm on the balls vanishing as the radius of the ball approaches zero (see Sec.1 for precise definitions).

Our purpose here is to extend their work in order to allow operators with lower order terms. Our proof is very different from the one given by M. Chicco in his paper [3] in the case of continuous coefficients where a very careful spectral analysis is performed by pointwise perturbing a smooth coefficients operator. This seems to be impossible in this case because the discontinuity of the coefficients. What we do is obtain a-priori estimates, using in part the work [2] and then we obtain the result by a standard approximation argument.

1. Some functional spaces.

We start this section by recalling the definitions of the spaces BMO and VMO.

We say that a locally integrable function f in $\mathbb{R}^n$ is in the space BMO if

$$\sup_B \fint_B |f(x) - f_B|\, dx = \|f\|_* < +\infty$$

where B ranges in the class of the balls in $\mathbb{R}^n$. Here f_B is the average of f over B: $\fint_B f(x)\, dx$.

For $f \in$ BMO and $r > 0$ we set

$$\sup_{\rho \le r} \fint_B |f(x) - f_B|\, dx = \eta(r) \tag{1.1}$$

where B ranges in the class of the balls with radius ρ.

We say that a function $f \in$ BMO is in the space VMO (see [8]) if $\lim_{r \to 0+} \eta(r) = 0$. We will refer to $\eta(r)$ as the VMO modulus of f.

We will need for further developments the following known property of the space VMO (see e.g. [8], [4]).

Theorem 1.1. *For $f \in$ BMO the following conditions are equivalent:*

(1) f is in VMO;

(2) f is in the BMO closure of the set of the uniformly continuous functions which belong to BMO;

(3) $\lim_{y\to 0} \| f(x-y) - f(x) \|_* = 0.$

By this theorem and a known result (see [4]) we have that if $f \in$ VMO, the usual mollifiers converge to f in the BMO norm. In other words, given any $f \in$ VMO with VMO modulus $\eta(r)$, it is possible to find a sequence of C^∞ functions $\{f_h\}$ converging to f in BMO as $h \to 0$ and with their VMO moduli $\eta_h(r) \leq \eta(r)$.

Moreover, for $f \in L^p(\Omega)$, we set

$$\sup_{|E|\leq\sigma} \int_E |f(x)|^p \, dx = \omega^p(\sigma) \text{ (}^1\text{)}.$$

Clearly $\omega(\sigma)$ is a decreasing function in $]0, |\Omega|]$ such that $\lim_{\sigma\to 0} \omega(\sigma) = 0$. We will refer to $\omega(\sigma)$ as the AC modulus of $|f|^p$.

2. Notations, assumptions and main result.

In Ω, a bounded open set of $\mathbb{R}^n$ $(n \geq 3)$, we consider the elliptic equation in nondivergence form

$$Lu = \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b_i(x) u_{x_i} + cu = f$$

and the associated Dirichlet problem

$$\text{(2.1)} \qquad \begin{cases} Lu = f & \text{in } \Omega \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \,,\ f \in L^p(\Omega) & \text{with } p \in]1, +\infty[\,. \end{cases}$$

On the coefficients of L we make the following assumptions

$$\text{(2.2)} \qquad \begin{cases} a_{ij}(x) \in \text{VMO} \cap L^\infty(\mathbb{R}^n) & i,j = 1,\dots,n \\ a_{ij} = a_{ji} & i,j = 1,\dots,n \quad \text{a.e. in } \Omega \\ \exists \lambda > 0 : & \\ \lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leq \lambda|\xi|^2 & \text{a.e. in } \Omega \quad \forall \xi \in \mathbb{R}^n \end{cases}$$

$$\text{(2.3)} \qquad \begin{cases} b_i \in L^t(\Omega) \quad i = 1,\dots,n & \text{where } t = n \text{ for } 1 < p < n, \\ & t > n \text{ for } p = n,\ t = p \text{ for } p > n. \end{cases}$$

(1) If $E \subseteq \mathbb{R}^n$ is Lebesgue measurable we set $|E|$ for its Lebesgue measure.

(2.4) $\begin{cases} c \in L^s(\Omega) & \text{where } s = \frac{n}{2} \text{ for } 1 < p < \frac{n}{2},\ s > \frac{n}{2} \text{ for } p = \frac{n}{2}, \\ & s = p \text{ for } p > \frac{n}{2}, \quad c \leq 0 \text{ a.e.in } \Omega \end{cases}$

Our purpose is to prove an existence and uniqueness result for problem (2.1).

First we want to prove the following theorem

Theorem 2.1. *Let* $\partial\Omega \in C^{1,1}$ *and assume (2.2), (2.3) and (2.4). Let* $q,\ p \in]1, +\infty[,\ q \leq p,\ f \in L^p(\Omega)$. *Then there exists a positive number* r_0 *depending on* $n,\ \lambda,\ p,$ *the* VMO *moduli of* a_{ij} *and the* AC *moduli of* $|b|^t = \left(\sum_{i=1}^{n} b_i^2\right)^{t/2}$ *and of* $|c|^s$ *such that for* $r \in]0, r_0]$, B_r *a ball with radius* r, *and for any* u *solution of the problem*

$$\begin{cases} Lu = f & \text{a.e. in } \Omega_r = \Omega \cap B_r \neq \emptyset \\ u \in W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r) \end{cases}$$

we have $u \in W^{2,p}(\Omega_r) \cap W_0^{1,p}(\Omega_r)$.
Furthermore there exists a positive constant C_0 *such that*

$$\| u_{x_ix_j} \|_{L^p(\Omega_r)} \leq C_0 \| f \|_{L^p(\Omega_r)} \qquad \forall i, j = 1, \ldots, n. \tag{2.5}$$

Here C_0 *depends on* $n,\ p,\ \partial\Omega,\ \lambda,$ *the* VMO *moduli of* $a_{i,j}$ $(i, j = 1, \ldots, n)$.

Proof. We start by observing that for the Dirichlet problem

$$\begin{cases} \widetilde{L}u = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_ix_j} = f - \sum_{i=1}^{n} b_i u_{x_i} - cu & \text{a.e. in } \Omega_r \\ u \in W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r), \quad f \in L^q(\Omega_r), \end{cases} \tag{2.6}$$

F. Chiarenza, M. Frasca and P. Longo in [2] have proved the inequality

$$\|u_{x_ix_j}\|_{L^q(\Omega_r)} \leq C_1 \left\| f - \sum_{i=1}^{n} u_{x_i} - cu \right\|_{L^q(\Omega_r)}.$$

By Sobolev's lemma it follows

$$\begin{aligned} \|u\|_{W^{2,q}(\Omega_r)\cap W_0^{1,q}(\Omega_r)} &\leq C_1 \|f\|_{L^q(\Omega_r)} + \\ &+ C_1 S \Big(\| |b| \|_{L^t(\Omega_r)} + \|c\|_{L^s(\Omega_r)} \Big) \|u\|_{W^{2,q}(\Omega_r)\cap W_0^{1,q}(\Omega_r)} \end{aligned} \tag{2.7}$$

where S is the Sobolev's constant.
Fix $r_0 > 0$ so small that

$$\| |b| \|_{L^t(\Omega_{r_0})} + \|c\|_{L^s(\Omega_{r_0})} < \frac{1}{2C_1 S}\,.$$

Then from (2.7) we obtain for any $r \in]0, r_0]$

$$\|u\|_{W^{2,q}(\Omega_r)\cap W_0^{1,q}(\Omega_r)} \le C_0 \|f\|_{L^q(\Omega_r)}\,. \tag{2.8}$$

Define, for $r < r_0$

$$T : W^{2,q}(\Omega_r) \to L^q(\Omega_r)$$

for any $q \le p$ by setting

$$Tv = f - \sum_{i=1}^{n} b_i v_{x_i} - cv\,.$$

We now consider the Dirichlet problem

$$\begin{cases} \tilde{L}u = Tu \\ u \in W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r)\,, \end{cases}$$

with v given in $W^{2,q}(\Omega_r)$.
For this problem the existence of a unique solution u is known by theorems 4.3 and 4.4 in [2].
Then we can define

$$Fv : W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r) \to W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r)$$

by setting $Fv = u$.
The operator F is a contraction in $W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r)$ for any $q \le p$. Indeed, let

$$F(v_1) = u_1\,,\ F(v_2) = u_2\,,\ \text{for } v_1,\ v_2 \in W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r)\,.$$

Using (2.7) we have

$$\|F(v_1) - F(v_2)\|_{W^{2,q}(\Omega_r)\cap W_0^{1,q}(\Omega_r)} = \|u_1 - u_2\|_{W^{2,q}(\Omega_r)\cap W_0^{1,q}(\Omega_r)} \le$$
$$\le C_1 S \left(\| |b| \|_{L^t(\Omega_r)} + \|c\|_{L^s(\Omega_r)} \right) \cdot \|v_1 - v_2\|_{W^{2,q}(\Omega_r)\cap W_0^{1,q}(\Omega_r)}\,.$$

Then F has an unique fixed point in all the $W^{2,q}(\Omega_r) \cap W^{1,q}(\Omega_r)$ spaces with $q \le$p.
Let z be the fixed point in $W^{2,p}(\Omega_r) \cap W_0^{1,p}(\Omega_r) \subseteq W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r)$.
By this inclusion z is also a fixed point in $W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r)$ for any $q \le p$; because by assumption u is a fixed point in $W^{2,q}(\Omega_r) \cap W_0^{1,q}(\Omega_r)$ the uniqueness of the fixed point implies $z = u$. □

We now obtain the same result in Ω. More precisely, we show the following theorem

Theorem 2.2. *Let $\partial\Omega \in C^{1,1}$ and assume (2.2), (2.3), (2.4). Let $q,\ p \in]1,+\infty[,\ q \le p,\ f \in L^p(\Omega)$. Then for any u solution of the problem*

$$\begin{cases} Lu = f & \text{a.e. in } \Omega \\ u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \end{cases}$$

we have $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.
Furthermore it exists a positive constant C_3 such that

$$\|u\|_{W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)} \le C_3 \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right) . \tag{2.9}$$

Here C_3 depends on $n,\ p,\ \partial\Omega,\ \lambda$, on the VMO *moduli of $a_{i,j},\ i,j = 1,\dots,n$, on the L^t and L^s norm respectively of $b_i,\ i = 1,\dots,n$ and c and their* AC *moduli.*

Proof. We consider a covering of $\overline{\Omega}$ by N balls $B_{r_k},\ k = 1,\dots,N,\ r_k \in]0,r_0]$ (where r_0 is the same of the previous theorem).
We consider a partition of the unity α_k associated with this covering of $\overline{\Omega}$. Then $u = \sum_{k=1}^{N} \alpha_k u$ in $\overline{\Omega}$, with $\alpha_k u \in W^{2,q}(\Omega_{r_k}) \cap W_0^{1,q}(\Omega_{r_k})$, where: $\Omega_{r_k} = \Omega \cap B_{r_k}$.
We have

$$\begin{aligned} L(\alpha_k u) &= \\ &= \alpha_k Lu + u\big(L(\alpha_k) - c\alpha_k\big) + 2\sum_{i,j=1}^{n} a_{ij}(\alpha_k)_{x_i} u_{x_j} \qquad \text{a.e. in } B_{r_k} . \end{aligned} \tag{2.10}$$

By assumption $Lu \in L^p(\Omega_{r_k})$ and by Sobolev's lemma the terms on the right hand side of (2.10) belong to $L^\tau(\Omega_{r_k})$ with $\tau > q$. Then, by theorem 2.1, we have that the solutions of (2.10) belong $W^{2,\tau}(\Omega_{r_k}) \cap W_0^{1,\tau}(\Omega_{r_k})$.
If $\tau = p$, recalling $u = \sum_{k=1}^{N} \alpha_k u$, we have $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.
If $\tau < p$ the same result is obtained iterating this procedure a finite number of times.
Finally the estimate (2.9) is obtained by using (2.5). More precisely we have

$$\|\alpha_k u\|_{W^{2,p}(\Omega_{r_k})\cap W_0^{1,p}(\Omega_{r_k})} \le C_0 \|L(\alpha_k u)\|_{L^p(B_{r_k})} , \tag{2.11}$$

$k = 1,\dots,N$ and by (2.10)

$$\begin{aligned} \|\alpha_k u\|_{W^{2,p}(\Omega_{r_k})\cap W_0^{1,p}(\Omega_{r_k})} &\le C_0\Big\{ \|\alpha_k Lu\|_{L^p(\Omega_{r_k})} + \\ &+ \Big\| u \sum_{i,j=1}^{n} a_{ij}(\alpha_k)_{x_i x_j} \Big\|_{L^p(\Omega_{r_k})} + \Big\| \sum_{i=1}^{n} b_i(\alpha_k)_{x_i} u \Big\|_{L^p(\Omega_{r_k})} + \\ &+ 2\Big\| \sum_{i,j=1}^{n} a_{ij}(\alpha_k)_{x_i} u_{x_j} \Big\|_{L^p(\Omega_{r_k})} \Big\} = C_0\big\{I_1 + I_2 + I_3 + I_4\big\} . \end{aligned} \tag{2.12}$$

The conclusion will follow majorizing I_1,——, I_4.
Let us confine ourselves, for simplicity, to the case $1 < p < n/2$, the other cases being similar.
We have:

$$I_1 \leq \max_{\Omega_{r_k}} |\alpha_k| \, \|f\|_{L^p(\Omega_{r_k})} , \tag{2.13}$$

$$I_2 \leq n^2 \lambda \max_{i,j} \max_{\Omega_{r_k}} \left|(\alpha_k)_{x_i x_j}\right| \, \|u\|_{L^p(\Omega_{r_k})} . \tag{2.14}$$

The integrals I_3 and I_4 are estimated using Sobolev's lemma and the Nirenberg–Gagliardo's estimate.
We have for any $\varepsilon_k > 0$, $k = 1, \ldots, N$

$$\begin{aligned} I_3 &\leq S \max_i \max_{\Omega_{r_k}} |(\alpha_k)_{x_i}| \sum_{i=1}^n \|b_i\|_{L^n(\Omega_{r_k})} \|u_{x_i}\|_{L^p(\Omega_{r_k})} \leq \\ &\leq S \max_i \max_{\Omega_{r_k}} |(\alpha_k)_{x_i}| \sum_{i=1}^n \|b_i\|_{L^n(\Omega_{r_k})} \cdot \\ &\quad \cdot \left(\sum_{i,j} \varepsilon_k \|u_{x_i x_j}\|_{L^p(\Omega_{r_k})} + c(\varepsilon_k) \|u\|_{L^p(\Omega_{r_k})} \right) . \end{aligned} \tag{2.15}$$

$$\begin{aligned} I_4 &\leq 2n\lambda \max_i \max_{\Omega_{r_k}} |(\alpha_k)_{x_i}| \sum_{i=1}^n \|u_{x_i}\|_{L^p(\Omega_{r_k})} \leq \\ &\leq 2n\lambda \max_i \max_{\Omega_{r_k}} |(\alpha_k)_{x_i}| \left(\sum_{i,j} \varepsilon_k \|u_{x_i x_j}\|_{L^p(\Omega_{r_k})} + c(\varepsilon_k) \|u\|_{L^p(\Omega_{r_k})} \right) . \end{aligned} \tag{2.16}$$

Finally observing that

$$\|u\|_{W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)} \leq \sum_{k=1}^N \|\alpha_k u\|_{W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)}$$

and using (2.13),——,(2.16) from (2.12) we obtain the estimate (2.9). □

Theorem 2.3. (Uniqueness) *Let $\partial\Omega \in C^{1,1}$. Assume (2.2), i) $b_i \in L^t(\Omega)$ with $t > n$ for $1 < p \leq n$, $t = p$ for $p > n$ and ii) $c \in L^s(\Omega)$ with $s = n$ for $1 < p \leq n$, $s = p$ for $p > n$, $c \leq 0$ a.e. in Ω.*
Then the solution of the Dirichlet problem

$$\begin{cases} Lu = 0 & a.e.\ in\ \Omega \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \end{cases}$$

is 0 in Ω.

Proof. The function 0 belongs to $L^n(\Omega)$. By theorem 2.2 it follows that $u \in W^{2,n}(\Omega) \cap C^0(\overline{\Omega})$; hence, recalling the Pucci-Alexandroff maximum principle, the thesis follows. □

Theorem 2.4. (Existence) *Let $\partial\Omega \in C^{1,1}$. Assume (2.2), i) $b_i \in L^t(\Omega)$ with $t > n$ for $1 < p \leq n$, $t = p$ for $p > n$ and ii) $c \in L^s(\Omega)$ with $s = n$ for $1 < p \leq n$, $s = p$ for $p > n$, $c \leq 0$ a.e. in Ω. Then the Dirichlet problem (2.1) has a (unique) solution u. Furthermore it exists a positive constant C_4 such that*

$$\|u\|_{W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)} \leq C_4 \|f\|_{L^p(\Omega)} . \tag{2.17}$$

Here the constant C_4 depends on n, p, $\partial\Omega$, λ, on the VMO *moduli of a_{ij}, $i, j = 1, \ldots, n$, on the L^t and L^s norms respectively of b_i, $i = 1, \ldots, n$, and c and their* AC *moduli.*

Proof. First we prove the (2.17). Then the existence result will follow by a standard approximation argument.
We prove now estimate (2.17) by contradiction.
If (2.17) is not true, there exists a sequence of operators

$$\left\{ L^{(k)} = \sum_{i,j=1}^{n} a_{ij}^{(k)}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^{(k)}(x) \frac{\partial}{\partial x_i} + c^{(k)} \right\}$$

verifying assumptions (2.2), i) and ii) with the VMO moduli and the L^∞ norms of $a_{ij}^{(k)}$, $k \in$ N, uniformly bounded and the L^t and L^s norms respectively of $b_i^{(k)}$ and $c^{(k)}$, $k \in$ N, and their AC moduli uniformly bounded, and a sequence of functions $\{u^{(k)}\}$, $u^{(k)} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying

$$\|u^{(k)}\|_{W^{2,p}(\Omega)} = 1 \quad , \quad \lim_k \|L^{(k)} u^{(k)}\|_{L^p(\Omega)} = 0 .$$

We start by observing that it is possible to find subsequences of $\{a_{ij}^{(k)}\}$, $\{b_i^{(k)}\}$, $\{c^{(k)}\}$, which we relabel as $\{a_{ij}^{(k)}\}$, $\{b_i^{(k)}\}$ and $\{c^{(k)}\}$, such that $\{a_{ij}^{(k)}\}$

converges a.e. in $\mathbb{R}^n$ to a function α_{ij} verifying assumptions (2.2) (see theorem 4.4 of [2]), $\{b_i^{(k)}\}$, and $\{c^{(k)}\}$ weakly converging to $\beta_i \in L^t(\Omega)$ and to $\gamma \in L^s(\Omega)$ with t and s verifying assumptions i) and ii).
Now we set

$$L^{(\alpha)} = \sum_{i,j=1}^{n} \alpha_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \beta_i(x)\frac{\partial}{\partial x_i} + \gamma .$$

There exists a subsequence of $\{u^{(k)}\}$, weakly converging to a function $u^{(\alpha)} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and then $\{\|u^{(k)}\|_{L^p(\Omega)}\}$ converges to $\|u^{(\alpha)}\|_{L^p(\Omega)}$.
Suppose, for simplicity $1 < p < n$. Since for $\zeta \in L^{p'}(\Omega)$, $p' = p/(p-1)$, we have

$$\int_\Omega \left| \left(L^{(k)}u^{(k)} - L^{(\alpha)}u^{(\alpha)}\right) \zeta \right| dx \leq$$

$$\leq \int_\Omega \left| \sum_{i,j=1}^{n} \left(u^{(k)}_{x_i x_j} - u^{(\alpha)}_{x_i x_j}\right) \alpha_{ij}\zeta \right| dx + \sum_{i,j=1}^{n} \|u^{(k)}_{x_i x_j}\|_{L^p(\Omega)} \cdot$$

$$\cdot \left\| \left(a_{ij}^{(k)} - \alpha_{ij}\right)\zeta \right\|_{L^{p'}(\Omega)} + \sum_{i=1}^{n} \|u^{(k)}_{x_i} - u^{(\alpha)}_{x_i}\|_{L^q(\Omega)} \cdot \|\beta_i \zeta\|_{L^{q'}(\Omega)} +$$

$$+ \sum_{i=1}^{n} \|b_i^{(k)} - \beta_i\|_{L^t(\Omega)} \|\zeta\|_{L^{p'}(\Omega)} \cdot \|u^{(k)}_{x_i} - u^{(\alpha)}_{x_i}\|_{L^q(\Omega)} +$$

$$+ \sum_{i=1}^{n} \int_\Omega \left| u^{(\alpha)}_{x_i} \zeta \left(b_i^{(k)} - \beta_i\right)\right| dx + \|c^{(k)} - \gamma\|_{L^n(\Omega)} \|\zeta\|_{L^{p'}(\Omega)} \cdot$$

$$\cdot \|u^{(k)} - u^{(\alpha)}\|_{L^t(\Omega)} + \|u^{(k)} - u^{(\alpha)}\|_{L^t(\Omega)} \|\gamma\zeta\|_{L^{t'}(\Omega)} +$$

$$+ \int_\Omega \left| \left(c^{(k)} - \gamma\right) u^{(\alpha)} \zeta \right| dx$$

where $1 < q < np/(n-p)$, $q' = q/(q-1)$, $1 < t < np/(n-2p)$ and $t' = t/(t-1)$, we have that $\{L^{(k)}u^{(k)}\}$ converges weakly in $L^p(\Omega)$ to $L^{(\alpha)}u^{(\alpha)}$.
Hence $L^{(\alpha)}u^{(\alpha)} = 0$ a.e. in Ω and by theorem 2.3 $u^{(\alpha)} = 0$.
Thus $\|u^{(k)}\|_{L^p(\Omega)}$ converges to zero, which, on account of (2.9), contradicts $\|u^{(k)}\|_{W^{2,p}(\Omega)} = 1$. □

REFERENCES

[1] F. Chiarenza – M. Frasca – P. Longo, *Interior $W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients,* to appear in Ricerche di Matematica.

[2] F. Chiarenza – M. Frasca – P. Longo, *$W^{2,p}$ solvability of the Dirichlet problem for non divergence elliptic equations with VMO coefficients,* to appear in Trans. Amer. Math. Soc.

[3] M. Chicco, *Solvability of the Dirichlet problem in $H^{2,p}(\Omega)$ for a class of linear second order elliptic partial differential equations,* Bollettino U.M.I. (4) 4 (1971), pp. 374-387.

[4] J.B. Garnett, *Bounded analitic functions,* Academic Press, New York (1981).

[5] D. Gilbarg – N.S. Trudinger, *Elliptic partial differential eqautions of second order,* 2^{nd} edition, Springer – Verlag, Berlin (1983).

[6] D. Greco, *Nuove formole integrali di maggiorazione per le soluzioni di una equazione lineare di tipo ellittico ed applicazioni alla teoria del potenziale,* Ricerche di Matematica 5 (1956), pp. 126-149.

[7] A.I. Koshelev, *On boundedness in L^p of solutions of elliptic differential equations,* Mat. Sbornik 38 (1956), pp. 359-372.

[8] D. Sarason, *Functions of vanishing mean oscillation,* Trans. Amer. Math. Soc., 207 (1975), pp. 391-405.

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