DYNAMICS OF POLYNOMIALS IN FINITE AND INFINITE BENZ PLANES

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The classical Benz planes, that is, Möbius, Minkowski, and Laguerre planes, can be coordinatized [cf. 1], respectively, by the field $\mathbb{C}$ of complex numbers, the ring of "double numbers" $z = x + yj$ ($x, y \in \mathbb{R}$) where an element $j \notin \mathbb{R}$, with $j^2 = 1$ is adjoined, and the ring of "dual numbers" $z = x + ye$ where an element $e \notin \mathbb{R}$ with $e^2 = 0$ is adjoined to $\mathbb{R}$. When the field $\mathbb{R}$ is replaced by any other field, in our case finite prime fields $F_p$ ($p$ a prime), one also obtains coordinate structures for corresponding Benz planes. The dynamics of polynomials of degree at least 2 in the classical Möbius plane has attracted much attention recently because there fractal structures make their appearance. The question posed in this context has been for which values of $z$ the sequence $P_{n+1}(z) = P_O(P_n(z))$ is bounded if $P_O(z)$ is a function. This gave rise to the determination of Julia and Mandelbrot sets for such functions [cf.2]. In this paper we will restrict ourselves to the cases of Minkowski and Laguerre planes and to functions $P_O$ that are polynomials of degree at least 2, with coefficients from the ground field.

1. Minkowski planes.

We are working now in the ring $(F, j)$ of double numbers over the field $F, j^2 = 1$. Let $z = x + yj, x$ and $y$ in $F$. We use the notation $x = Fi(z)$ [$Fi(z)$ is $Re(z)$ is $F = \mathbb{R}$] and $y = Im(z)$. Then we have the following

**Theorem 1.1.** If $P$ is a polynomial of degree at least 2 over $(F, j)$ with coefficients from $F$, then $Fi(Px \pm yj) + Im(P(x \pm yj)) = P(x \pm y)$. 
Proof. First we will deal with the special case $P(z) = ax^k, z = x \pm yj; a, x, y \in F, k \geq 2$.

Then we obtain

$$P(z) = ax^k \pm akx^{k-1}yj + \ldots,$$

$$Ft(P(z)) + Im(P(z)) = ax^k \pm akx^{k-1}y + \ldots = P(x + y),$$

in view of $j^r = 1$ for every even $r$. Since $P(z)$ is the sum of terms of the type $ax^k$, the assertion follows.

Corollary. If $z = x + yj, \zeta = \xi + \eta j$ with $x, y, \xi, \eta \in F$, then $P(z) = P(\zeta)$ if and only if $x \pm y = \xi \pm \eta$.

We assume now $F = \mathbb{R}$. It happens often that the sequence $P_{n+1} = P_O(P_n(z))$ is bounded for all natural $n$ and all real values of $z$ within an interval, say, $s < z < t$ (details can be found in [3]). Then Theorem 1.1 implies that the sequence is bounded also for all values of $z = x + yj$ such that $s < x + y < t$ and $s < x - y < t$. If we plot the values $z = x + yj$ in an orthonormal coordinate system as points $(x, y)$, then we obtain

Theorem 1.2. If the sequence $P_{n+1}(z) = P_O(P_n(z))$ is bounded for all points in the real interval $s < z < t$, then it is also bounded for all points within the square whose diagonal is the segment $[s, t]$ on the $x$-axis.

This can also be expressed by saying that this square is the ”filled in Julia set” for the polynomial $P_O(z)$.

Now let $F = F_p$, the field of $p$ elements, $p$ prime. Then the sequence $P_{n+1} = P_O(P_n(z))$, having now only a finite number of potential values ($p^2$ to be exact), has to be periodic, although possibly with a pre-period. As a consequence of Theorem 1.1 we have now

Theorem 1.3. Let $z = x + yj; x, y \in F_p$. The period length of $P_n(x + y)$ divides the period length of $P_n(z)$.

Proof. Let $u$ be the period length of $z$, and $v$ that of $x + y$. In the periodic part of the sequence $P_{n+1}(z) = P_O(P_n(z))$, there is a $P_n$ such that $P_n(z) = P_{n+u}(z) = a + bj$, say, and therefore, by Theorem 1.1, also $P_{n+u}(x + y) = a + b$. However, if $x' \neq x$ and $y' = x + y - x'$, then $P_n(x + y) = P_n(x' + y') = P_{n+u}(x + y)$, although $P_n(x + yj)$ may be distinct from $P_{n+u}(x' + y'j)$. Thus every period of $P_n(z)$ is also a (possibly multiple) period of $P_n(x + y)$, but not necessarily vice versa. As a consequence, $v$ has to divide $u$.

If a ”limit” is defined as a period of length 1, we get the following
Corollary. The sequence \( P_n(x + yj) \) can yield a limit only if \( P_n(x + y) \) has a limit.

2. Laguerre planes.

We are now working in the ring \((F, e)\) of dual numbers over the field \( F, e^2 = 0 \). Let \( z = x + ye, x \) and \( y \) in \( F \). Again, we use the notation \( x = Fi(z) \) and \( y = Im(z) \). Then, in analogy with Theorem 1.1, we have

**Theorem 2.1.** If \( P \) is a polynomial of degree at least 2 over \((F, e)\) with coefficients from \( F \), then \( Fi(P(x \pm ye)) = P(x) \).

**Proof.** First suppose that \( P(z) = az^k, z = x \pm ye; a, x, y \in F \). Then we obtain

\[
P(z) = ax^k \pm akx^{k-1}ye, \quad Fi(P(z)) = ax^k = P(x),
\]

in view of \( e^r = 0 \) for every \( r > 1 \). Since \( P(z) \) is the sum of terms of the type \( az^k \), the assertion follows.

**Corollary.** If \( z = x + ye, \z = \z + \eta e, x, y, \z, \eta \in F \), then \( P(z) = P(\z) \) if and only if \( x = \z \).

We assume \( F = \mathbb{R} \). If \( P_n(z) \) is bounded for all \( n \) and all real values of \( z \) within the interval \([s, t]\), then Theorem 2.1 implies that the sequence \( Fi(P_n(x + ye)) \) is bounded also for all values \( z = x + ye \) with \( s < x < t \). If we plot the values of \( z \) in an orthonormal coordinate system as points \((x, y)\), then we obtain

**Theorem 2.2.** If the sequence \( P_n(z) \) is bounded for all points in the real interval \( s < z < t \), then \( Fi(P_n(z)) \) is also bounded within the infinite strip \( s < Re(z) < t \), for all values of \( Im(z) \).

This result applies only to the real part of \( P_n(z) \). The next theorem will tell us more.

**Theorem 2.3.** If \( P_n(z) \) is bounded for a given \( z = x + ye \) with \( y \neq 0 \), then it is also bounded for every \( z' = x + y'e \).

**Proof.** Let \( y' = yq \). Then we claim that \( Im(P_n(z')) = qIm(P_n(z)) \). Again it suffices to prove the assertion for \( P_0(z) = z^k \). We use induction. \( Im(P_0(z)) = Im(x + ye)^k = kx^{k-1}y, \) and \( Im(P_0(z')) = kx^{k-1}y' = qIm(P_0(z)) \). Now assume that \( P_n(z) = a + be \) and \( P_n(z') = a + qbe \). Then \( P_{n+1}(z) = a^k + ka^{k-1}b \) and \( P_{n+1}(z') = a^k + ka^{k-1}qb \), and the assertion is proved because if \( Im(P_n(z)) \) is bounded, so is \( Im(P_n(z')) \), and the real part is the same for \( z' \) as for \( z \) and hence bounded.
This shows that the filled in Julia set now consists of infinite vertical stripes (containing also their points on the real axis). However, there may be points $x$ on the real axis for which $P_n(x)$ is bounded, but in all of the $P_n(x + ye)$ with nonzero $y$ the real part is bounded in view of Theorem 2.2, but the imaginary part is not.

Now let $F = F_p, p$ a prime. Then the sequence $P_n(z)$ has to be periodic, possibly with a pre-period. Theorem 2.1 implies

**Theorem 2.4.** Let $z = x + ye, x, y \in F_p$. The period length of $P_n(x)$ divides the period length of $P_n(z)$.

The proof is analogous to that of Theorem 1.3.

**Corollary.** The sequence $P_n(x + ye)$ can yield a limit only if $P_n(x)$ has a limit.

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**REFERENCES**


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