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DYNAMICS OF POLYNOMIALS IN FINITE AND INFINITE BENZ PLANES

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The classical Benz planes, that is, Möbius, Minkowski, and Laguerre planes, can be coordinatized [cf. **1**], respectively, by the field $\mathbb{C}$ of complex numbers, the ring of "double numbers" $z = x + yj$ $(x, y \in \mathbb{R})$ where an element $j \notin \mathbb{R}$, with $j^2 = 1$ is adjoined, and the ring of "dual numbers" $z = x + ye$ where an element $e \notin \mathbb{R}$ with $e^2 = 0$ is adjoined to $\mathbb{R}$. When the field $\mathbb{R}$ is replaced by any other field, in our case finite prime fields F_p (p a prime), one also obtains coordinate structures for corresponding Benz planes. The dynamics of polynomials of degree at least 2 in the classical Möbius plane has attracted much attention recently because there fractal structures make their appearance. The question posed in this context has been for which values of z the sequence $P_{n+1}(z) = P_O(P_n(z))$ is bounded if $P_O(z)$ is a function. This gave rise to the determination of Julia and Mandelbrot sets for such functions [cf.**2**]. In this paper we will restrict ourselves to the cases of Minkowski and Laguerre planes and to functions P_O that are polynomials of degree at least 2, with coefficients from the ground field.

1. Minkowski planes.

We are working now in the ring (F, j) of double numbers over the field $F, j^2 = 1$. Let $z = x + yj, x$ and y in F. We use the notation $x = Fi(z)$ [$Fi(z)$ is $Re(z)$ is $F = \mathbb{R}$] and $y = Im(z)$. Then we have the following

Theorem 1.1. *If P is a polynomial of degree at least 2 over (F, j) with coefficients from F, then $Fi(Px \pm yj)) + Im(P(x \pm yj)) = P(x \pm y)$.*

Proof. First we will deal with the special case $P(z) = az^k, z = x \pm yj;\ a, x, y \in F,\ k \geq 2$.
Then we obtain

$$P(z) = ax^k \pm akx^{k-1}yj + \ldots,$$

$$Fi(P(z)) + Im(P(z)) = ax^k \pm akx^{k-1}y + \ldots = P(x+y),$$

in view of $j^r = 1$ for every even r. Since $P(z)$ is the sum of terms of the type az^k, the assertion follows.

Corollary. *If $z = x + yj$, $\zeta = \xi + \eta j$ with $x, y\, \xi, \eta \in F$, then $P(z) = P(\zeta)$ if and only if $x \pm y = \xi \pm \eta$.*

We assume now $F = \mathbb{R}$. It happens often that the sequence $P_{n+1} = P_O(P_n(z))$ is bounded for all natural n and all real values of z within an interval, say, $s < z < t$ (details can be found in [**3**]). Then Theorem 1.1 implies that the sequence is bounded also for all values of $z = x+yj$ such that $s < x+y < t$ and $s < x - y < t$. If we plot the values $z = x + yj$ in an orthonormal coordinate system as points (x, y), then we obtain

Theorem 1.2. *If the sequence $P_{n+1}(z) = P_O(P_n(z))$ is bounded for all points in the real interval $s < z < t$, then it is also bounded for all points within the square whose diagonal is the segment $[s, t]$ on the x-axis.*

This can also be expressed by saying that this square is the "filled in Julia set" for the polynomial $P_O(z)$.

Now let $F = F_p$, the field of p elements, p prime. Then the sequence $P_{n+1} = P_O(P_n(z))$, having now only a finite number of potential values (p^2 to be exact), has to be periodic, although possibly with a pre-period. As a consequence of Theorem 1.1 we have now

Theorem 1.3. *Let $z = x + yj;\ x, y \in F_p$. The period lenght of $P_n(x + y)$ divides the period lenght of $P_n(z)$.*

Proof. Let u be the period lenght of z, and v that of $x + y$. In the periodic part of the sequence $P_{n+1}(z) = P_O(P_n(z))$, there is a P_n such that $P_n(z) = P_{n+u}(z) = a + bj$, say, and therefore, by Theorem 1.1, also $P_{n+u}(x + y) = a + b$. However, if $x' \neq x$ and $y' = x + y - x'$, then $P_n(x+y) = P_n(x'+y') = P_{n+v}(x+y)$, although $P_n(x+yj)$ may be distinct from $P_{n+v}(x' + y'j)$. Thus every period of $P_n(z)$ is also a (possibly multiple) period of $P_n(x + y)$, but not necessarily vice versa. As a consequence, v has to divide u.

If a "limit" is defined as a period of length 1, we get the following

Corollary. *The sequence $P_n(x+yj)$ can yield a limit only if $P_n(x+y)$ has a limit.*

2. Laguerre planes.

We are now working in the ring (F, e) of dual numbers over the field $F, e^2 = 0$. Let $z = x + ye$, x and y in F. Again, we use the notation $x = Fi(z)$ and $y = Im(z)$. Then, in analogy with Theorem 1.1, we have

Theorem 2.1. *If P is a polynomial of degree at least 2 over (F, e) with cofficients from F, then $Fi(P(x \pm ye)) = P(x)$.*

Proof. First suppose that $P(z) = az^k$, $z = x \pm ye$; $a, x, y \in F$. Then we obtain

$$P(z) = ax^k \pm akx^{k-1}ye, \; Fi(P(z)) = ax^k = P(x),$$

in view of $e^r = 0$ for every $r > 1$. Since $P(z)$ is the sum of terms of the type az^k, the assertion follows.

Corollary. *If $z = x + ye$, $\zeta = \xi + \eta e, x, y, \xi\eta \in F$, then $P(z) = P(\zeta)$ if and only if $x = \xi$.*

We assume $F = \mathbb{R}$. If $P_n(z)$ is bounded for all n and all real values of z within the interval $[s, t]$, then Theorem 2.1 implies that the sequence $Fi(P_n(x + ye))$ is bounded also for all values $z = x + ye$ with $s < x < t$. If we plot the values of z in an orthonormal coordinate system as points (x, y), then we obtain

Theorem 2.2. *If the sequence $P_n(z)$ is bounded for all points in the real interval $s < z < t$, then $Fi(P_n(z))$ is also bounded within the infinite strip $s < Re(z) < t$, for all values of $Im(z)$.*

This result applies only to the real part of $P_n(z)$. The next theorem will tell us more.

Theorem 2.3. *If $P_n(z)$ is bounded for a given $z = x + ye$ with $y \neq 0$, then it is also bounded for every $z' = x + y'e$.*

Proof. Let $y' = yq$. Then we claim that $Im(P_n(z')) = qIm(P_n(z))$. Again it suffices to prove the assertion for $P_O(z) = z^k$. We use induction. $Im(P_O(z)) = Im(x + ye)^k = kx^{k-1}y$, and $Im(P_0(z')) = kx^{k-1}y' = qIm(P_0(z))$. Now assume that $P_n(z) = a + be$ and $P_n(z') = a + qbe$. Then $P_{n+1}(z) = a^k + ka^{k-1}b$ and $P_{n+1}(z') = a^k + ka^{k-1}qb$, and the assertion is proved because if $Im(P_n(z))$ is bounded, so is $Im(P_n(z'))$, and the real part is the same for z' as for z and hence bounded.

This shows that the filled in Julia set now consists of infinite vertical stripes (containing also their points on the real axis). However, there may be points x on the real axis for which $P_n(x)$ is bounded, but in all of the $P_n(x+ye)$ with nonzero y the real part is bounded in view of Theorem 2.2, but the imaginary part is not.
Now let $F = F_p, p$ a prime. Then the sequence $P_n(z)$ has to be periodic, possibly with a pre-period. Theorem 2.1 implies

Theorem 2.4. *Let $z = x + ye$, $x, y \in F_p$. The period lenght of $P_n(x)$ divides the period lenght of $P_n(z)$.*

The proof is analogous to that of Theorem 1.3.

Corollary. *The sequence $P_n(x+ye)$ can yield a limit only if $P_n(x)$ has a limit.*

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