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ISOMETRIES IN GALOIS SPACES

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1. We determine the group of isometries of certain spherical spaces which are embedded in Galois Spaces. Concerning other results of similar type see [2], [3], [4], [5], [6].

2. Suppose that M and W are sets with $M \neq \emptyset$ and that d is a mapping from $M \times M$ into W. Then (M, W, d) is called a distance space and $d\,(x, y)$ the distance of x, y. We say that $f : M \to M$ is an isometry in case

$$d\,(x, y) = d\,(f\,(x), f\,(y))$$

holds true for all $x, y \in M$. An isometry needs not to be injective ([1]). However, the set of all bijective isometries of (M, W, d) is a group (under the permutation product) which we denote by $I\,(M, W, d)$. For every group G there exists a distance space (M, W, d) such that

$$G \cong I\,(M, W, d)$$

([1]).

3. Suppose that F is a Galois field $GF(q)$ with $2 \nmid q$. Let $n \geq 2$ be an integer and let $G = (g_{ij})$ be a symmetric $(n+1) \times (n+1)$-matrix over F such that $\det G \neq 0$. By

$$\sum (F, n+1, G) \tag{1}$$

we denote the distance space

$$(F^{n+1}, F, d_G)$$

such that

(2) $$d_G(x,y) := (x-y)^2$$

with the scalar product

(3) $$vw := \sum_{i,j=1}^{n+1} g_{ij} v_i w_j$$

for $v, w \in F^{n+1}$. Every isometry f of (1) must be bijective and of form

$$f(x) = x\Pi + a$$

for all $x \in F^{n+1}$ with matrices

$$\Pi = \begin{pmatrix} a_{11} & \dots & a_{1,n+1} \\ \vdots & & \\ a_{n+1,1} & \dots & a_{n+1,n+1} \end{pmatrix}$$

and

$$a = (a_1 \dots a_{n+1})$$

over F such that

(4) $$\Pi G \Pi^T = G.$$

A consequence of (4) is

$$\det \Pi \in \{1, -1\}.$$

4. Suppose that

$$\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}$$

are elements of $F = GF(q)$ such that

$$\varepsilon_1 = 1 \quad \text{and} \quad \varepsilon_i^2 = 1$$

for $i = 2, \dots, n+1$. The set of points of the n-dimensional spherical geometry of signature $\varepsilon_1, \dots, \varepsilon_{n+1}$ is then defined by

$$S_{\varepsilon_1,\dots,\varepsilon_{n+1}}(F) := \{x \in F^n \mid x^2 = 1\} \tag{5}$$

with the scalar product

$$vw = \sum_{i=1}^{n+1} \varepsilon_i v_i w_i \tag{6}$$

of F^{n+1}. Let $\sum$ be the $(n+1)$-dimensional affine space over F and let $\sum^{\nu+1}$ be a $(\nu+1)$-dimensional affine subspace of $\sum$ passing through the origin. Then

$$\sum\nolimits^{\nu+1} \cap S_{\varepsilon_1,\dots,\varepsilon_{n+1}}(F)$$

is called a ν-dimensional subspace of the spherical space $S_{\varepsilon_1,\dots,\varepsilon_{n+1}}(F)$. Fundamental will be now the distance space

$$(S_{\varepsilon_1,\dots,\varepsilon_{n+1}}(F),\ F,\ d(x,y) = xy) \tag{7}$$

in the following three cases

A) $\varepsilon_1 = \dots = \varepsilon_{n+1}$ (Möbius case),

B) $\varepsilon_1 = \dots = \varepsilon_n = -\varepsilon_{n+1}$ (Minkowski case),

C) $n+1$ even and $\varepsilon_1 = \dots = \varepsilon_{\frac{n+1}{2}} = -\varepsilon_{\frac{n+3}{2}} = \dots = -\varepsilon_{n+1}$ (Plücker case).

5. We define a $(n+1) \times (n+1)$-matrix G by

$$g_{ij} = \delta_{ij} \cdot \varepsilon_i, \tag{8}$$

where δ_{ij} denotes the Kronecker symbol

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}.$$

Theorem. *Let Δ be one of the distance spaces (7) in cases A, B, C. Then every isometric of Δ must be bijective. The group of isometries of Δ consists exactly of the mappings*

$$f(x) = x\Pi \tag{9}$$

for $x \in S_{\varepsilon_1 \dots, \varepsilon_{n+1}}(F)$ *with*

(10) $$\Pi G \Pi^T = G.$$

This group is the group of isometries of space (1), leaving invariant 0, restricted on $S_{\varepsilon_1 \dots, \varepsilon_{n+1}}(F)$.

Proof. (a) Suppose that $\varphi : F^{n+1} \to F^{n+1}$ is an isometry of (1) leaving invariant 0 in case (8). Because of

$$d_G(0, x) = d_G(0, \varphi(x))$$

we have

(11) $$x^2 = [\varphi(x)]^2.$$

Hence $x^2 = 1$ implies $[\varphi(x)]^2 = 1$. The restriction

$$f = \varphi \,|\, S_{\varepsilon_1, \dots, \varepsilon_{n+1}}(F)$$

thus maps S into S. It is f injective since φ is bijective. Take an element

$$y \in S := S_{\varepsilon_1, \dots, \varepsilon_{n+1}}(F).$$

Since φ is bijective there exists $x \in F^{n+1}$ with $y = \varphi(x)$. Now $y^2 = 1$ implies $x^2 = 1$ according to (11) and f is hence a bijection of S. Moreover is f an isometry of (7) because of

(12) $$2d\,(x, y) = d_G\,(0, x + y) - d_G(0, x) - d_G\,(0, y).$$

We namely observe that φ is linear according to section 3 and that hence (12) implies

$$d\,(x, y) = d\,(\varphi(x),\, \varphi\,(y)).$$

Let now f be an arbitrary isometry of (7).

(b) In case A we consider the $n+1$ points

$$E_1 = (1, 0, \dots, 0),\ E_2 = (0, 1, 0, \dots, 0), \dots, E_{n+1} = (0, \dots, 0, 1).$$

Observe

$$E_i \in S_{1,1,\dots,1}$$

for $i = 1, \ldots, n+1$. Hence $A_i := f(E_i) \in S$. Now

$$A_i A_j = f(E_i) f(E_j) = E_i E_j = \delta_{ij}$$

and

$$h(x) := x_1 A_1 + \ldots + x_{n+1} A_{n+1}$$

is hence an isometry of (1) for the special G under consideration. Let p be the restriction of h on S. Then $p^{-1} f$ is an isometry of S leaving invariant $E_1, \ldots, E_{n+1}$. Suppose that x is a point of S and that $y := p^{-1} f(x)$. Then

$$x_i = x E_i = p^{-1} f(x) \cdot p^{-1} f(E_i) = y_i$$

for $i = 1, \ldots, n+1$. Hence $x = y$ and thus $p^{-1} f = \mathrm{id}$, i.e. $f = p$.

(c) In case B we consider the $n+1$ points

$$E_1 = (1, 0, \ldots, 0), \ldots, E_n = (0, \ldots, 0, 1, 0),\ E = (-1, -1, 0, \ldots, 0, 1).$$

Obviously, $E, E_i \in S$. Observe

$$E_i E_j = \delta_{ij}$$

and

$$E^2 = 1,\ E E_1 = -1 = E E_2,\ E E_i = 0 \quad \text{for} \quad i = 3, \ldots, n.$$

Put $A := f(E)$ and $A_i := f(E_i)$. The $A_1, \ldots, A_n$ are obviously linearly independent because of

$$\delta_{ij} = E_i E_j = f(E_i) f(E_j) = A_i A_j.$$

Also, $A_1, \ldots, A_n, A$ are linearly independent. This is a consequence of

$$A^2 = 1,\ A A_1 = -1 = A A_2,\ A A_i = 0 \quad \text{for} \quad i = 3, \ldots, n:$$

$$A =: \alpha_1 A_1 + \ldots + \alpha_n A_n$$

namely implies $0 = A A_i = \alpha_i$ for $i = 3, \ldots, n$ and

$$A = -A_1 - A_2$$

which together with $A^2 = 1$ leads to a contradiction. Now

$$h(x) := x_1 A_1 + \ldots + x_n \cdot A_n + x_{n+1} \cdot (A + A_1 + A_2)$$

is an isometry of (1) for the present

$$G=\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{pmatrix}.$$

Let p be the restriction of h on S. Then $p^{-1}f$ is an isometry of S leaving invariant $E_1, \dots, E_n, E$. Suppose that x is a point of S and that $y := p^{-1}f\,(x)$. Then

$$x_i = xE_i = p^{-1}f\,(x) \cdot p^{-1}f\,(E_i) = y_i$$

for $i = 1, \dots, n$ and

$$-x_1 - x_2 + x_{n+1} = xE = yE = -y_1 - y_2 + y_{n+1},$$

i.e. $x_{n+1} = y_{n+1}$. Hence $x = y$ and thus $p^{-1}f = \text{id}$, i.e. $f = p$.
(d) In case C we consider the $n + 1$ points

$$E_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{i,n+1}),$$
$$F_i$$

for $i = 1, 2, \dots, k := \frac{n+1}{2}$, where the first and the second component of F_i is -1 and where all the other components are 0 up the $(k+1)$-th component which is supposed to be equal to 1. Obviously, $E_i, F_i \in S$. Define

$$D_i := \begin{cases} E_i & \text{for} \quad i = 1, \dots, k \\ F_{i-k} + E_1 + E_2 & \phantom{\text{for}} \quad i = k+1, \dots, 2k \end{cases}.$$

Observe

(13) $$D_i D_j = \varepsilon_i \delta_{ij}$$

for $i, j \in \{1, \dots, n + 1\}$. Put now

$$A_i = f\,(E_i) \text{ and } B_i = f\,(F_i)$$

for $i = 1, \dots, k$. Because of (13) and $E_i, F_i \in S$ we get

$$A_i A_j = E_i E_j = \varepsilon_i \delta_{ij},$$
$$B_i B_j = F_i F_j = \varepsilon_1 + \varepsilon_2 + \varepsilon_{k+i}\delta_{ij},$$
$$A_i B_j = E_i F_j = -\varepsilon_1 \delta_{i1} - \varepsilon_2 \delta_{i2}.$$

Put

$$H_i := \begin{cases} A_i & \text{for} \quad i = 1, \ldots, k \\ B_{i-k} + A_1 + A_2 & \phantom{\text{for}} \quad i = k+1, \ldots, 2k \end{cases}.$$

Hence

$$H_i H_j = \varepsilon_i \delta_{ij}.$$

It is now

$$h(x) := x_1 H_1 + \ldots + x_{n+1} H_{n+1}$$

an isometry of (1) for the present G. Let p be the restriction of h on S. Then $p^{-1} f$ is an isometry of S leaving invariant $E_1, \ldots, F_k$. Suppose that x is a point of S and that $y := p^{-1} f(x)$. Then

$$\varepsilon_i x_i = x E_i = p^{-1} f(x) \cdot p^{-1} f(E_i) = \varepsilon_i y_i$$

for $i = 1, \ldots, k$ and

$$-x_1 - \varepsilon_2 x_2 + \varepsilon_{k+i} x_{k+i} = x F_i = y F_i = -y_1 - \varepsilon_2 y_2 + \varepsilon_{k+i} y_{k+i}$$

for $i = 1, \ldots, k$. Hence $x = y$ and thus $f = p$.

Remarks:

1. With the same arguments the Theorem can be carried over to the general case $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}$, where $\varepsilon_1 = \varepsilon_2 = 1$.
2. The spherical geometry of case B is the de Sitter-world over $GF(q)$. The group of isometries is here the Lorentzgroup of $[GF(q)]^{n+1}$ of those isometries which leave invariant the origin 0.

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