LE MATEMATICHE
Vol. XLVII (1992) – Fasc. II, pp. 205-211

THE q-PERFECT GRAPHS – 2

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1. Introduction.

Let q be a positive integer. Many graphs admit a partial coloring with q colors and a clique partition such that each of the cliques is *strongly colored* , that is: contains the largest possible number of different colors. If a graph G and all its induced subgraphes have this property, we say that G is *q-perfect* (Lovasz). In a previous paper [4], the specific properties for the case $q = 2$ were investigated. Here, we study some graphs which are q-perfect for other values of q, and more specially the *balanced graphs*.

2. Some characterization theorems for q-perfect graphs.

Let G be a *simple* graph (no loops, no multiple edges); we denote by $\alpha(G)$ the stability number, by $\theta(G)$ the least number of cliques which cover the vertex set, by $\omega(G)$ the maximum size of a clique, and by $\chi(G)$ the chromatic number.

Let q be a positive integer. A *partial q-coloring* of G is a set of q pairwise disjoint stable sets $S_1, S_2, \dots, S_q$ each one corresponding to a *color* ; some of the vertices may have no color. The largest possible number of colored vertices in a partial q-coloring is denoted by $\alpha_q(G)$. A partial q-coloring with $\alpha_q(G)$ colored vertices is *optimal.*

Let $M = (C_1, C_2, \dots,)$ be a partition of the vertex set $V(G)$ into cliques; by definition, the *q-norm* of M is:

$$B_q(M) = \sum_{j \geq 1} \min\{|C_j|, q\}.$$

We denote $\theta_q(G)$ the minimum q-norm for the clique partitions of G. If $B_q(M) = \theta_q(G)$, we say that M is *q-optimal.*

For every clique partition $M = (C_1, C_2, \ldots)$ and for every partial q-coloring $(S_1, S_2, \ldots, S_q)$, we have

$$(1) \qquad |\bigcup_{i=1}^{q} S_i| = \sum_j |C_j \cap \bigcup S_i| \leq \sum_j \min\{|C_j|, q\} = B_q(M).$$

Hence $\alpha_q(G) \leq \theta_q(G)$. If every subgraph G_A of G satisfies $\alpha_q(G_A) = \theta_q(G_A)$, we say that G is *q-perfect*. Clearly $\alpha_1(G) = \alpha(G)$, $\theta_1(G) = \theta(G)$, and a graph G is 1-perfect if and only if G is perfect.

For a hypergraph $H = (E_1, E_2, \ldots, E_m)$ on a set X of vertices, $\nu(H)$ denotes the maximum number of pairwise disjoint edges, and $\tau(H)$ denotes the least cardinality of a *transversal set* of H (set of vertices which meets all the edges). Clearly, $\nu(H) \leq \tau(H)$, and if $\nu(H) = \tau(H)$, the hypergraph H is said to have the *König property*. For a set $A \subset X$, let H/A denote the partial hypergraph of H defined by the edges contained in A; if $H \setminus A$ has the König property for all A, we say that H is a *mengerian hypergraph*; for various classes of mengerian hypergraphs, see [3].

Theorem 1. *Let q be an integer ≥ 2; let G_0 be a graph without K_{q+2}. Then G_0 is q-perfect if and only if the two following conditions hold:*

(i) every subgraph G with $\omega(G) \leq q$ is q-colorable;

(ii) for every subgraph G, the hypergraph G^{q+1} of the $(q+1)$-cliques of G has the König property.

Proof. 1. Let G_0 be a K_{q+2}-free q-perfect graph. We have (i) because otherwise, G_0 would contain a subgraph G with

$$\alpha_q(G) < |V(G)|,\ \omega(G) \leq q.$$

Since $\omega(G) \leq q$, the q-norm $B_q(M)$ of any q-optimal clique partition M is equal to $|V(G)|$, and consequently:

$$\theta_q(G) = |V(G)|.$$

So, $\alpha_q(G) < \theta_q(G)$, a contradiction.

We have (ii), because for a subgraph G of G_0, a q-optimal partition M contains exactly $\nu(G^{q+1})$ classes of size $q+1$, and , consequently,

$$\nu(G^{q+1}) = |V(G)| - B_q(M) = |V(G)| - \theta_q(G).$$

Because of (i), the subgraph G is q-colorable iff it can be obtained from G_0 by removing a transversal set of G_0^{q+1}, and , consequently,

$$\theta_q(G) = \alpha_q(G) = |V(G)| - \tau(G^{q+1}).$$

Thus, $\nu(G^{q+1}) = \tau(G^{q+1})$.

2. Assume that the graph G_0 satisfies (i) and (ii). Then, for every subgraph G of G_0,

$$\alpha_q(G) = |V(G)| - \tau(G^{q+1}) = |V(G)| - \nu(G^{q+1}) = B_q(M) = \theta_q(G)$$

Thus, G_0 is q-perfect.

Corollary. *A K_4-free graph G is 2-perfect if and only if the two following conditions hold:*

(i') G does not contain an induced C_{2k+1} with $k \geq 2$;
(ii') the triangle-hypergraph G^3 is mengerian.

(From Theorem 1 with $q = 2$.)

When $\omega(G)$ is not specified, another characterization can be obtained by the minimax theorem of linear programming. Let G be a graph on $\{x_1, x_2, \ldots, x_n\}$; denote by G^{q+1} a hypergraph on $V(G)$ whose edges are the maximal cliques of G having cardinality $\geq q+1$; let $E'_1, E'_2, \ldots, E'_m$ denote its edges, and put $E''_i = \{x_i\}$ for $i = 1, 2, \ldots, n$.

Denote by A the incidence matrix of the hypergraph

$$H = (E'_1, E'_2, \ldots, E'_m, E''_1, E''_2, \ldots, E''_n).$$

Theorem 2. *Let q be an integer ≥ 2. A graph G_0 is q-perfect if an only if the two following conditions hold:*

(i) every subgraph G of G_0 with $\omega(G) \leq q$ is q-colorable
(ii) for every subgraph G of G_0 if A denotes the incidence matrix of the hypergraph $H = (E'_1, \ldots, E'_m, E''_1, \ldots, E''_n)$ obtained from G^{q+1} by adding a loop at each vertex, and if $q = (q, q, \ldots, q, 1, 1, \ldots, 1)$ denotes the $(m+n)$-dimensional vector with the first m coordinates equal to q, the linear problem:

$$\textit{maximize} \sum_{j=1}^{n} t_j = \mathbf{1}.\mathrm{t}$$

$$\text{(I)} \qquad \mathrm{t} = (t_1, t_2, \ldots t_n) \geq \mathbf{0}$$

$$\mathbf{A}^*\mathbf{t} \le \mathbf{q}$$

and its dual:

$$\textit{minimize } q\sum_{i=1}^{m} y'_i + \sum_{j=i}^{n} y''_j = \mathbf{q}.\mathbf{y}.$$

(II) $$\mathbf{y} = (y'_1, y'_2, \ldots, y'_m, y"_1, y"_2, \ldots, y"_n) \ge 0$$

$$\mathbf{Ay} \ge \mathbf{1}$$

have both integral solutions.

Proof. Remark that each coordinate of an integral solution t of (I) is equal either to 0 or to 1; furthermore, t is the characteristic vector of a set T with $\omega(G_T) \le q$, $|T|$ maximum; hence if (i) holds, we have:

$$\mathbf{1}.\mathbf{t} = \alpha_q(G).$$

Remark also that an integral solution $\mathbf{y}$ of (II) is the characteristic vector of a clique covering of G; this clique covering gives also a q-optimal clique partition whose q-norm is equal to $\mathbf{q}.\mathbf{y}$, so $\mathbf{q}.\mathbf{y} = \theta_q(G)$.

1. Assume that G_0 is q-perfect. We have (i), because otherwise some subgraph G with $\omega(G) \le q$ would satisfy $\alpha_q(G) < |V(G)|$, and, as in the proof of Theorem 1, we get:

$$\theta_q(G) = |V(G)| > \alpha_q(G)$$

A contradiction.

We have also (ii), because, by (i), we get:

$$\max\{\mathbf{1}.\mathbf{t} \,/\, \mathbf{t} \in \mathbb{N}^{\mathbf{n}},\ \mathbf{A}^*\mathbf{t} \le \mathbf{q}\} = \alpha_q(G) = \theta_q(G) =$$

$$= \min\{\mathbf{q}.\mathbf{y} \,/\, \mathbf{y} \in \mathbb{N}^{\mathbf{m}},\ \mathbf{Ay} \ge \mathbf{1}\}.$$

Hence, (I) and (II) have both integral solutions.

2. Assume that (i) and (ii) hold true. Let G be a subgraph of G_0; let t^0 be an integral solution of (I), and let $\mathbf{y}^0$ be an integral solution of (II). We have:

$$\alpha_q(G) = \sum_{j=1}^{n} t_j^0 = \max \mathbf{1}.\mathbf{t} = \min \mathbf{y}.\mathbf{q} = \mathbf{y}^0.\mathbf{q} = \theta_q(G),$$

(because of (i) and of the Minimax Theorem).

Hence, G_0 is q-perfect.

3. The balanced graphs.

It seems difficult to characterize the structure of the graphs which are q-perfect for all q. By a theorem of Greene and Kleitman [17], the comparability graphs have this property, by a theorem of Greene [16], the cocomparability graphs have also this property. Later, Cameron [8] gave a statement including these two results, and independently, we proved in [4] the q-perfectness for a new class of perfect graphs: the balanced graphs.

We introduced in [2] the *balanced hypergraphs* to extend the properties of the totally unimodular matrices; a hypergraph $H = (E_1, E_2, \dots E_m)$ on $X = \{x_1, x_2, \dots, x_n\}$ is *balanced* if every cycle of odd lenght $k \geq 3$, say $\{x_1, E_1, x_2, \dots x_k, E_k, x_1)$ has an edge E_i containing three of the $x_j's$. The recognition of a balanced hypergraph can be done in polynomial time (Conforti, Cornuéjols, Rao, [10]). A graph G is *balanced* if the hypergraph $H(G)$ of the maximal cliques in G is balanced. It is well known that *every balanced graph G is perfect.*
Furthermore,

Theorem 3. *Every balanced graph is q-perfect for all $q \geq 1$.*

Proof. Let G be a balanced graph and let $q \geq 1$; it suffices to show that the conditions (i) and (ii) of Theorem 2 hold. The condition (i) follows from the fact that G is perfect. To show (ii), recall a theorem of Fulkerson, Hoffman, Oppenheim [14], which states that the incidence matrix A of a balanced hypergraph of order n satisfies, for $m = m(H)$ and $\mathbf{q} \in \mathbb{N}^m$,

$$\max\{\mathbf{1.t} \,/\, \mathbf{t} \in \mathbb{N}^n,\ \mathbf{A^*t} \leq \mathbf{q}\} = \min\{\mathbf{q.y} \,/\, \mathbf{y} \in \mathbb{N}^m,\ \mathbf{Ay} \geq \mathbf{1}\}.$$

Since the hypergraph H obtained from $H(G)$ by adding a loop at each vertex is balanced, the condition (ii) follows.

Though they do not have a simple easy characteristic property in the context of Graph Theory, the balanced graphs include many well known classes: interval graphs, line graphs of bipartite multigraphs, cacti (whose blocks are cliques), intersection graphs of a family of directed paths in a tree (with directed edges), and , more generally, all the intersection graphs of unimodular hypergraphs; also the intersection graphs of a family of balls in a tree and all the intersection graphs of totally balanced hypergraphs occuring in location problems (Tamir, Lubiw, etc..., see [3], Chap. 5).

Let H_k $k \geq 3$ be a graph with $2k$ vertices consisting of the union of a clique $\{a_1, a_2, \dots, a_k\}$ and a stable set $(b_1, b_2, \dots, b_k)$, together with a hamilton cycle $[a_1, b_1, a_2, , \dots, b_k, a_1]$; H_k is sometimes called a *k-sun* (or *k-trampoline*), and

H_3 is called the Hajos graph. Farber [12] has shown that $H(G)$ is totally balanced iff G is triangulated (every cycle ≥ 4 has a chord) and no subgraph is isomorphic to a k-sun with $k \geq 3$. By a similar argument, we obtain:

A triangulated graph is balanced if and only if it does not contain an induced k-sun with k odd ≥ 3.

Let G be a balanced graph and let C be a clique of G. We say that a vertex x *dominates* C if $x \notin C$ and $C \cup \{x\}$ is also a clique. For a balanced graph, the prohibited configurations are the odd cycles μ such that: *each edge* $e \in \mu$ *is contained in a clique* C_e *with* $C_e \cap \mu = \{e\}$, *and which is dominated by no vertex in* μ. These cycles are called *unbalanced*. It is easy to show that if an unbalanced cycle μ is a triangle, the only minimal configuration associated with μ is the Hajos graph (plus, eventually, additional edges having no endpoints on μ); in general, one can suppose that the cliques C_e are triangles, but apparently, there is no simple description for all minimal prohibited configurations.

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