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ON GRAPHS THAT ARE CRITICAL WITH RESPECT TO THE PARAMETERS: DIAMETER, CONNECTIVITY AND EDGE-CONNECTIVITY

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In graph theory, the term *critical* is usually used with respect to a specified graph parameter P and applies when the graph G under consideration has the property P but alteration of G (such as vertex deletion, edge deletion or edge addition) results in a graph not having property P. In this paper, we consider graphs which are critical with respect to parameter P under the single operation of: deleting a vertex; deleting an edge; adding an edge. In particular, we focus on the graph parameters: diameter, connectivity and edge-connectivity. We review important results and mention many open problems.

1. Introduction.

Graph theory can be conveniently used to model large complex systems, such as: computer networks, electronic circuits, communication networks, assembly production lines, pipeline networks and traffic networks, whose proper performance requires the correct functioning and interaction of the system many components. In the graph model of such systems vertices represent components and edges (arcs, when orientation is a factor) represent the interactions between the components. This provides us with a structural model of the system being studied. Non structural information such as cost, capacity, efficiency and reliability can be incorporated into the graph theoretic model by assigning appropriate weights to the vertices and edges of the graph.

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In many applications, particularly in the area of network design, the graph theoretical problem that arises is the following:

> **Let $\mathscr{G}(n, P)$ denote the class of graphs on n vertices satisfying a set of properties P. Given a performance measure M, the problem is to characterize the optimal, with respect to M, members of $\mathscr{G}(n, P)$.**

The set of properties P reflect the network requirements such as: reliability, efficiency, capacity, and throughput and are usually expressed in terms of bounds on certain graph parameters such as: connectivity, diameter and degree. M could refer to cost, output, or a graph parameter such as: number of edges, number of vertices, connectivity and diameter.

For given n, P and M a number of graph optimization problems arise; for a detailed discussion we refer to Caccetta [10] and Caccetta and Vijayan [17]. In the characterization of the class $\mathscr{G}(n, P)$ it is often fruitful to study a restricted class of graphs the so called *"critical graphs"*.

The term *critical* is used in the literature in several ways. Usually it is used with respect to a specified graph parameter P and applies when the graph G under consideration has the property P but alteration of G (such as vertex deletion, edge deletion or edge addition) results in a graph not having property P. The resultant critical class of graphs has more structure than the general class and this structure can be utilized to yield a considerable amount of useful information. Often there is no loss of generality and quite a lot to gain, in considering this class of graphs.

The critical graphs considered here arise under the single operation of : deleting a vertex, deleting an edge, adding an edge. Such graphs have been considered by many authors for a number of parameters including: minimum degree, connectivity, edge-connnectivity, diameter, chromatic number and various covering numbers (vertex, edge, clique, etc.). It is not our intention here to review all this work in detail, however, we mention the following important references. Under the single operation of edge-deletion the parameters that have been studied include; connectivity and edge-connectivity (Dirac [24], Halin [32-34], Mader [41-45], Bollobás [5], Cai [18-20], Budayasa et. al. [8,9]); diameter (Anunchuen and Caccetta [3], Caccetta and Häggkvist [11], Fan [26], Füredi [27], Gliviak [28], Glivjak and Plesnik [31], Glivjak et. al. [30], Plesnik [48]); chromatic-index (Yap [50]); and the vertex covering number (see Lovász and Plummer [40]). Under the single operation of edge-addition the parameters that have been studied include: diameter (Caccetta and Smyth [12-16], Ore [47]); k-extendability (Anunchuen and Caccetta [2]). Under the single operation of vertex deletion the parameters that have been studied include: connectivity (Chartrand et. al. [21], Entringer [25], Hamidoune [35], Krol and Veldman [36]); edge-connectivity (Cozzens and Wu [22-23]); diameter (Boals and Ali [4], Gliviak [29] and Plesnik [48]).

In this expository paper we focus on graphs critical with respect to the graph parameter connectivity, edge-connectivity and diameter. These parameters are considered important in the area of network design as they provide measures of network efficiency and reliability. For example, in a communication network the minimum number of link (centre) failures required to destroy communication between at least two centres in the network corresponds to the edge (vertex)-connectivity of the graph representing the network. The maximum number of links over which a message between any two centres in the network must travel corresponds to the diameter of the graph representing the network.

Section 2 considers diameter critical graphs whilst Section 3 considers graphs that are critical with respect to connectivity and edge-connectivity. We conclude this introduction with some basic graph theoretic terminology.

For the most part we use standard graph theoretic notation and terminology. Thus G is a simple uindirected graph with vertex set $V(G)$, edge set $E(G)$, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. The complement of G is denoted by $\overline{G}$.

Let G and H be graphs. We denote the union and intersection of G and H by $G \cup H$ and $G \cap H$, respectively. The join of two disjoint graphs G and H, denoted by $G \vee H$, is the graph obtained from $G \cap H$ by joining each vertex of G to each vertex of H.

The *complete* graph on n vertices is denoted by K_n. The *complete bipartite* graph with bipartitioning sets of order n and m is denoted by $K_{n,m}$; that is $K_{n,m} = \overline{K}_n \vee \overline{K}_m$. The *cycle* on n vertices is denoted by C_n. The *distance* $d_G(x, y)$ between the two vertices x and y is defined as the length of the shortest (x, y)-path in G; if there is no path connecting x and y we define $d_G(x, y)$ to be infinite. The *diameter* $d(G)$ of a graph G is defined as:

$$d(G) = \max_{x,y \in V(G)} \{d_G(x, y)\}.$$

The *connectivity* $k(G)$ of G is defined as the minimum number of vertices whose deletion results in either a disconnected graph or else the trivial graph K_1. Similarly the *edge-connectivity* $K'(G)$ of G is defined as the minimum number of edges whose deletion results in a disconnected graph or else the trivial graph K_1.

2. Diameter critical graphs.

The diameter of a graph is an important graph theoretic parameter with considerable application. The problem of characterizing graphs with a prescribed diameter is very much unresolved. Indeed, not even graphs of diameter 2 have been completely characterized. Usually, a number of graph parameters (such as: order, minimum degree, maximum degree and connectivity) are studied with the objective of determining the relationship between the parameters. The typical approach is to fix some of the parameters and see how the others vary. The book by Bollobás [5] contains an excellent account of such work. The recent review by Caccetta [10] details some of the results since the publication of Bollobás' book.

A well know elementary result states that if $d(G) \geq 3$, then $d(\overline{G}) \leq 2$. Bloom et. al. [7] proved that $d(G) = 2$ if and only if $\overline{G}$ is non-empty and not spanned by a double star. They used this result to derive a number of necessary conditions for a graph to have diameter 2. Recently, Achuthan et. al. [1] considered the problem of characterizing the class:

$$\mathscr{G}(n, x, y) = \{G : |V(G)| = n,\ d(G) = x \text{ and } d(\overline{G}) = y\}.$$

Some useful results were obtained.

Many authors have studied graphs whose diameter changes by the deletion/addition of edges and by the deletion of vertices. In this section we discuss some of the important results and conjectures in this area. We consider these so called critical graphs in terms of the operation: edge deletion, edge addition, and vertex deletion.

2.1 Edge Deletion.

Let G be a graph having diameter k. G is said to be (k,t)-*critical* if for any $E' \subseteq E(G)$, $d(G - E') > k$ if and only if $|E'| \geq t$. Denote the class of (k,t)-critical graphs by $\mathscr{G}(k,t)$. $(k,1)$-critical graphs do exist. For example, $C_{2k} \in \mathscr{G}(k,1), C_{2k+1} \in \mathscr{G}(k,1), K_{n,m} \in \mathscr{G}(2,1), K_n \in \mathscr{G}(1,1)$ and the well known Petersen graph is in the class $\mathscr{G}(2,1)$. In fact, any graph of diameter k having girth (length of smallest cycle) at least $k+2$ is in the class $\mathscr{G}(k,1)$.

The class $\mathscr{G}(k,1)$ was first studied by Glivjak [28], Glivjak et. al. [30], Glivjak and Plesnik [31] and Plesnik [48]. We mention briefly some of the more important results obtained by these authors.

In [31] it was proved that given any graph H there exists a graph $G \in \mathscr{G}(k,1)$ that contains H as an induced subgraph. More specifically, if

$H \in \mathcal{G}(d,1)$, then the graph G^* obtained by adding to each vertex of H a path of length k containing k new vertices is in the class $\mathcal{G}(d+2k,1)$. This along with its converse was established in [48]. Note that the graph G^* has connectivity 1. Graphs with higher connectivity were also constructed. In fact, for any given integer $k \geq 1$ and $r \geq 2$ there exists an r-regular, r-connected graph $G \in \mathcal{G}(k,1)$. The graphs constructed in [48] have a large number of vertices. This suggests the following problem:

Problem 2.1. *Given integers $k \geq 2$ and $r \geq 3$, for what value of n does there exist an r-regular(-semiregular), r-connected garph $G \in \mathcal{G}(k,1)$ of order n?*

The same question could be asked with an edge-connectivity condition and also a minimum degree condition.

One interesting question is that of determining the number of edged of a graph $G \in \mathcal{G}(k,1)$. More specifically, since a tree of diameter k is $(k,1)$-critical the following problem is of interest.

Problem 2.2. *For $k \geq 2$ and $n \geq k+1$, determine*

$$e_{\max}(k) = \max\left\{|E(G)| : G \in \mathcal{G}(k,1)) \text{ and } |V(G)| = n\right\}.$$

Also, for what value of m does there exist a $(k,1)$-critical graph on n vertices having m edges?

Plesnik [48] proved that $e_{\max}(k) > \frac{3}{4}\binom{n}{2}$. In the same paper he made the following well known conjecture for $k=2$ which was made, independently by Simon and Murty (private communication).

Conjecture 2.1. *Let $G \in \mathcal{G}(2,1)$ be a graph on n vertices. Then*

$$|E(G)| \leq \lfloor \frac{1}{4}n^2 \rfloor$$

with equality holding if and only if $G \cong K_{\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2} \rceil}$.

This conjecture has been studied by: Caccetta and Häggkvist [11] who proved the bound $.27n^2$; Fan [26] who improved the bound to $.2532n^2$; and very recently by Füredi [27] who established the conjecture for extremely large n.

Füredi [27] made the following conjecture:

Conjecture 2.2. *Let G be a graph of order n satisfying:*

(i) every pair of vertices is joined by at least ℓ disjoint paths of length ≤ 2, and

(ii) the deletion of any edge of G destroys property (i).
Then

$$|E(G)| \leq (\ell - 1)(n - \ell + 1) + \lfloor \frac{1}{4}(n - \ell + 1)^2) \rfloor.$$

The complete tripartite graph with parts of sizes $\ell - 1$, $\lfloor \frac{1}{2}(n - \ell + 1) \rfloor$ and $\lceil \frac{1}{2}(n - \ell + 1) \rceil$ satisfies the condition (i) and (ii) above and has $(\ell - 1)(n - \ell + 1) + \lfloor \frac{1}{4}(n - \ell + 1)^2 \rfloor$ edges.

A number of other conjectures can be found in [27]. We noted earlier that any graph of diameter k having girth at least $k + 2$ is in the class $\mathscr{G}(k, 1)$. Gliviak [28] proved that for any integer $k \geq 2$ and for any graph H of girth at least $k + 2$, there exist a graph $G \in \mathscr{G}(k; 1)$ having girth at least $k + 2$ containing H as an induced subgraph. Also, the author established a number of estimates for the minimum and the maximum degree as well as the size of such graphs. For the particular case of $k = 2$, Glivjak et. al. [30] characterized (2,1)-critical graphs of girth at least four.

The class of $\mathscr{G}(k, t)$, $t \geq 2$ has only been studied by Kys [38] and Anunchuen and Caccetta [3]. Kys [38] made the following conjecture:

Conjecture 2.3. $\mathscr{G}(k, t) = \emptyset$ *for* $k \geq 2$, $t \geq 2$.

In his paper Kys [38] proved the conjecture for about half the cases, namely: $k = 2$; $k = 3$; $k = 4$, $t \geq 3$; and for $k \geq 2$, $t \geq k$. Recently, Anunchuen and Caccetta [3] proved the conjecture for: $k \geq 2$, $t \geq 3$; and for $k = 4$ and 5. This leaves unresolved the case $k \geq 6$ and $t = 2$. It would be interesting to resolve this remaining case.

Finally, we mention one variation of $(k, 1)$-critical graphs studied by Kys [37]. A graph G of diameter k is said to be *strongly* $(k, 1)$*-critical* if for every edge $e = xy$ of G, $d_{G-xy}(u, v) > k$ if and only if $u = x$ and $v = y$. Observe that C_4 is strongly (2,1)-critical. Kys [37] conjectured that:

Conjecture 2.4. *For every integer* $k \geq 2$*, there exists a strongly* $(k, 1)$*-critical graph.*

In his paper, Kys [37] established this conjecture for the case $k = 2$, by proving that given any graph H of girth at least 4 there exists a strongly (2,1)-critical graph G containing H as an induced subgraph. Further, examples for $k = 3, 4$ and 6 were given.

Strongly $(k, 1)$-critical graphs are related to $(k, 1)$-critical graphs having girth at least $k + 2$ according to the following result of Kys [37]. A graph G is strongly $(k, 1)$-critical if and only if $G \in \mathscr{G}(k, 1)$, has girth at least $k + 2$ and for every pair of non-adjacent vertices x and y there are at least two edge-disjoint (x, y)-paths of length at most k.

2.2 Edge Addition.

Now we consider graphs whose diameter decreases with the addition of an edge between any pair of non-adjacent vertices. A graph G of diameter $D \geq 2$ is said to be *D-critical* if $d(G+e) < D$ for every edge e of $\overline{G}$. D-critical graphs can be conveniently studied by considering the distance decomposition of a graph.

Let G be a graph of diameter $D \geq 2$. Then $V(G)$ can be partitioned into $D+1$ non-empty sets $L_0, L_1, \ldots, L_D$, such that $L_0 = \{u\}$ and L_i, $i = 1, 2, \ldots, D$, denotes the set of vertices of G at distance i from u. u is called a *peripheral vertex*; in fact any vertex x of G having a vertex at distance D is a peripheral vertex of G.

Ore [47] introduced the concept of D-critical graphs and observed that a graph G is D-critical if and only if every peripheral vertex u gives rise to a distance decomposition $L_0 = \{u\}$, $L_1, \ldots, L_D$ such that $|L_D| = 1$ and every vertex of L_i, $i = 0, 1, \ldots, D-1$, is adjacent to every other vertex in L_i and L_{i+1}. Thus a D-critical graph is completely determined once $|L_i| = n_i$, $i = 0, 1, \ldots, D$, are specified. The sequence $(n_0, n_1, \ldots, n_D)$ is called a *vertex sequence.*

The above simple observation motivated Ore [47] to introduce a connectivity requirement. Let $\mathscr{G}_e(n, D, K)$ $[\mathscr{G}_v(n, D, K)]$ denote the class of D-critical K-edge [respectively, K-vertex]-connected graphs on n vertices. The problem that arises is the following:

Problem 2.3. *Characterize the classes $\mathscr{G}_e(n, D, K)$ and $\mathscr{G}_v(n, D, K)$. In particular, determine*

$$f_x(n, D, K) = \{|E(G)| : G \in \mathscr{G}_x(n, D, K)\}, \qquad x = e, v.$$

Ore [47] completely characterized the class $\mathscr{G}_v(n, D, K)$. In particular he showed that the edge-maximal members of this class have a very simple structure. For $D \geq 4$ this simple structure is described by the requirement that an edge-maximal graph must have vertex sequence $(1, K, n_2, n_3, \ldots, n_{D-2}, K, 1)$ with $n_i = K$ for all $i \geq 2$ except possibly one or a consecutive pair. Note that the connectivity constraint imposes the condition $n_i \geq K$ for $i = 1, 2, \ldots, D-1$.

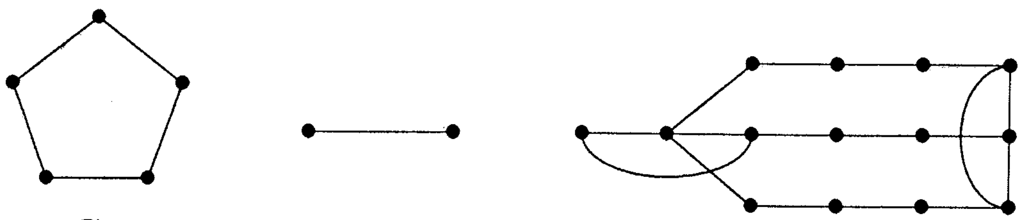
The edge-minimal members of $\mathscr{G}_v(n, D, K)$ were completely characterized by Caccetta and Smyth [12]. In the same paper, the edge-minimal members of $\mathscr{G}_e(n, D, K)$ were completely characterized for sufficiently large n; the problem for "small" n is open and appears difficult. As one would expect, for large n, the edge-minimal members of $\mathscr{G}_e(n, D, K)$ and $\mathscr{G}_v(n, D, K)$ coincide. The characterization of the edge-minimal graphs was in terms of vertex sequences. Unlike the edge-maximization problem, the specification of the vertex arrangement depends, not on the maximum concentration of vertices, but rather on, the maximum spreading of vertices.

The edge-maximal members of $\mathscr{G}_e(n, D, K)$ were completely characterized by Caccetta and Smyth [15]. The edge-connectivity requirement can only be satisfied if there are at least K edges between the vertices in L_i and L_{i+1}, $i = 0, 1, \ldots, D-1$. That is, $n_i n_{i+1} \geq K$ for $i = 1, 2, \ldots, D-1$. This requires that $n_{i-1} + n_i + n_{i+1} \geq K + 1$ for $i = 1, 2, \ldots, D-1$. For $D \geq 6$, $K \geq 8$, it was proved in [15], that the vertex sequence of an edge maximal graph $G \in \mathscr{G}_e(n, D, K)$ takes the form $(1, K, n_2, \ldots, n_{D-1}, K, 1)$ with $n_i n_{i+1} \geq K$ for $i = 2, 3, \ldots, D-3$, and $n_{i-1} + n_i + n_{i+1} = K + 1$ for all i, $3 \leq i \leq D-3$ except possibly one which must be $i = 3$ or $i = D-3$. The structure of edge-maximal graphs dependes heavily (see Caccetta and Smyth [13]) on the values of n and D.

The question of determining the function $f_x(n, D, K)$ defined in Problem 2.3 is very much open, though some results have been obtained by Caccetta and Smyth [14, 16].

2.3 Vertex deletion.

In this section we consider graphs whose diameter changes when a vertex is deleted. Consider the graphs displayed in Figure 2.1.



G_1 G_2 G_3

Figure 2.1

Clearly $d(G_1) < d(G_1 - v)$ for every $v \in V(G_1)$, $d(G_2) > d(G_2 - u)$ for every $u \in V(G_2)$, and $d(G_3 - a) < d(G_3) < d(G_3 - w)$ for every $w \in V(G_3) \setminus \{a\}$. Thus the diameter could increase or decrease when a vertex is deleted. This motivates the following definition.

A graph G is said to be *v-critical,v^+-critical, v^--critical* if $d(G - u)$ is not equal to, greater than and less than, respectively, $d(G)$ for every $u \in V(G)$. Observe that G_1 is v^+-critical, G_2 is v^--critical, and G_3 is v-critical. Note also that G_3 is not v^+-critical.

Gliviak [29] proved that a v-critical graph G has at most two elements in the set $\{u \in V(G) : d(G - u) < d(G)\}$. An immediate consequence of this is that a graph G is v^--critical if and only if $G \cong K_2$ or $\overline{K}_2$. Thus only v- and v^+-critical graphs are of interest.

v-critical graphs of diameter 2 were first studied by Glivjak et. al. [30]. They completely characterized the triangle free v-critical graphs of diameter 2 having order at most 10 and gave constructions for higher orders. The only other work on v- and v^+-critical graphs that we are aware of is that of Bosis and Ali [4], Gliviak [29], Glivjak and Plesnik [31] and Plesnik[48].

Gliviak [29] established a number of results concerning v- and v^+-critical graphs. In particular, he proved that a graph G with $\delta(G) \geq 2$ having girth at least $d(G) + 3$ is v-critical. Further, he proved that every graph H is an induced subgraph of a v-critical graph G of diameter $D \geq 2$ that is not v^+-critical. The graph G is given in Figure 2.2.

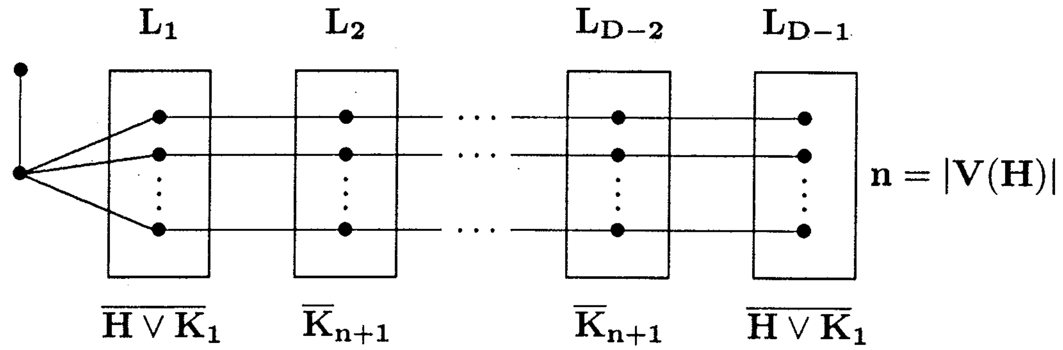


Figure 2.2

Note that all vertices in L_i, $i = 2, 3, \ldots, D - 2$ have degree 2 in G and are joined to exactly one vertex in L_{i-1} and one vertex in L_{i+1}.

Boals and Ali [4] proved that every graph H is an induced subgraph of a v^+-critical graph G of diameter $D \geq 2$; the graph G constructed is given in

Figure 2.3.
The graph given in Figures 2.2 and 2.3 have $nD+1$ and $(n+1)(D-1)+2$ vertices, respectively. It would be of interest to determine the smallest graphs having the required property.

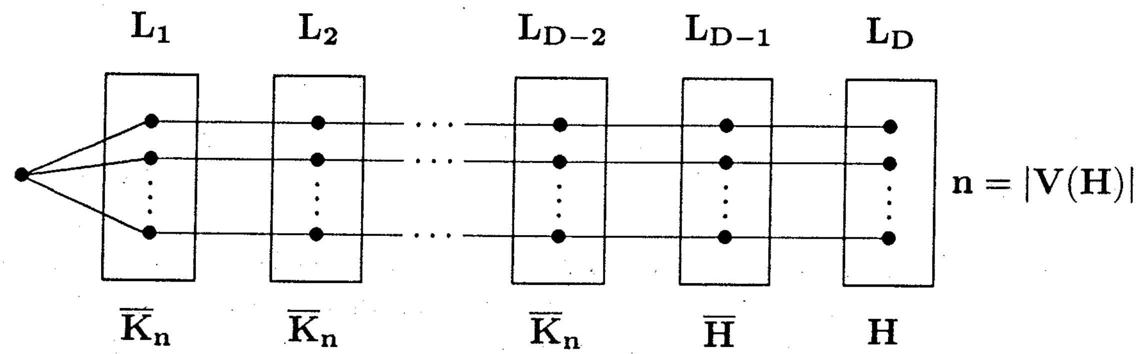


Figure 2.3

Problem 2.4. *Given a graph H and an integer $D \geq 2$ find a v^+-critical graph containing H as induced subgraph and having as few vertices as possible. The same question can be asked for v-critical graphs.*

There is one nice caracterization theorem concerning v-critical graphs due to Gliviak [29]. To describe this result we need the following concept. A *branch* at a cut vertex u of G is a maximal subgraph of G containing u such that u is not a cut vertex. The result is the following. A v-critical graph G of diameter $D \geq 2$ is either a path of lenght D or a block with at most two branches which are paths each of lenght at most $\lfloor \frac{1}{2}D \rfloor$ for odd $D \geq 5$ and $\lfloor \frac{1}{2}D \rfloor - 1$ otherwise. A consequence of this is that every v^+-critical graph is a block. This latter result was also proved by Boals and Ali [4]. We note that the path lenght specified in the Gliviak's result can be achieved – construction were given in [29].

We conclude this section by stating some problems and conjectures.

Problem 2.5. *Characterize v-critical and (v^+)-critical graphs with prescribed:*
(a) connectivity;
(b) edge-connectivity;
(c) minimum degree;
(d) number of edges.

The following two conjectures were made by Boals and Ali [4].

Conjecture 2.5. *If G is a v^+-critical graph of diameter D, then $d(G - x) \leq 2D - 1$ for every $x \in V(G)$.*

Conjecture 2.6. *If G is a v^+-critical graph of diameter 2, then $\Delta(G) \leq \frac{1}{2}|V(G)|$.*

Conjecture 2.5 was proved for the cases $D = 2$ and 3.

3. Critically connected graphs.

In this section we consider graphs that are critical with respect to the graph parameters connectivity κ and edge-connectivity κ'. A graph G is said to be *P-vertex-critical*, $P = \kappa$ or κ', if $P(G - v) < P(G)$ for every vertex v of G. Similarly, G is said to be *P-edge-critical*, $P = \kappa$ or κ', if $P(G - e) < P(G)$ for every edge e of G. Let $\mathscr{C}_v(n, P)$ and $\mathscr{C}_e(n, P)$, respectively, denote the class of P-vertex-critical and P-edge-critical graphs on n vertices.

The main problem that araise concerns the characterization of the classes $\mathscr{C}_v(n, \kappa)$, $\mathscr{C}_v(n, \kappa')$, $\mathscr{C}_e(n, \kappa)$ and $\mathscr{C}_e(n, \kappa')$. Of particular interest is the problem of determing the number of edges that a graph in each of these classes can have. It is very well knonw that for any graph G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. Further, there exists a class of graphs (sometimes referred to as Harary graphs) on n vertices with $\kappa = \kappa' = \delta$ having $\lceil \frac{1}{2} n\delta \rceil$ edges. This edge-minimal class belongs to each of $\mathscr{C}_v(n, \kappa)$, $\mathscr{C}_v(n, \kappa')$, $\mathscr{C}_e(n, \kappa)$ and $\mathscr{C}_e(n, \kappa')$. However, the non edge-minimal graphs are not so easily described.

The graphs in Figure 3.1 show that these classes are not necessarily identical: clearly G_1, G_2, G_3 and G_4 belong only to the class $\mathscr{C}_v(10, \kappa(G_1))$, $\mathscr{C}_v(10, \kappa'(G_2))$, $\mathscr{C}_e(10, \kappa(G_3))$ and $\mathscr{C}_e(10, \kappa'(G_4))$, respectively.

The class $\mathscr{C}_e(n, \kappa)$ was first studied by Dirac [24] who obtained a characterization for the case $\kappa = 2$. He established that a graph $G \in \mathscr{C}_v(n, \kappa = 2)$ has $\delta(G) = 2$ and $\varepsilon(G) \leq 2n - 4$. He also proved that G is 3-colourable; which means that the vertex set of G can be coloured by using three colours such that no two adjacent vertices receive the same colour. A similar result was obtained by Plummer [49].

Halin characterized (see [34]) the edge-maximal graphs of $\mathscr{C}_e(n, \kappa = 3)$ for $n \geq 8$. He established that the complete bipartite graph $K_{3,n-3}$ is the unique edge-maximal graph. For the more general case, Mader [42] proved that for $n > 3k - 2$ the complete bipartite graph $K_{k,n-k}$ is the only edge-maximal member of $\mathscr{C}_e(n, \kappa = k)$. Further, for $k+1 \leq n \leq 3k-2$, Cai [18] characterized the edge-maximal members of $\mathscr{C}_e(n, \kappa = k)$

Halin [32] proved that if G is in the class $\mathscr{C}_e(n, \kappa)$ then $\delta(G) = \kappa(G)$. In a later paper, Halin [33] established that every graph $G \in \mathscr{C}_e(n, \kappa)$ has more

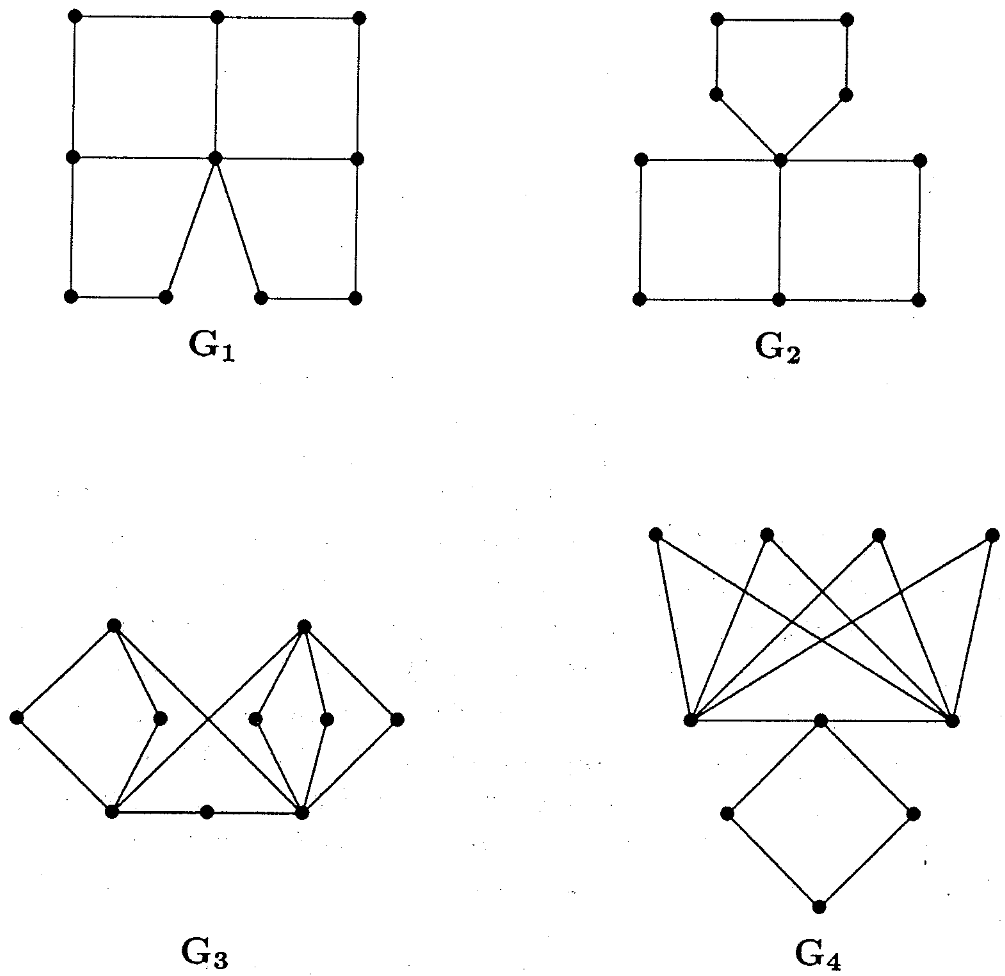


Figure 3.1

than $\frac{1}{2}\sqrt{\Delta(G)}$ vertices of degree $\delta(G)$. He also established a better bound when $\kappa = 3$. More specifically he proved that if $G \in \mathscr{C}_e(n, \kappa = 3)$ then G contains at least $\frac{2}{5}(n+3)$ vertices of degree 3. Bollobás [6] improved and extended this result to the general case. In particular, he proved that every $G \in \mathscr{C}_e(n, \kappa = k)$ contains at least $\frac{(k-1)n+2}{2k-1}$ vertices of degree $\delta(G) = k$.

The above results are about all that is knonw about the class $\mathscr{C}_e(n, \kappa)$. This leaves open a number of extremal problems including the interesting problem of characterizing the edge-maximal members of $\mathscr{C}_e(n, \kappa)$.

We next consider the class $\mathscr{C}_v(n, \kappa)$. A consequence of the definition of vertex-criticality is that if $G \in \mathscr{C}_v(n, \kappa = k)$, then $\kappa(G - u) = k - 1$ for every vertex u of G. An early result of Chartrand et. al. [21] established that $\delta(G) \leq \frac{3}{2}k - 1$ for $G \in \mathscr{C}_v(n, \kappa = k)$, $k \geq 2$. Hamidoune [35] proved that

G contained two such vertices with degree at most $\frac{3}{2}k - 1$, and this is the best possible.

Entringer [25] characterized the edge-maximal graphs of $\mathscr{C}_v(n, \kappa = 2)$; these graphs turn out to be unique for $n \geq 3$ except $n = 11$. Krol and Veldam [36] considered the subclass $\mathscr{A}_v(n, \kappa)$ of $\mathscr{C}_v(n, \kappa)$ consisting of those graphs in which every vertex is adjacent to a vertex of degree κ. They characterized the edge-maximal members of $\mathscr{A}_v(n, \kappa)$ for $\kappa \geq 3$. In particular, they showed that for $\kappa = 3$ the edge-maximal members of $\mathscr{A}_v(n, \kappa)$ coincide with the edge-maximal members of $\mathscr{C}_v(n, \kappa)$. They conjectured:

Conjecture 3.1. *The edge-maximal members of $\mathscr{A}_v(n, \kappa)$ coincide with the edge-maximal members of $\mathscr{C}_v(n, \kappa)$ for $\kappa \geq 3$.*

Note that Entringer's characterization of class $\mathscr{C}_v(n, \kappa = 2)$ demonstrates that the above conjecture is not valid for $\kappa = 2$, hence the condition $\kappa \geq 3$.

The above few results are all that is known about the class $\mathscr{C}_v(n, \kappa)$.

The class $\mathscr{C}_v(n, \kappa')$ has been studied by Cozzens and Wu [22, 23]. They considered only the subclass $\mathscr{C}'(n, \kappa')$ consisting of those graphs for which $\kappa'(G-v) = \kappa'(G)-1$ for every vertex v of G. Cozzens and Wu [22] established that the well known Harary graphs are the edge-minimal members of $\mathscr{C}'_v(n, \kappa')$ for $\kappa' \geq 2$. They also showed that the problem of finding an edge-minimal critical spanning subgraph H in a given graph G is NP-complete. In a latter paper Cozzens and Wu [23] characterized the edge-maximal members of the subclass $\mathscr{A}'_v(n, \kappa')$ of $\mathscr{C}'_v(n, \kappa')$ consisting of those graphs in which every vertex is adjacent to a vertex of degree κ'. As these are the only results, the problems of characterizing $\mathscr{C}'_v(n, \kappa')$ and the edge-maximal members of $\mathscr{C}'_v(n, k')$ have hardly been addressed.

Finally, we consider the class $\mathscr{C}_e(n, \kappa')$ for which there are many results. Lick [39] proved that if $G \in \mathscr{C}_e(n, \kappa' = k)$ then $\delta(G) = k$. Mader [41] extended this result by showing that such a G must have at least $k+1$ vertices of degree κ. In a later paper [43], he proved that the number of vertices of degree κ in a graph $G \in \mathscr{C}_e(n, \kappa' = k)$ is at least $\lfloor \frac{n}{k+1} \rfloor + k$ for odd $k \geq 5$, $\lfloor \frac{2n}{k+1} \rfloor + k - 2$ for odd $k \geq 7$, and at least $\lfloor \frac{n-1}{2k+1} \rfloor + k + 1$ for even k. These results have recently been improved by Cai [20], who established the bound: $\lfloor \frac{n}{k+1} \rfloor + k$ for even $k \geq 4$ and $\lfloor \frac{2n}{k+1} \rfloor + k - 2$ for even $k \geq 10$. In a forthcoming paper (personal communication), Mader gives the exact formula.

Mader [41] proved that for $n \geq 3k$ the edge maximal graphs of $\mathscr{C}_e(n, \kappa' = k)$ coincide with the edge-maximal graphs of the subclass $\mathscr{A}_e(n, \kappa' = k)$ of $\mathscr{C}_e(n, \kappa' = k)$ consisting of those graphs in which every vertex is adjacent to a vertex of degree κ'. The corresponding result for $n < 3k$ was proved by Cai [19] and, independently, by Budayasa et. al. [8].

Budayasa et. al. [9] determined completely the size spectrum of the class $\mathscr{C}_e(n, \kappa' = k)$.

We conclude this section by noting a generalization of critically connected graphs introduced by Maurer and Slater [46]. A graph G is called *t-critically k-connected* or simply *(t,k)-critical* if $\kappa(G - V') = k - |V'|$ for all $V' \subseteq V(G)$ with $|V'| \leq t$. For example, K_{n+1} is (n, n)-critical. The k-regular bipartite graph on $2k + 2$ vertices is $(2k, k)$-critical. Mader [44] conjectured that this bipartite graph is the only $(2k, k)$-critical graph. (t, k)-critical graphs have been studied in [44 - 46]. The following conjecture was stated in Maurer and Slater [46].

Conjecture 3.2. *If G is a (t, k)-critical graph with*

$$1 \leq k \leq \left\lfloor \frac{1}{2} t \right\rfloor \quad \text{or} \quad k = t,$$

then $G \cong K_{t+1}$.

Mader [45] has considered this conjecture and has proved that if a non-complete (t, k) graph exists it must contain less than $3(t - 1)$ vertices.

REFERENCES

[1] N. Achuthan, N.R. Achuthan and L. Caccetta, *On the Nordhaus-Gaddum class problems,* Australasian Journal of Combinatorics 2 (1990), pp. 5-27.

[2] N. Anunchuen and L. Caccetta, *On critically k-extendable graphs,* Australasian Journ. of Combinatorics 6 (1992), in press.

[3] N. Anunchuen and L. Caccetta, *On strongly edge-critical graphs of given diameter,* submitted for publication.

[4] A. Boals and H. Ali, *Vertex diameter critical graphs,* Congressus Numerantium 72 (1990), pp. 193-198.

[5] B. Bollobás, *Extremal graph theory,* Academic Press, London, (1978).

[6] B. Bollobás, *On graphs with equal edge-connectivity and minimum degree,* Discrete Mathematics 28 (1979), pp. 321-323.

[7] G.S. Bloom, J.W. Kennedy and L.V. Quintas, *Characterization of graphs of diameter two,* Amer. Math. Monthly 94 (1987), pp. 37-38.

[8] K. Budayasa, L. Caccetta and K. Vijayan, *The characterization of edge-maximal critically k-edge-connected graphs,* Australasian Journ. of Combinatorics 5 (1992), pp. 179-207.

[9] K. Budayasa, L. Caccetta and K. Vijayan, *On critically k-edge connected graphs with prescribed size,* submitted for publication.

[10] L. Caccetta, *Graph theory in network design and analysis,* Recent Studies in Graph Theory, V.R. Kulli ed., Vishwa International, India, (1989), pp. 29-63.

[11] L. Caccetta and R. Häggkvist, *On diameter critical graphs,* Discrete Mathematics 28 (1979), pp. 223-229.

[12] L. Caccetta and W.F. Smyth, *Diameter critical graphs with minimum number of edges,* Congressus Numerantium 61 (1988), pp. 143-153.

[13] L. Caccetta and W.F. Smyth, *Redistribution of vertices for maximum edge count in K-edge-connected D-critical graphs,* Ars Combinatoria 26B (1988), pp. 115-132.

[14] L. Caccetta and W.F. Smyth, *Graphs of maximum diameter,* Discrete Mathematics 102 (1992), pp. 121-141.

[15] L.Caccetta and W.F. Smyth, *A characterization of edge-maximal diameter critical graphs,* submitted for publication.

[16] L. Caccetta and W.F. Smyth, *The construction of K-connected D-critical graphs of given order and size,* submitted for publication.

[17] L. Caccetta and K. Vijayan, *Applications of graph theory,* Ars Combinatoria 23B (1987), pp. 21-77.

[18] M.C. Cai, *Minimally k-connected graphs of low order and maximal size,* Discrete Mathematics 41 (1982), pp. 229-234.

[19] M.C. Cai, *The maximal size of graphs with at most k-edge-disjoint paths connecting any two adjacent vertices,* Discrete Mathematics 85 (1990), pp. 43-52.

[20] M.C. Cai, *A remark on the number of vertices of degree k in a minimal k-edge-connected graph,* Discrete Mathematics 104 (1992), pp. 221-226.

[21] G. Chartrand, A. Kaugars and D.R. Lick, *Critically n-connected graphs,* Proc. of the Amer. Math. Soc. 32 (1972), pp. 63-68.

[22] M.B. Cozzens and S.Y. Wu, *On minimum critically n-edge-connected graphs,* SIAM J. Alg. Disc. Math. 8 (1987), pp. 659-668.

[23] M.B. Cozzens and S.Y. Wu, *Maximum critical n-edge-connected graphs,* Journal of Graph Theory 13 (1989), pp. 559-568.

[24] G.A. Dirac, *Minimally 2-connected graphs,* J. Reine Angew. Math. 228 (1967), pp. 204-216.

[25] R.C. Entringer, *Characterization of maximum critically 2-connected graphs,* J. of Graph Theory 2 (1978), pp. 319-327.

[26] G. Fan, *On diameter 2-critical graphs,* Discrete Mathematics 67 (1987), pp. 235-240.

[27] Z. Füredi, *The maximal number of edges in a minimal graph of diameter 2,* Journal of Graph Theory 16 (1992), pp. 81-98.

[28] F. Gliviak, *On certain edge-critical graphs of a given diameter,* Matematicky Casopis 25 (1975), pp. 249-263.

[29] F. Gliviak, *Vertex-critical graphs of given diameter,* Acta Mathematica Academiae Scientiarum Hungaricae 27 (1976), pp. 255-262.

[30] F. Glivjak, P. Kys and J. Plesník, *On irriducible graphs of diameter two without triangles,* Matematicky Casopis 19 (1969), pp. 149-157.

[31] F. Glivjak and J. Plesník, *On the existence of certain overgraphs of given graphs,* Acta F.R.N. Univ. Comen - Mathematica 23 (1969), pp. 113-119.

[32] R. Halin, *A theorem on n-connected graphs,* Journal of Comb. Theory 7 (1969), pp. 150-154.

[33] R. Halin, *On the structure of n-connected graphs,* Recent Progress in Combinatorics, W.T. Tutte ed., Academic Press, New York (1969), pp. 91-102.

[34] R. Halin, *Studies on minimally n-connected graphs,* Combinatorial Mathematics and Its Applications, D.J.A. Welsh ed., Academic Press, New York (1971), pp. 129-136.

[35] Y.O. Hamidoune, *On critically h-connected simple graphs,* Discrete Mathematics 32 (1980), pp. 257-262.

[36] J.J. Krol and G. J. Veldam, *On maximum critically k-connected graphs,* Discrete Mathematics 52 (1984), pp. 225-234.

[37] P. Kys, *Diameter strongly critical graphs,* Universitas Comeniana Acta Mathematica Universitatis Comenianae 37 (1980), pp. 71-83.

[38] P. Kys, *Diameter k-critical graphs,* Universitas Comeniana Acta Mathematica Universitatis Comenianae 38 (1981), pp. 63-85.

[39] D.R. Lick, *Minimally n-line connected graphs,* J. Reine Angew. Math. 252 (1972), pp. 178-182.

[40] L. Lovász and M.D. Plummer, *Matching Theory,* Annals of Discrete Mathematics 19 (1986), pp. 445-456.

[41] W. Mader, *Minimale n-fach kantenzusammenhängende graphen,* Math. Ann. 191 (1971), pp. 21-28.

[42] W. Mader, *Uber minimal n-fach zusammenhängende, unendliche graphen und ein extremal problem,* Arch. Math. (Basel) 23 (1972), pp. 553-560.

[43] W. Mader, *On k-critically connected graphs,* Progress in Graph Theory, J.A. Bondy and U.S.R. Murty eds., Academic Press, Canada, (1984), pp. 389-398.

[44] W. Mader, *Uber (k+1)-kritisch (2k+1)-fach zusammenhängende graphen,* J. Combinatorial Theory 45B (1988), pp. 45-57.

[45] S. Maurer and P.J. Slater, *On k-critical n-connected graphs,* Discrete Mathematics 20 (1977), pp. 255-262.

[46] O. Ore, *Diameter in graphs,* Journal of Comb. Theory 5 (1968), pp. 75-81.

[47] J. Plesník, *Critical graphs of given diameter,* Acta Fac. Rerum Natur. Univ. Comenian Math. 30 (1975), pp. 71-93.

[48] M.D. Plummer, *On minimal blocks,* Trans. of the Amer. Math. Soc. 134 (1968), pp. 85-94.

[49] H.P. Yap, *Some topics in graph theory,* London Math. Soc. (Lectures Note series 108), Cambridge University Press, (1986).

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