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SOME OF MY FAVOURITE PROBLEMS IN VARIOUS BRANCHES OF COMBINATORICS

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1. Problems of Catania Conference in 1989.

At the meeting in Catania in 1989, I gave a talk entitled " On Some of my favourite problems in graph theory and block designs ". I stated twenty problems and first of all I want to review those problems where important new results have been found.

1. Problem 1 was the old conjecture of Faber, Lovász and myself. The chromatic number of the union of n edge disjoint complete graphs of size n, is n. I offered, and offer, five hundred dollars for a proof or disproof. Jeff Kahn recently proved that the chromatic number is $< n(1 + o(1))$. I gave him a consolation prize of one hundred dollars and hope that he or somebody else will soon collect the five hundred dollars with n colors.

J. Kahn, *Coloring nearly disjoint hypergraphs*, J.C.T. Ser. A 59 (1992), 31-39 (see also J. Kahn and P.D. Seymour, *A fractional version of the Erdös-Faber-Lovász conjecture*, Combinatorica 12 (1992), 159-160.

2. Problem 2 was due to Du, Shu and myself. Consider n vertex disjoint triangles $T_i, 1 \leq i \leq n$, and a Hamiltonian cycle containing all the $3n$ vertices but not using any of the edges of the triangles. This gives a $G(3n; 6n)$. We conjectured that this graph is three-chromatic. This conjecture was very recently proved Fleischner and Stiebitz.

H. Fleishner and M. Stiebitz,*A solution to a colouring problem of P. Erdös*, Discrete Math. 101 (1992), 39-48.

3. In Problem 3, I conjectured that every subgraph of $\varepsilon n 2^n$ edges of the n-dimensional cube contains a c_6. This conjecture was disproved by Fan Chung. My old conjecture that every subgraph of $(\frac{1}{2}+\varepsilon)n2^{n-1}$ edges of the n-dimensional cube contains a c_4 remains open. Fan Chung's paper contains many further interesting problems, but I have to refer to her paper.

F.R.K. Chung, *Subgraphs of a hypercube containing no small even cycles*, J. Graph Theory 16 (1992), 273-286.

4. In Problem 6 I ask (among other questions) whether it is true that every graph of $(2k+1)n$ vertices, the smallest odd cycle of which has size $\geq 2k+1$, can be made bipartite by omitting at most n^2 edges. This is still open for every $k \geq 1$. I also ask whether such a graph can have at most n^{2k+1} cycles of size $2k+1$. A slightly weaker result has recently been proved by E. Györi and very recently Györi proved the conjecture for $k = 2$ and $n > n_0(k)$. This sharper result will appear soon.

E. Györi, *On the number of C_n 's in a triangle free graph*, Combinatorica 9 (1989), 101-102.

5. In Problem 20 Lovász and I ask: Let $m(n)$ be the smallest integer for which there is a family of sets $A_i, 1 \leq i \leq m(n), |A_i| = n, |A_i \cap A_j| \leq 1$ for every $1 \leq i < j \leq m(n)$, and if $\mathscr{S}$ is a set with $|\mathscr{S} \cap A_i| \geq 1$ (i.e., $\mathscr{S}$ meets all the $A's$) then $|\mathscr{S}| \geq n$. We thought that perhaps $m(n)/n \to \infty$ (i.e., if every blocking set has size $\geq n$ must we have $m(n)/n \to \infty$). Jeff Kahn disproved this. He proved the existence of a family for wich $m(n) < cn$. The best value of c is not yet known.

2. Some new problems.

6. Hajnal Galvin and I asked in our old triple paper the following question: Is it true thet if G has chromatic number $\aleph_1$, we can always color the edges by $\aleph_0$ colors so that for any division of the vertices into $\aleph_0$ classes, there always is a class which contains edges of all the colors. We could never settle this problem even if instead of $\aleph_0$ colors we only use two colors. Many generalizations are clearly possible. Hajnal and I recently tried to pose a related finite problem: Let $f(k,\ell)$ be the smallest integer for which if G has chromatic number $\geq f\ (k,\ell)$. One can color the edges of G by k colors, so that for any division of the vertices into ℓ classes, one of the classes will contain edges of all the colors.

Thomassen easily showed that $f(k;\ell)$ exists and the exact determination of $f(k,\ell)$ will perhaps not be very difficult. Nevertheless Hajnal and I were led to some related problems which seem interesting and are perhaps difficult. Is it true that to every k there is an $f(k)$ for which if the chromatic number of G is $\geq f(k)$ then G contains an odd cycle whose vertices induce a graph of chromatic number $\geq k$. To our surprise and regret we could get nowhere with this problem. We formulated another related question: Is it true that for every r there is a $g(r)$ so that if G has chromatic number $\geq g(r)$, then G has r edge disjoint odd cycles on the same vertex set? We could not even prove that $g(2)$ exists. We could not even get a positive answer to our questions if we assume that G has chromatic number $\aleph_1$. We were somwhat doubtful about the existence of $g(2)$ but are fairly sure that $f(k)$ exists.

P. Erdös, F. Galvin and A. Hajnal, *On set systems having large chromatic number and not containing prescribed subsystems*, Infinite and finite sets Colloq. Keszthely 1973 Math. Soc. Bolyai Vol. 10, 425-513, North Holland Amsterdam 1975.

7. I conjectured and Gyárfás proved that every k chromatic graph contains at least $[\frac{k-1}{2}]$ odd cycles of different lenght. Equality holds for the complete graph of k vertices. Special cases were proved earlier by Bollobás and Shelah.

The following conjecture is probably very much more difficult: Is it true that every triangle free k-chromatic graph contains more than $k^{2-\varepsilon}$ odd cycles of different lenght? In fact, does it contain $k^{2-\varepsilon}$ cycles of different lenght?

Perhaps the following question of Hajnal and myself is both interesting and difficult: Let $G(n)$ be an edge critical k-chromatic graph of n vertices (i.e., the omission of any edge decreases the chromatic number). Is it true that our $G(n)$ contains an odd cycle of length $f(n;k)$ where $f(n;k)$ tends to infinity as n tends to infinity? If true it would be interesting to estimate $f(n;k)$.

A. Gyárfás, *Graphs with k odd cycle length*, Discrete Math. 103 (1992), 41-48.

8. Faudree, Schelp and I considered the following problem: Let G be a graph, every vertex of which has degree $\geq k$ and the girth of G is > 4. We showed that the number of cycles of different length is $> ck^2$. Perhaps if the girth is $> 2s$ then the number of cycles of different lenght is $> c_s k^s$. We are very far from being able to prove this conjecture. We could prove only that if the girth is > 6 then the number of cycles of different length is $> ck^{5/2}$.

9. Faudree and I considered the following problem: Let $G(n)$ be a graph of n vertices. Let $3 \leq a_1 < a_2 < \ldots < a_k \leq n$ be the lengths of cycles occuring in $G(n)$. Let now $G(n)$ run through all the graphs of n vertices and consider the set of sequences belonging to at least one of the graphs $G(n)$. Denote by $f(n)$ the number of distinct sequences. Clearly $f(n) \leq 2^{n-2}$ but we expect that $f(n)$ is very much smaller. In fact we conjecture $f(n) < (2-\varepsilon)^n$, $f(n)^{1/n} \to c < 2$. We proved $f(n) > 2^{n/4(1-\varepsilon)}$.

10. In an old paper I ask the following question. Find all critical four chromatic graphs which are the union of bipartite graphs and disjoint edges. $K(4)$ is of course such a graph. Gyárfás found a graph of 20 vertices satisfyng this condition but we could not characterise all of them.

11. Is it true that every graph of minimal degree ≥ 3 contains two cycles of length e_1 and e_2 satisfyng

$$e_1 < e_2 \leq e_1 + 2? \tag{1}$$

If true, this is the best possible as is shown by bipartite graphs. I could not show (1) even if every vertex of G has degree $\geq k$.

12. Here is an old problem of Brandon Mc Kay and myself, which I completely forgot.

Let $G(n; cn^2)$ be a graph, the largest trivial subgraph of which has size less than $\alpha_c \log n$ (following a notation of Bollobás we call a complete or empty graph *trivial*). Is it true that there is an ε so that for every $t < \varepsilon n^2$ our graph has an induced subgraph which contains exactly t edges? We only proved this for $t < \varepsilon(\log n)^2$. Perhaps our conjecture was too optimistic.

13. Noga Alon and Bollobás asked: Let $G(n)$ be a graph the largest trivial subgraph of which has size $< c \log n$. Then such a graph has at least cn^2 induced subgraphs, no two of which have the same number of edges and vertices. Perhaps $n^{5/2}$ is the correct order of magnitude but V. T. Sós and I could only prove $n^{3/2}$. V. T. Sós, Faudree and I thought taht such a $G(n)$ has an induced subgraph of size $c_1 n$ and at least $[c_2\sqrt{n}]$ vertices of different degree.

14. Several years ago Rothschild and I posed the following problem: Let $G(n; cn^2)$ be a graph of n vertices and cn^2 edges. Assume that every edge is contained in a triangle. Is it then true that there is an edge which is contained in many triangles. More precisely let $g(n, c)$ be the largest integer for which there is at least one edge which is contained in $g(n, c)$ triangles. Szemerédi proved that his regularity lemma implies that for every $c > 0$, $g(n; c) \to \infty$ and Noga

Alon and Trotter observed that for $c < \frac{1}{4}$ $g(n;c) < k(c)n^{1/2}$, for $c > \frac{1}{4}$ it is easy to see that $g(n;c) > \alpha_c n$. More precisely, we can pose the following problem : Let $g(n;k)$ be the largest integer for which for every graph of n vertices and k edges every edge of which is contained in at least one triangle there is an edge which is contained in at least $g(n;k)$ triangles. It is well known and easy to see that for $k \geq [\frac{n^2}{4}] + 1$, $g(n;k) > cn$ and here we can drop the assumption that every edge is contained in a triangle (by the well-known theorem of Turán, $k \geq [\frac{n^2}{4}] + 1$ implies the existence of a triangle). It is not difficult to prove that for $k > \frac{n^2}{4} - cn$

(1) $$g(n;k) > f(c) \cdot n.$$

The construction of Alon-Trotter gives that as $c \to 0$, $f(c) \to 0$. Thus (1) in some sense is best possible. It would of course be interesting what happens to $g(n;k)$ if $k = \frac{n^2}{4} - n^\alpha$ for $1 < \alpha < 2$. I think it would be interesting to prove that for every c and sufficiently large n

(2) $$g(n;c) > \log n.$$

Rothschild and I expected that a very much stronger result that (2) will hold.

The following slight modification of our question also seems interesting: Denote by $e(n;k)$ the smallest integer for which every graph $G(n;e(n;k))$ every edge of which is contained in at least one triangle has an edge which is contained in at least k triangles. Estimate $e(n;k)$ as well as you can. Szemerédi's result implies that for every k, $e(n;k) = \sigma(n^2)$. Ruzsa observed that

(3) $$e(n;2) > cnr_3n$$

where $r_3(n)$ is the largest integer for which there is a set of $r_3(n)$ integers not containing an arithmetic progression of three terms. Roth proved more than forty years ago that

$$r_3(n) < cn/ \log\log n$$

and Heath-Brown and Szemerédi proved

$$r_3(n) < n/(\log n)^{1/4}.$$

Felix Behrend proved ($\exp t = e^t$))

$$r_3(n) > n \exp(-c(\log n)^{1/2})$$

It would be very nice to get a better inequality for $r_3(n)$ and to get a better upper and lower bound for $e(n;2)$ and more generally for $e(n;k)$ it is not at all clear to me how $e(n;k)$ depends on k. I offer a prize of one thousand dollars for clearing up these questions.

R.L. Graham, B.L. Rothschild, J.H. Spencer, *Ramsey Theory*, Wiley, Interscience Series.

15. Another old question of Rothschild and myself states as follows: Let $G\ (n)$ be a graph of n labelled vertices. Denote by $f(g(n))$ the maximum number of ways we can color the edges of our $G(n)$ by two colors so that there should be no monochromatic triangle. We conjectured

$$f(G(n)) \leq 2^{[n^2/4]}.$$

Equality for Turán graph, i.e., for the complete bipartite garph of $[\frac{n}{2}]$ white and $[\frac{n+1}{2}]$ black vertices. To our disappointment we could not prove our conjecture; perhaps we overlooked a simple argument. Clearly many generalizations would be possible.

16. An old and forgotten problem of Bondi and myself states: Let $h(n)$ be the smallest integer for which every $G(n; n + h(n))$ contains two cycles of the same length. We proved

$$c_1 n^{1/2} < h(n) < c_2(n \log n)^{1/2}. \tag{1}$$

Perphas $h(n)$ can be determined exactly but we could not improve (1).

3. Some problems on combinatorial number theory and geometry.

17. Let $x_1, x_2, \ldots, x_n$ be n distinct points, no five on a line. Many years ago I conjectured that the number of distinct lines containing four or more points is $o(n^2)$. In fact, perhaps it is $< cn^{3/2}$. An example of B. Grunbaum shows that if true, $cn^{3/2}$ is best possible except for the value of c.

Purdy and I have the following question: Denote by $f_k(n)$ the maximum number of distinct lines with k or more points. Trivially $f_2(n) = \binom{n}{2}$ and it is well-known that $f_3(n) = \frac{n^2}{6} - cn$ (see the Orchard problem). We would like to prove that

$$\lim f_k(n)/n^2 = c_k$$

exists for every k and determine c_k for $k > 3$.

S. Burr, B. Grunbaum, N.J.A. Sloane, *The orchard problem*, Geometriae Dedicata 2 (1974), 394-429.

18. Denote by $d(x_i; x_j)$ the distance between x_i and x_j. Assume that if two distances differ, they differ by at least one. I conjectured then that

$$D(x_1, \dots, x_n) > cn \tag{1}$$

where $D(x_1, \dots, x_n)$ is the diameter of $x_1, \dots, x_n$. If $x_1, \dots, x_n$ are in three-dimensional space, then probably

$$D(x_1, \dots, x_n) > cn^{2/3} \tag{2}$$

and (2) is best possible. Assume now that our points are in any Euclidean space and that all the $\binom{n}{2}$ distances are different. Then trivially, $D(x_1, \dots, x_n) > (1 + o(1))\binom{n}{2}$, but I conjecture that in fact

$$D(x_1, \dots, x_n) \geq (1 + o(1))n^2. \tag{3}$$

I can prove (3) only if the points are on a line and (3) is best possible. So far I could not even prove (3) if our points are in the plane.

19. Let there be given n points in the plane in general position, i.e., no three on a line and no four on an circle. Let $f(n)$ be the smallest integer so that our points determine at least $f(n)$ distinct distances. In a forthcoming paper we prove

$$f(n) < n \exp(\log n)^{1/2}$$

and we conjecture $f(n) / n \to \infty$.

P. Erdöz, Z. Füredi, J. Pack, I. Ruzsa, *The grid revisted*, to appear.

20. Here is a Pizier type problem formulated recently by Nesetril, Rödl and myself, but first I must explain the terminology. In 1983 at the international congress in Warsaw, Pizier asked me the following problem: A sequence of integers $a_1 < a_i < \dots < a_t$ is called independent if all the sums $\sum_{i=1}^{m} \varepsilon_i a_i, \varepsilon_i = 0$ or 1 are distinct, e. g., the powers of 2 are independent. Pizier wanted a necessary and sufficient condition that an infinite sequence A should be the union of finite number of independent sequences. He conjectured that the following condition is necessary and sufficient: There should be an absolute constant $\varepsilon > 0$ so that every n and every subsequence $a_1, \dots, a_n$ of $A, a_1, a_2, \dots, a_n$ should contain a subsequence of $\varepsilon\ n$ terms which is independent. The condition is clearly necessary; the problem is whether it is sufficient. Pizier conjectured this because he observed that if true, it would imply that his condition on Sidon sets is both necessary and sufficient. Nesetrill, Rödl and I wrote a long paper on Pizier type theorems but we could not decide about Pizier's problem. We write a second paper on this subject and here is a new Pizier type problem which is perhaps interesting: Let $\mathscr{S}$ be an infinite set of points. Assume that there is an absolute constant $\varepsilon > 0$ so that among any n of our points there always are $\varepsilon\ n$ no three on a line. Is it then true that our set is the union of a finite number of sets $\{A_i\}$ where no three points in A_i are on a line? Trivially our condition is necessary. Is it also sufficient? Another Pizier type problem which is stated in our paper is: Let $\mathscr{S}$ be an infinite sequence of integers. Assume that there is an absolute constant ε so that among any n numbers of our sequence there is a subsequence of $\varepsilon . n$ terms which contains no arithmetic progression of three terms. Is it then true that our sequence is the union of a finite number of sequences, no one of which contains an arithmetic progression of three terms? If true, clearly many generalizations are possible.

One of my first serious problems which I posed more than sixty years ago, and which is still open, states as follows: Denote by $f(n)$ the largest set of integers $1 \leq a_1 < a_2 < \ldots < a_k \leq n$ which is independent. I conjectured

$$\max f(n) \leq \frac{\log n}{\log 2} + C \tag{1}$$

for some absolute constant C. I offer five hundred dollars for a proof or disproof. Leo Moser and I proved

$$f(n) < \frac{\log n}{\log 2} + \frac{\log \log n}{2 \log 2} + C \tag{2}$$

and as far as I know (2) is still the best upper bound for $f(n)$. Perhaps $f(n) < \dfrac{\log n}{\log 2} + 3$ and Conway and Guy constructed for infinitely many n sequences satisfying $f(2^n) = n + 2$. My problem seems to have nothing to do with the conjecture of Pizier.

P. Erdös, J. Nesetril and V. Rödl, *On Pizier type problems and results*, Mathematics of Ramsey Theory, Algorithms and Combinatorics 5, Springer Verlag, 214-231.

21. Last year I made the following silly conjecture: Let $a_1 < a_2 < \ldots$ be the sequence of integers of the form $2^\alpha 3^\beta$. Is it true that every integer n can be

written as the sum of distinct a's no one of which divides any other? I mistakenly thought that this was a nice and difficult conjecture but Jansen and several others found a simple proof by induction.
If $n = 2m$, write m in the required form and multiply all the summands by 2. If m is odd, define $3^\beta \leq m < 3^{\beta+1}$ and write $m - 3^\beta = 2n$ as the sum of even a's of the required form.

I then thought to prove that perhaps all integers n can be written in the form

$$(1) \qquad n = b_1 + b_2 + \ldots + b_{\ell'} \quad b_1 < b_2 < \ldots < b_\ell < 2b_1$$

where all the b's are composed of 2, 3 and 5. I could not prove (1). Probably if $n > n_0(\varepsilon)$, n can be written in the form (1) with $b_\ell < (1+\varepsilon)b_1$, but I could prove nothing. Perhaps I again overlooked a trivial argument. If we insist that all the b's are of the form $2^\alpha 3^\beta$, then surely almost all integers cannot be written in the form (1), even if b_{ℓ/b_1} can be a large constant independent of n.

22. A very old problem of mine states as follows: Let $1 \leq a_1 < a_2 < \ldots a_t \leq n$ be a sequence of integers for which no $k+1$ of the a's are pairwise relatively prime. I conjecture that $\max t$ is given if the a's are the multiples of the first k primes. To my great surprise Lovan Khachatrian, currently at the University of Bielefeld, found a counterexample. Probably my conjecture remains true if $n > n_0(k)$, perhaps $n > (1+c)p_k^2$ will suffice. Another related conjecture states as follows: Let $b_1 < b_2 < \ldots < b_t \leq n$ be the sequence of integers, all prime factors of the b's are greater than p_k. Assume further $(b_1, b_j) > 1$. Then we obtain $\max t$ by taking the $b's$ which are multiples of p_{k+1}. After the ingenious counterexampel of Khachatrian, I am no longer sure whether the conjecture is correct.

23. Here is a recent problem of Sárközy and myself: Let $a_1 < a_2 < \ldots < a_k \leq n$ be a sequence of integers. assume that the product a_i, a_j is never squarefree. When is k maximal? Our obvious guess was: k is maximal if the $a's$ are the even numbers and the odd non-squarefree numbers. Perhaps this is easy but we could not prove it.

Assume next that $a_i a_j + 1$ is never squarefree. This is satisfied if $a_i \equiv 7 \pmod{25}$. Is this sequence maximal?

Problem 23 is perhaps not serious mathematics, but I think that problem 24 due to Burr and myself will lead to interesting developments.

24. Divide the squares into two classes in an arbitrary way. Is it true that there is an absolute constant c so that for every $n > n_0$ n can always be written as a sum of distinct squares of the same class. The answer is surely affirmative and n_0 should be determined, but of course the squares can be replaced by k-th powers etc. If we live we hope to investigate these problems further.

REFERENCES

[1] S. Burr, P. Erdös, *A Ramsey type property in additive number theory,* Glasgow Math. J. 27 (1985), pp. 5-10.

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