LE MATEMATICHE
Vol. XLVII (1992) – Fasc. II, pp. 241-258

PATH-DECOMPOSITIONS

KATHERINE HEINRICH

A path-decomposition of a graph is a partition of its edges into subgraphs each of which is a path or a union of paths (a linear forest). We survey known results when the graph and the linear forest are of prescribed types, and when the decomposition satisfies further additional properties.

1. Introduction.

The basic question concerning graph decomposition is: *When can the edges of a given graph be partitioned into a specified collection of subgraphs?* Specifically, let G be a graph and $\mathscr{F} = \{F^1, F^2, \ldots, F^r\}$ be a family of graphs. An $\mathscr{F}$*-decomposition* of G (also called a $(G, \mathscr{F})$*-design*) is a partition of the edge -set of G into the r subgraphs $F^1, F^2, \ldots, F^r$. In the case when all subgraphs in $\mathscr{F}$ are isomorphic to the graph F we will refer to the design as (G, F)*-design*. To date the most far-reaching result is that of Wilson [71] who proved that there is a (K_n, F)-design for all sufficiently large n satisfying $| V(F) | \leq n, n\,(n-1) \equiv 0$ (mod $2 \mid E\,(F) \mid$) and $n-1 \equiv 0$ (mod d), where d is the greatest common divisor of the degrees of the vertices in F. (Note that except for the "size" of n these are precisely the easily determined necessary conditions for the existence of such a design.) Several surveys of $(G, \mathscr{F}))$-designs have been written and the reader is referred to Akiyama and Kano [1], Bermond and Sotteau [10], Chung and Graham [18], Harary and Robinson [26], Rodger [58] and the outstanding book of Bosák [13].

This research was supported in part by the Natural Sciences and Engineering Research Council of Canada under grant A 7829

The particular question we will consider in this paper is: For what G and $\mathscr{F}$, where the subgraphs in $\mathscr{F}$ are linear forests, does a $(G, \mathscr{F})$-design exist? (A *linear forest* is a graph each component of which is a path.) Throughout we will always assume the graph being considered is simple unless it is stated otherwise.

What follows is a list of notation that will be used:

G^λ	The graph G in which every edge has multiplicity λ.
DG	The graph G with each edge replaced by two oppositely directed arcs.
mG	m edge-disjoint copies of G.
$E(G)$	The set of edges in the graph G.
$V(G)$	The set of all vertices in G.
$\chi'(G)$	The edge-chromatic number of G.
$\Delta(G)$	The maximim degree in G.
$\delta(G)$	The minimum degree in G.
$G \cup H$	The edge -disjoint union of the graphs G and H.
K_n	The complete graph on n vertices.
P_k	The path with k vertices and $k-1$ edges.
$\vec{P}_k$	The directed path P_k.
C_k	The cycle with k vertices.
$K_{(m,n)}$	The complete multipartite graph with m parts each of size n.
$K_{a_1,a_2,\ldots,a_m}$	The complete multipartite graph with m parts of sizes $a_1, a_2, \ldots$ $\ldots, a_m$.
Z_m	The cyclic group of order m.

2. (G, F)-designs, where F is a linear forest.

A (G, F)-design when F is a path is usually referred to as a path-decomposition. Certainly the most well known path-decomposition is the (K_{2n}, P_{2n})-design which is easily derived from the $(K_{2n}), C_{2n})$-design (given by Walecki [51, page 162]) on deleting a vertex.

Theorem 2.1. *There exists a (K_{2n}, P_{2n})-design for all values of n.*

Proof. With the vertices of K_{2n} labelled by Z_{2n}, the paths are $F^i = [i, i+1, i+2n-1, i+2, \ldots, i+n+2, i+n-1, i+n+1, i+n]$, $1 \leq i \leq n$.

Many years later Hell and Rosa [30] defined a *balanced graph design* to be a (K_n^λ, F)-design with the additional property that each vertex of K_n^λ lies in the same number of copies of F (hence the term balanced). They focussed their attention on the case when $F \cong P_k$. It is not difficult to show the existence of all (K_n^λ, P_n)-designs (which are necessarily balanced).

Lemma 2.2. *There exists a* (K_{2n+1}^2, P_{2n+1})*-design for all values of* n.

Proof. With the vertices of K_{2n+1}^2 labelled by Z_{2n+1}, the paths are $F^i = [i, i+1, i+2n, i+2, i+2n-1, \ldots, i+n-1, i+n+2, i+n, i+n+1]$, $1 \le i \le 2n+1$.

Corollary 2.3. *Ther exists a balanced* (K_n^λ, P_n)*-design if and only if* λn *is even.*

Partial results on the existence of balanced path-decompositions were obtained by Hell and Rosa [30], Hung and Mendelsohn [40], and Lawless [46, 47]. Soon after complete solutions were independently presented by Huang [36] and Hung and Mendelsohn [41].

Theorem 2.4. *(Huang; Hung and Mendelsohn) There exists a balanced* (K_n^λ, P_k)*-design if and only if*

$$\lambda n\,(n-1) \equiv 0 \pmod{2\,(k-1)}$$

$$\lambda k\,(n-1) \equiv 0 \pmod{2\,(k-1)}$$

$$\lambda\,(n-1) \equiv 0 \pmod{k-1}.$$

Huang [37] also gave necessary and sufficient conditions for the existence of balanced $(K_n^\lambda, P_2 \cup P_3)$-designs, where $P_2 \cup P_3$ is a linear forest.

Others were meanwhile considering the more general question of the existence of (K_n^λ, F)-designs, particularly for small graphs F. When F is the path P_k it is easy to see that the only necessary condition for the existence of the design is $\lambda n\,(n-1) \equiv 0 \pmod{2\,(k-1)}$. The existence of these designs is trivially observed when $k = 2$. We will prove the result for the case $k = 3$ and, although the proof is quite simple, it serves to illustrate the recursive technique that is frequently used.

Theorem 2.5. *There exists a* (K_n^λ, P_3)*-design if and only if* $\lambda n\,(n-1) \equiv 0$ *(mod 4).*

Proof. Only two cases need to be considered:
(a) $\lambda = 1$ and $n \equiv 0, 1 \pmod 4$ and
(b) $\lambda = 2$ and $n \equiv 2, 3 \pmod 4$.

In all constructions $V(K_{2m+1}^\lambda) = Z_{2m+1}$ and $V(K_{2m}^\lambda) = Z_{2m-1} \cup \{\infty\}$. We begin with the small designs.

(K_4, P_3)-design: $F^i = [\infty, i, i+1]$, $1 \leq i \leq 3$
(K_5, P_3)-design: $F^i = [i, i+1, i+3]$, $1 \leq i \leq 5$
(K_6^2, P_3)-design: $F^i = [i, \infty, i+1]$, $F^{i+5} = [i, i+1, i+2]$, $F^{i+10} = [i, i+2, i+4]$, $1 \leq i \leq 5$
(K_7^2, P_3)-design: $F^i = [i, i+1, i+2]$, $F^{i+7} = [i, i+2, i+4]$, $F^{i+14} = [i, i+3, i+6]$, $1 \leq i \leq 7$.

Also note that $C_4 \cong K_{2,2}$ is the union of two paths of lenght two.

(a) If $n = 4t$, write $K_{4t} = tK_4 \cup \binom{t}{2}K_{4,4}$. Since each subgraph has a decomposition into paths of lenght two so too does K_{4t}. In the case $n = 4t+1$ add a new vertex, ∞, and replace each of the $t\,(K_4, P_3)$-designs with a (K_5, P_3)-design on the original four vertices and the vertex ∞.

(b) If $n = 4t+2$, write $K_{4t+2}^2 = (t-1)K_4^2 \cup K_6^2 \cup \binom{t-1}{2}K_{4,4}^2 \cup (t-1)K_{4,6}^2$. Again, as each subgraph has a decomposition into paths of lenght two so too does K_{4t+2}^2. In the case $n = 4t+3$ add a new vertex, ∞, as was done in part (a), noting that a (K_7^2, P_3)-design is also required.

Bermond and Schönheim [9] gave constructions for (K_n, P_4)-designs ($n \equiv 0, 1 \pmod 3$) and Huang and Rosa [39] constructed (K_n, P_k)-designs, $2 \leq k \leq 9$, for all possible values of n. Bermond, Huang, Rosa and Sotteau [8] also constructed all (K_n, P_5)-designs and all $(K_n, P_2 \cup P_3)$-designs. To construct $(K_n, 2P_3)$-designs is a straightforward exercise. In fact using the recursive method illustrated in Theorem 2.5, it is not difficult (although it does take a little time) to construct all (K_n^λ, F)-designs, where F is a linear forest on at most six vertices.

Finally in 1963, Tarsi [65] completely settled the question of the existence of (K_n^λ, P_k)-designs.

Theorem 2.6. *(Tarsi) A (K_n^λ, P_k)-design exists if and only if $\lambda n\,(n-1) \equiv 0$ (mod $2\,(k-1)$) and $n \geq k$.*

His method is quite different from either of the two we have so far described. For $k \leq n-2$, when $\lambda\,(n-1)$ is even, K_n^λ is Eulerian. Tarsi produced an Eulerian walk in which any two occurrences of a particular vertex are separated by at least $k-1$ distinct vertices all different from it. This path can then be "broken" into copies of P_k. When $\lambda\,(n-1)$ is odd a collection of paths of lenght k is first chosen so that no edge occurs in them more than λ times and each vertex of K_n^λ is an endpoint of exactly one of them. When these paths are deleted from the graph what remains is Eulerian and again an appropriate Eulerian walk is described. The case $k = n$ and $k = n-1$ are treated separately (but note Corollary 2.3 provides a solution in the case $k = n$).

Let us now return to the paper of Hell and Rosa. The main goal of that paper was to find all $(K_n^\lambda, \frac{n}{3}P_3)$-designs ($n \equiv 0 \pmod 3$), where $\frac{n}{3}P_3$ is a linear forest with each component a path of lenght 2. The question of the existence of such designs arose from a recreational mathematics problem posed by Dudeney [19, Problem #272] in which he asks the reader to arrange nine schoolboys so that on each of six days they walk in three rows of three and on no occasion does any boy walk twice beside any other. (The naming of the problem as the "handcuffed prisoners problem" is a result of an alternate formulation also given by Dudeney [20].)

To simplify notation, we will refer to a $(K_n^\lambda, (n/k)P_k)$-design (where necessarily $n \equiv 0 \pmod k$), as a *resolvable* (K_n^λ, P_k)-design; meaning that we have a (K_n^λ, P_k)-design with the property that the paths can be partitioned into vertex-disjoint spanning subgraphs of K_n^λ. Each spanning subgraphs is referred to as a *factor*. (Such designs are necessarily balanced).

Hell and Rosa [30] and Nakamura, Kimura and Suganuma [53] obtained partial results on the existence of these designs for general k. The second authors considered in particular the case $k = 3$ and in an unpublished manuscript (in Japanese) they describe an investigation of the general case. Huang and Rosa [38] made a thorough study of the resolvable (K_9, P_3)-designs showing that there are precisely 334 of them which are not isomorphic. Fortunately, this calculation was repeated by Ollerenshaw and Bondi [55] who determined that in fact there are 332 non-isomorphic designs. The case $k = 3$ was finally resolved by Horton [33].

Theorem 2.7. *(Horton) There exists a resolvable (K_n^λ, P_3)-design if and only if $n \equiv 0 \pmod 3$ and $\lambda(n-1) \equiv 0 \pmod 4$.*

In that paper Horton also showed that for n "sufficiently large" the necessary conditions for the existence of a (K_n, P_k)-design (obtained by simple counting) are also sufficient. The case $k \geq 4$ was later resolved by Bermond, Heinrich and Yu [6].

Theorem 2.8. *(Bermond, Heinrich and Yu) There exists a resolvable (K_n^λ, P_k) -design ($k \geq 4$) if and only if $n \equiv 0 \pmod k$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.*

There are two somewhat surprising features of the proof. First, it is not particularly long and second, it does not contain the case $k = 3$. The proof relies on recursive ideas as described earlier and depends critically on the following lemma which appears to be extremely fruitful and which has been of use in other resolvable decomposition problems.

Lemma 2.9. *Let H be a multipartite graph with $V(H) = \cup_{i=1}^k V_i$ so that for $1 \leq i < j \leq k$, the bipartite subgraph on vertex-set $V_i \cup V_j$ with* *bipartition* (V_i, V_j), *is* $t((i,j))$*-regular, where* t *is a mapping from the set* $\{(i,j) : 1 \le i < j \le k\}$ *to the non-negative integers. Let* $G(H)$ *be a graph with* $V(G(H)) = \{1, 2, \ldots, k\}$ *in which the edge* ij *has multiplicity* $t((i,j))$. *There is a resolvable* (H, P_k)*-design, if there is a* $(G(H), P_k)$*-design.*

Proof. Suppose there is a $(G(H), P_k)$-design with paths $P(1), P(2), \ldots, P(m)$, where $m = \sum_{1 \le i \le j \le k} \frac{t((i,j))}{(k-1)}$. Each $P(i)$ yields an appropriate factor of H as follows. To each edge pq of $P(i)$ associate a 1-factor $F_{pq}(i)$ from the bipartite subgraph with vertex-set $V_p \cup V_q$, so that $\cup_{i=1}^{m} F_{pq}(i)$ is the bipartite subgraph on the vertex-set $V_p \cup V_q$ (noting that $F_{pq}(i)$ is empty if pq is not an edge of $P(i)$). Clearly $\cup_{pq \in P(i)} F_{pq}(i)$ is a factor of H, each component of which is a path of lenght $k-1$.

We next consider (G, F)-designs where G is other than a complete graph, and begin with $F \cong P_3$. Such designs were completely determined Kotzig [44] but we will present the proof of Caro and Schönheim [14]. (Note that the existence of a (G, P_3)-design is equivalent to the existence of a one-factor in the line graph of G, a question which had already been considered and resolved (for example, see [31]).)

Theorem 2.10. *(Caro and Schönheim) There is a* (G, P_3)*-design if and only if each component of G has an even number of edges.*

Proof. The necessity of the condition is obvious. The sufficiency is proved using induction on the number of edges in the graph and we need only consider the case when G is connected. If G has only two edges, then $G \cong P_3$. If G is a cycle of even lenght, there is a (G, P_3)-design. If G has no cut-edge and is not a cycle, choose an edge e of G with an adjacent edge f so that $G - \{e, f\}$ is connected and apply the induction hypothesis to obtain a $(G - \{e, f\}, P_3)$-design.

If G has a cut-edge e, then $G - e$ has two components H_1 and H_2, where H_1 has an even number of edges and H_2 an odd number. If H_1 is not a single vertex, then by induction we have an (H_1, P_3)-design and an $(H_2 \cup e, P_3)$-design. If H_1 is a single vertex, let f be an edge of H_2 which is adjacent with e. Then if f is not a cut-edge of H_2 we have an $(H_2 - f, P_3)$-design and $ef \cong P_3$. If f is a cut-edge, then $H_2 - f \cong H_3 \cup H_4$ and we either have an (H_3, P_3)-design and an (H_4, P_3)-design which together with the path ef give the decomposition, or an $(H_3 \cup e, P_3)$-design and a $(H_4 \cup f, P_3)$-design.

Corollary 2.11. *There is a* (K_n^λ, P_3)*-design if and only if* $\lambda n(n-1) \equiv 0$ *(mod 4).*

Corollary 2.12. *There is a* $(K_{n,m}^\lambda, P_3)$*-design if and only if* λmn *is even and* $mn \ge 2$.

Truszczyński [67] considered the question of the existence of $(K^{\lambda}_{n,m}, P_k)$-designs.

Theorem 2.13. *(Truszczyński) If $m \geq n$ and either λ is even or m and n are even, there is a $(K^{\lambda}_{m,n}, P_k)$-design if and only if $\lambda mn \equiv 0$ (mod $k-1$), $m \geq \lceil \frac{k}{2} \rceil$ and $n \geq \lceil \frac{k-1}{2} \rceil$.*

In that paper the case when λ and at least one of m and n are odd was also studied. In particular Truszczyński showed (using a simple counting argument) that if λ is odd, $m \geq n$, and either: m and k are even, n is odd and $k-1 > \lambda n$; m is odd, n and k are even and $k-1 > \lambda m$; or m and n are odd, and $k-1 > \lambda n$, then there is no $(K^{\lambda}_{m,n}, P_k)$-design. This discovery led him to conjecture that except in these cases there exists a $(K^{\lambda}_{m,n}, P_k)$-design if and only if $\lambda mn \equiv 0$ (mod $k-1$), $m \geq \lceil \frac{k}{2} \rceil$ and $n \geq \lceil \frac{k-1}{2} \rceil$.

Ushio [69] and Ushio and Tsuruno [70] studied resolvable $(K_{a_1,a_2,\ldots a_m}, P_k)$ -designs.

Theorem 2.14. *(Ushio) There exists a resolvable $(K_{m,n}, P_3)$-design if and only if $m+n \equiv 0$ (mod 3), $m \leq 2n \leq 4m$ and $3mn \equiv 0$ (mod $2(m+n)$).*

Theorem 2.15. *(Uschio and Tsuruno) There exists a resolvable $(K^{\lambda}_{(m,n)}, P_3)$-design if and only if $mn \equiv 0$ (mod 3) and $\lambda(m-1)n \equiv 0$ (mod 4).*

Some time later Yu [72] substantially extend this work.

Theorem 2.16. *(Yu) For $k \geq 4$, there exists a $(K^{\lambda}_{(m,n)}, P_k)$-design if $mn \equiv 0$ (mod k), $\lambda(m-1)kn \equiv 0$ (mod $2(k-1)$) and either $n \equiv 0$ (mod k) or $m \equiv 0$ (mod k).*

Corollary 2.17. *If k is a prime, there exists a $(K^{\lambda}_{(m,n)}, P_k)$-design if and only if $mn \equiv 0$ (mod k) and $\lambda(m-1)kn \equiv 0$ (mod $2(k-1)$).*

Using the above results Yu is also able to show that the necessary conditions are sufficient when $m = 2, 3$.

Corollary 2.18. *(Yu) (a) There exists a $(K^{\lambda}_{n,n}, P_k)$-design if and only if $2n \equiv 0$ (mod k) and $\lambda kn \equiv 0$ (mod $2(k-1)$).*
(b) There is a $(K^{\lambda}_{n,n,n}, P_k)$-design if and only if $3n \equiv 0$ (mod k) and $\lambda n \equiv 0$ (mod $k-1$).

Caro and Schönheim [14] gave necessary and sufficient conditions for the existence of a (T, P_k)-design, where T is a tree.

We now turn our attention to the existence of (K^{λ}_n, F)-designs where F is a linear forest other than a path or the disjoint spanning union of isomorphic paths and begin by considering (G, F)-designs. Bialostocki and Roditty [11] gave necessary and sufficient conditions for a $(G, 3P_2)$-design showing that with a finite number of exceptions, the design exists if and only if the number of edges in G is divisible by 3 and $\Delta(G) \leq \frac{|E(G)|}{3}$. This result was improved substantially by Alon [2].

Theorem 2.19. *(Alon) There exists a (G, tP_2)-design if and only if $|E(G)| \equiv 0 \pmod t$ and $t\chi'(G) \leq |E(G)|$.*

Proof. The necessity is obvious. For the sufficiency, first observe that if in some graph there are two disjoint matchings of sizes m and n, where $m > n$, then there exist disjoint matchings of sizes $m - 1$ and $n + 1$. Since, G has an edge-partitioning into $\chi'(G)$ matchings, applying the observation to pairs of matchings yields the result.

Corollary 2.20. *There exists a (K_n^λ, tP_2)-design if and only if $\lambda n(n-1) \equiv 0 \pmod{2t}$ and $t \leq \lfloor \frac{n}{2} \rfloor$.*

For the case $\lambda = 1$ this result was proven much earlier by Folkman and Fulkerson [23]. The design is *cyclic* if there is an n-cycle σ which permutes the vertices and also maps the set of all matchings onto itself. Recently Rees [56] has shown that in fact it is always possible to find cyclic (K_n^λ, tP_2)-designs when $t < \frac{n}{2}$. The case when $t = \frac{n}{2}$ (n even) was dealt with earlier by Hartman and Rosa [27] who showed the designs exist except when n is at least 8 and a power of 2, in which case they do not exist.

Alon's result also settles the question for complete bipartite graphs.

Corollary 2.21. *There exists a $(K_{n,m}^\lambda, tP_2)$-design if and only if $\lambda nm \equiv 0 \pmod t$ and $t \leq \min\{m, n\}$.*

Favaron, Lonc and Truszczyński [21] considered other subgraphs; in particular they considered the linear forest $P_2 \cup P_3$.

Theorem 2.22. *(Favaron, Lonc and Truszczyński) There exists a $(G, P_2 \cup P_3)$-design if and only if $|E(G)| \equiv 0 \pmod 3$, $\Delta(G) \leq \frac{2}{3}|E(G)|, c(G) \leq \frac{|E(G)|}{3}$ (where $c(G)$ is the number of components of G with an odd number of edges), the edges of G cannot be covered by two adjacent vertices and G is not one of six specified graphs.*

Corollary 2.23. *There exists a $(K_n^\lambda, P_2 \cup P_3)$-design if and only if $n \geq 5$ and $\lambda n(n-1) \equiv 0 \pmod 6$.*

Kotzig [45] claimed that if G is a 3-regular graph, there exists a (G, P_4)-design if and only if G has a 1-factor. (To see this observe that the number of paths of length 3 is the size of a 1-factor.) He asked for necessary and sufficient conditions for a (G, P_{2n+2})-design when G is $(2n+1)$-regular. The only result we have along these lines is that of Jacobson, Truszcyński and Tuza [42]. (We do note, however, that Jünger, Reinelt and Pulleyblank [43] have asked if there exists a (G, P_4)-design whenever G is a simple planar 2-edge connected bipartite graph satisfying $4|\,E(G)| \equiv 0 \pmod 3$.)

Theorem 2.24. *(Jacobson, Truszcyński and Tuza) There exists a (G, P_5)-design for every 4-regular bipartite graph G.*

Closely related to this is the work of Bondy [12] on perfect path double covers of graphs. He conjectured that for every graph G there is a decomposition of G^2 into paths in which every vertex is the end-vertex of exactly two paths (paths of length zero are assumed to have two identical end-vertices). This conjecture was proven by Li [48]. Bondy further defined a k-regular path double cover of a graph G which in our terminology is a (G^2, P_k)-design, calling it *perfect* if every vertex is the end-vertex of exactly two paths. He asked for which G and k such designs exist.

Theorem 2.25. *(Bondy) If G is a connected simple graph, there exists a (G^2, P_3)-design if and only if G is not a tree with a 1-factor.*

Theorem 2.26. *(Bondy) If G is a connected simple graph, there exists a perfect (G^2, P_3)-design if and only if G is unicyclic and each vertex of degree 2 lies on the cycle.*

Bondy extended Kotzig's conjecture by conjecturing that if G is a simple k-regular graph, then there exists a perfect (G^2, P_{k+1})-design. In support of this conjecture he proved its validity in the case $k = 3$.

Turning now to other linear forests, Ruiz [63] has shown that there is a (K_{2n}, F)-design for any linear forest F with n edges. His result is proven by choosing the "right" set of edges from the path F^1 (as described in Walecki's construction given in Theorem 2.1) to yield a subgraph isomorphic to F in which the differences on each edge are distinct, and then using it to generate the design. The work of Ruiz was generalized by Martinova [52] who used the same idea to show that there is a $(K_{n,n}, F)$-design where F is any linear forest on n edges.

3. $(G, \mathscr{F})$-designs, where each $F \in \mathscr{F}$ is a linear forest.

When Tarsi [65] obtained his results on (K_n^λ, P_k)-designs, he was easily able to extend the work to address the following more general question:
Given a collection of paths with the property that each has at most n vertices

and the sum of the lengths is $\binom{n}{2}$*, is there a decomposition of* K_n *into precisely these paths?*

This question (in the case $\lambda = 1$) was also posed by Slater [64] who called a graph G *path-arboreal* if for a multiset $\{a_i : 1 \leq i \leq r\}$ of r positive integers satisfying $2 \leq a_i \leq |V(G)|$ and $\sum_{i=1}^{r}(a_i - 1) = |E(G)|$, there exists a $(G, \mathscr{F})$-design, where $\mathscr{F} = \{P_{a_i} : 1 \leq i \leq r\}$.

Theorem 3.1. *(Tarsi) If* $(n-1)\lambda$ *is even, and* $\{a_i : 1 \leq i \leq r\}$ *is a multiset of* r *positive integers satisfying* $2 \leq a_i \leq n-2$ *and* $\sum_{i=1}^{r}(a_i - 1) = \lambda\binom{n}{2}$*, then there exists a* $(K_n^\lambda, \mathscr{F})$*-design, where* $\mathscr{F} = \{P_{a_i} : 1 \leq i \leq r\}$.

Note that this theorem not only imposes conditions on the parity of λ and n but restricts the lengths of the paths. Improvements were obtained by Ng [54], who considered only the case $\lambda = 1$ and n odd. His main result allows all but Hamilton paths, although he did under certain conditions also construct designs in which some of the paths were Hamiltonian.

Theorem 3.2. *(Ng) If* n *is odd and* $\{a_i : 1 \leq i \leq r\}$ *is a multiset of* r *positive integers satisfying* $2 \leq a_i \leq n-1$ *and* $\sum_{i=1}^{r}(a_i - 1) = \binom{n}{2}$*, then there exists a* $(K_n, \mathscr{F})$*-design, where* $\mathscr{F} = \{P_{a_i} : 1 \leq i \leq r\}$.

Truszczyński [67] investigated the question: *Is* $K_{m,n}^\lambda$ *path-arboreal?*

Theorem 3.3. *(Truszczyński) If* $\{a_i : 1 \leq i \leq r\}$ *is a multiset of* r *positive integers satisfying* $a_i \geq 2, \sum_{i=1}^{r}(a_i - 1) = \lambda mn$*, and either*

1. $m > n$ *and* $a_i \leq 2n, i = 1, 2, \ldots, r$,
2. $m = n = 2$ *and* $a_i \leq 4, i = 1, 2, \ldots, r$, *or*
3. $m = n \geq 2$ *and* $a_i \leq 2n - 4, i = 1, 2, \ldots, r$, *then there exists a* $(K_{m,n}^\lambda, \mathscr{F})$*-design, where* $\mathscr{F} = \{P_{a_i} : 1 \leq i \leq r\}$.

Fink and Straight [22] define a graph G to be *path-perfect* if there exists a $(G, \mathscr{F})$-design, where $|E(G)| = \binom{n}{2}$ and $\mathscr{F} = \{P_2, P_3, \ldots, P_n\}$. The first result they state is not difficult to verify.

Theorem 3.4. *The complete graph* $K_n, n \geq 2$*, is path-perfect.*

Proof. When n is odd, there is a (K_n, C_n)-design (Lucas [51]) and $C_n \cong P_{k+1} \cup P_{n-k+1}, 1 \leq k \leq \frac{n-1}{2}$. When n is even, there is a (K_n, P_n)-design (Theorem 2.1) and $P_n \cong P_{k+1} \cup P_{n-k}, 1 \leq k \leq \frac{n}{2} - 1$.

In that paper they also claimed that both $K_{r,2r-1}$ and $K_{r,2r+1}$ are path-perfect, and considered the existence od regular path-perfect graphs. Considerable progress on the question of path-perfect complete bipartite graphs has been made independently by Truszczyński [67] and Chilakamurri and Hamburger [16], although only in the case when s is even.

Theorem 3.5. *(Chilakamurri and Hamburger; Truszczyński) If* $s > t$ *and* s *is even,* $K_{s,t}$ *is path-perfect if and only if* $2st = n(n+1)$ *for some integer* n *and* $2t \geq n$.

More generally, Zaks and Liu [74] obtained results on decomposing complete bipartite graphs into r paths, one of each even length from 2 to $2r$, and one of each odd length from 1 to $2r - 1$.

Theorem 3.6. *(Zaks and Liu) There exists a* $(K_{n,n}, \{P_2, P_4, \ldots, P_{2n}\})$*-design for all* n.

Theorem 3.7. *(Zaks and Liu) There exists a* $(K_{n,n+1}, \{P_3, P_5, \ldots, P_{2n+1}\})$*-design for all odd* n

This latter result was extended by Chilakamurri [15] who showed that these designs also exist for all even n except $n = 2$, in which case the design does not exist.

Theorem 3.8. *(Chilakamurri) There exists a* $(K_{n,n+1}, \{P_3, P_5, \ldots, P_{2n+1}\})$*-design for all even* n *except* $n = 2$.

By being less restrictive on the linear forests in $\mathscr{F}$ (that is, by saying only that the subgraphs in the decomposition must be members of a set $\mathscr{F}$ of linear forest) Lonc [50], Favaron, Lonc and Truszczyński [21] and Truszczyński [68] have obtained several results.

Theorem 3.9. *(Lonc) If* $gcd(k,t) = 1$*, there exists an integer* $c = c(k,t)$ *so that for every graph G satisfying* $\delta(G) \geq c$*, there is a decomposition of G into paths, each of length* k *or* t.

Theorem 3.10. *(Favaron, Lonc and Truszczyński) If G is a connected graph and* $G \ncong K_3$*, then there is a decomposition of G into paths of lengths 2 and 3 if and only if G is not a tree with all vertices of odd degree.*

Theorem 3.11. *(Truszczyński) Every sufficiently large graph G satisfying* $|E(G)| \equiv 0 \pmod{k}$ *and* $\Delta(G) \leq \frac{2}{k}|E(G)|$ *has a decomposition into linear forests each containing precisely* k *edges.*

Note that these results do not specify the number of linear forests of each type. The only substantial result we have in which there are at least two types and the number of each is specified is due to Bermond, Heinrich and Yu [7] and concerns resolvable designs.

Theorem 3.12. *(Bermond, Heinrich and Yu) There is a resolvable decomposition of* K_n^λ *into* s *forests each isomorphic to* $\frac{n}{2}P_2$ *and* t *forests each isomorphic* *to* $\frac{n}{k}P_k$ *when* $st \neq 0$, *if and only if* $n \equiv 0 \pmod 2, n \equiv 0 \pmod k$ *and* $ks + 2t(k-1) = \lambda k(n-1)$.

4. Related questions.

In this section we very briefly mention an assortment of somewhat related questions. We begin with directed path-designs.

The only directed (G, F)-designs we will describe are those in which G is a symmetric directed graph (that is, G contains the arc (a, b) exactly when it contains the arc (b, a)). The first result is an immediate corollary to Theorem 2.1.

Theorem 4.1. *There exists a* $(DK_n, \vec{P}_n)$*-design when* n *is even.*

The case n odd is considerably more difficult and it was not until 1980 that Tillson [66] constructed examples of such designs. (The question had been raised by Bermond and Faber [5] who attributed it to E.G. Strauss.)

Theorem 4.2. *(Tillson) There exists a* $(DK_n, \vec{P}_n)$*-design when* n *is odd and* $n \neq 3, 5$, *in which cases the designs do not exist.*

The existence of $(DK_n, \vec{P}_k)$-designs has not otherwise been considered except in the case of resolvable $(DK_n, \vec{P}_k)$-designs. Yu [73] has completely settled the question of their existence provided $k \neq 3, 5$. (And it is no accident that we have the same constraint on path length as in the previous theorem.)

Theorem 4.3. *(Yu) There exists a resolvable* $(DK_n^\lambda, \vec{P}_k)$*-design,* $k \neq 3, 5$, *exactly when* $n \equiv 0 \pmod k$ *and* $n \equiv 1 \pmod{k-1}$.

Truseczyński [67] also studied directed designs.

Theorem 4.4. *(Truszczyński) If* $m \geq n$, *there exist a* $(K_{m,n}^\lambda, \vec{P}_k)$*-design if and only if* $2\lambda mn \equiv 0 \pmod{k-1}, m \geq \lceil \frac{k}{2} \rceil$ *and* $n \geq \lceil \frac{k-1}{2} \rceil$.

Heinrich and Nonay [29] considered the question of the existence of (K_n^2, P_n)-designs with the property that any two of the paths have exactly one edge in common. Designs for $n \leq 20$ (but $n \neq 4, 7$, in which case the designs do not exist) were given and recursive constructions described. The construction was later improved by Horton and Nonay [35] who described a more general "product" technique. (We remark that such designs were used as a tool in partially answering a question of Hering [32] who asked for (K_n^2, C_{n-1})-designs with this intersection property.) An analogous question can be asked in the directed case where we ask for $(DK_n, \vec{P}_k)$-designs in which any two paths have exactly one oppositely directed arc in common (see [29]).

A (G, F)-design is *self-complementary* if F is a self-complementary graph (that is, if $K_{|V(F)|}$ has a decomposition into exactly two copies of F) and if on replacing each copy of F in the (G, F)-design by its complement, the result is another (G, F)-design. (For a survey of self-complementary graph decompositions see [59].) The only self-complementary linear forest is the path P_4. Granville, Moisiadis and Rees [24] constructed self-complementary (K_n, P_4)-designs, showing that they exist exactly when $n \equiv 1 \pmod 3$.

Alspach [3] showed that the collection of all paths P_3 with vertices from an n-set can be partitioned into copies of K_n if and only if $n \equiv 0, 1$ (mod 4). The analogous result for directed paths of length 2 was obtained by Heinrich [28].

Horton [34] constructed a (K_{2n}, P_{2n})-design $\mathscr{A}$ with an associated (K_{2n}, nP_2)-design $\mathscr{B}$ with the property that each nP_2(1-factor) in the decomposition $\mathscr{B}$ has exactly one edge from each of the n paths in the decomposition $\mathscr{A}$ for all values of n satisfying $gcd\,(n, 30) = 1$.

Li [49] reports on progress made concerning the question of classifying all graphs with the property that their edges can be decomposed into paths each of length at least 3 (the question being attributed to W. R. Pulleyblank).

One variant of the "basic question" stated in Section 1 is obtained by either allowing some edges of G not to be used in the decomposition or by requiring that all edges be used but allowing some to be used more than once. More specifically, let G and F be graphs and define $p\,(G, F)$ to be the maximum number of edge-disjoint copies of F in G, and $c\,(G, F)$ to be the minimum number of copies of F required to contain every edge of G at least once. The first of these is called the *packing number* and the second the *covering number*. (There have been many papers on determining these numbers and the reader is referred to the (now somewhat old) survey articles by Beineke [4] and Harary [25].) The only results specifically on linear forests are due to Roditty [60, 61, 62] who determined the packing and coverings numbers in all cases when $G \cong K_n$ and F is any linear forest on at most 5 vertices or F is the path of lenght 5. (He also obtained results for several other forests.) We note that using some of the results od Section 2 and the recursive construction technique it is not difficult to show that $p\,(K_n, F) = \lfloor \frac{\binom{n}{2}}{|E(F)|} \rfloor$ and $c\,(K_n, F) = \lceil \frac{\binom{n}{2}}{|E(F)|} \rceil$, where F is any linear forest on at most 6 vertices.

M. L. Yu (personal communication) has recently asked the following question. Colour the edges of P_{2k+1} red and blue so that there are k edges of each colour, and colour the edges of K_n^2 red and blue so that there is a red and a blue copy of K_n. If $n\,(n-1) \equiv 0 \pmod{2k}$, is there a decomposition of the coloured complete graph into copies of the coloured path?

Finally there has been much study of path numbers, path-covering numbers and linear arboricity. The *linear arboricity* of a graph G is the minimum number of linear forests required to decompose the graph. Harary [25] conjectured that for any graph G the linear arboricity is between $\lceil \frac{\delta(G)}{2} \rceil$ and $\lceil \frac{\delta(G)+1}{2} \rceil$. The *path number* of G is the minimum number of paths required to partition $E(G)$, and the *path-covering number* is the minimum number of paths necessary to cover the edges of G. In 1960, Gallai conjectured by Chung [17] that the path-covering number is also at most $\lceil \frac{n}{2} \rceil$. These questions are of quite a different flavour than those we have so far considered and for that reason we will leave the reader to consult appropriate source. (Similar numbers have also been investigated for directed graphs and tournements.)

REFERENCES

[1] J. Akiyama and M. Kano, *Factors and factorizations of graphs - a survey,* J. Graph Theory 9 (1985), pp. 1-42.

[2] N. Alon, *A note on the decomposition of graphs into isomorphic matchings,* Acta Math. Hung. 42 (1983), pp. 221-223.

[3] B. Alspach, *A 1-factorization of the line graphs of complete graphs,* J. Graph Theory 6 (1982), pp. 441-445.

[4] L.W. Beineke, *A survey of packing and covering of graphs,* The Many Faces of Graph Theory, ed. G. Chartrand and S.A. Kapoor, Berlin, Heidelberg, New York, 1969.

[5] J.C. Bermond and V. Faber, *Decompositions of the complete directed graph into k-circuits,* J. Combinatorial Theory B 21 (1976), pp. 146-155.

[6] J.C. Bermond, K. Heinrich and M.L. Yu, *Existence of resolvable path designs,* Europ. J. Combinatorics 11 (1990), pp. 205-211.

[7] J.C. Bermond, K. Heinrich and M.L. Yu, *On resolvable mixed path designs,* Europ. J. Combinatorics 11 (1990), pp. 313-318.

[8] J.C. Bermond, C. Huang, A. Rosa and D. Sotteau, *Decompositions of complete graphs into isomorphic subgraphs with five vertices,* Ars Combinatoria 10 (1980), pp. 212-254.

[9] J.C. Bermond and J. Schönheim, *G-decompositions of K_n where G has four vertices or less,* Discrete Math. 19 (1977), pp. 113-120.

[10] J.C. Bermond and D. Sotteau, *Graph decompositions and G-designs,* Proc. 5th British Combinatorial Conf. (1975), pp. 53-72.

[11] A. Bialostocki and Y. Roditty, *$3K_2$-decompositions of a graph,* Acta Math. Acad. Sci. Hungar. 40 (1982), pp. 201-208.

[12] J.A. Bondy, *Perfect path double covers of graphs,* J. Graph Theory 14 (1990), pp. 259-272.

[13] J. Bosák, *Decompositions of Graphs,* Kluwer Academic, Dordrecht, 1990.

[14] Y. Caro and J. Schönheim, *Decompositions of trees into isomorphic subtrees,* Ars Combinatoria 9 (1980), pp. 119-130.

[15] K.B. Chilakamarri, *Decompositions of bipartite graphs into paths,* Amer. Math. Monthly 95 (1988), pp. 634-636.

[16] K.B. Chilakamarri and P. Hamburger, *A note on packing paths into complete bipartite graphs,* Bulletin of the Institute of Combinatorics and its Applications 4 (1992), pp. 32-34.

[17] F.R.K. Chung, *On the coverings of graphs,* Discrete Math. 30 (1980), pp. 89-93.

[18] F.R.K. Chung and R.L. Graham, *Recent results in graph decompositions, in: H.N.V. Temperley, ed.,* Combinatorics, London Mathematical Society Lecture Notes Series 52 (Cambridge University Press, Cambridge, 1981), pp. 103-124.

[19] H.E. Dudeney, *Amusements in Mathematics,* Nelson and Sons, 1917 (reprinted by Dover Publ., NewYork, 1958).

[20] H.E. Dudeney, *536 Puzzles and Curious Problems,* Charles Scribner's Sons, New York, 1967.

[21] O. Favaron, Z. Lonc and M. Truszczyński, *Decomposition of graphs into graphs with three edges,* Ars Combinatoria 20 (1985), pp. 125-146.

[22] J.F. Fink and H.J. Straight, *A note on path-perfect graphs,* Discrete Mathematics 33 (1981), pp. 95-98.

[23] J. Folkman and D.R. Fulkerson, *Edge colourings in bipartite graphs,* in Combinatorial Theory and its Applications, eds. R. Bose and T. Dowling, Univ. N. Carolina Press, Chapel Hill, 1969, pp. 561-577.

[24] A. Granville, A. Moisiadis and R. Rees, *On complementary decompositions of the complete graph,* Graphs and Combinatorics 5 (1989), pp. 57-61.

[25] F. Harary, *Covering and Packing in Graphs I,* Ann. N.Y. Acad. Sci. 175 (1970), pp. 198-205.

[26] F. Harary and R. W. Robinson, *Isomorphic factorizations X: Unsolved problems,* J. Graph Theory 9 (1985), pp. 67-86.

[27] A. Hartman and A. Rosa, *Cyclic one-factorizations of the complete graph,* Europ. J. Comb. 6 (1985), pp. 45-48.

[28] K. Heinrich, *Partitioning oriented 2-paths,* J. Graph Theory 7 (1983), pp. 359-367.

[29] K. Heinrich and G. Nonay, *Path and cycle decompositions of complete multigraphs,* Ann. Discrete Math. 27 (1985), pp. 275-286.

[30] P. Hell and A. Rosa, *Graph decompositions, Handcuffed prisoners and balanced P-designs,* Discrete Math. 2 (1972), pp. 229-252.

[31] R.L. Hemminger and L.W. Beineke, *Line graphs and line-digraphs,* Selected topics in Graph Theory edited by: L.W. Beineke and R.J. Wilson, Academic Press, London, 1978, pp. 271-305.

[32] F. Hering, *Block designs with cyclic block structure,* Ann. Discrete Math. 6 (1980), pp. 201-214.

[33] J.D. Horton, *Resolvable path designs,* J. Combinatorial Theory A 39 (1985), pp. 117-131.

[34] J.D. Horton, *Hamilton path tournament design,* Ars Combinatoria 27 (1989), pp. 69-74.

[35] J.D. Horton and G. M. Nonay, *Self-orthogonal Hamilton path decompositions,* Discrete Math. 97 (1991), pp. 251-264.

[36] C. Huang, *On handcuffed designs,* Dept. of C. and O. Research Report CORR75 - 10, University of Waterloo.

[37] C. Huang, *Balanced graph designs on small graphs,* Utilitas Mathematica 10 (1976), pp. 77-108.

[38] C. Huang and A. Rosa, *Complete classification of the solution to the problem of 9 prisoners,* Procedings of the Twenty-Fifty Summer Meeting of the Canadian Mathematical Congress (1971), pp. 553-562.

[39] C.Huang and A. Rosa, *Decomposition of complete graphs into trees,* Ars Combinatoria 5 (1978), pp. 23-63.

[40] S.H.Y. Hung and N.S. Mendelsohn, *Handcuffed designs,* Aequationes Math. 11 (1974), pp. 256-266.

[41] S.H.Y. Hung and N.S. Mendelsohn, *Handcuffed designs,* Discrete Math. 18 (1977), pp. 23-33.

[42] M.S. Jacobson, M. Truszczyński and Z. Tuza, *Decomposition of regular bipartite graphs,* Discrete Math. 89 (1991), pp. 17-27.

[43] M. Jünger, G. Reinelt and W.R. Pulleyblank, *On partitioning the edges of graphs into connected subgraphs,* J. Graph Theory 9 (1985), pp. 539-549.

[44] A. Kotzig, *From the theory of finite regular graphs of degree three and four,* Časopis Pěst. Mat. 82 (1957), pp. 76-92 (in Czech).

[45] A. Kotzig, *Problem #2 Proceding of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing,* Congressus Numerantium XXIV (1979), pp. 913-915.

[46] J.F. Lawless, *On the construction of handcuffed designs,* J. Combinatorial Theory A 16 (1974), pp. 76-86.

[47] J.F. Lawless, *Further results concerning the existence of handcuffed designs,* Aequationes Math. 11 (1974), pp. 97-106.

[48] H. Li, *Perfect path double covers in every simple graph,* J. Graph Theory 14 (1990), pp. 645-650.

[49] W. Li, *Graphs decomposable into paths of lengths at least three,* Ars Combinatoria 17 (1984), pp. 27-32.

[50] Z. Lonc, *Decompositions of graphs into trees,* J. Graph Theory 13 (1989), pp. 393-403.

[51] E. Lucas, *Récréations Mathématiques,* Volume II, Gauthiers-Villars, Paris, 1884: (republished by Librairie Scientifique et Technique, 1960).

[52] M. Martinova, *Isomorphic decompositions of complete multipartite graphs into linear forests,* Ars Combinatoria 24 (1987), pp. 193-198.

[53] G. Nakamura, H. Kimura and H. Suganuma, *On the handcuffed prisoner problem and the resolvable balanced path design,* Surikagaku October (1981), pp. 108-112.

[54] K.W.P. Ng, *On Path decompositions of Complete Graphs,* M. Sc. Thesis, Simon Fraser University, 1985.

[55] Dame Kathleen, Ollerenshaw and Sir Hermenn Bondi, *The nine prisoners problem,* Bull. Inst. Math. Appl. 14 (1978), No. 5-6, pp. 121-143.

[56] R. Rees, *Cyclic (0,1)-factorizations of the complete graph,* J. Comb. Math. Comb. Comp. 4 (1988), pp. 23-28.

[57] R. Rees and C.A. Rodger, *Subdesigns in complementary path decompositions and incomplete two-fold designs with block size four,* Ars Combinatoria, to appear.

[58] C.A. Rodger, *Graph decompositions,* Le Matematiche XLV (1990), pp. 119-140.

[59] C.A. Rodger, *Self-complementary graph decompositions,* preprint.

[60] Y. Roddity, *Packing and covering of the complete graph with a graph G of four vertices or less,* J. Combinatorial Theory A 34 (1983), pp. 231-243.

[61] Y. Roditty, *The packing and covering of the complete graph I: The forests of order five,* Internat. J. Math. Sci. 9 (1986), pp. 277-282.

[62] Y. Roditty, *The packing and covering of the complete graph II: The trees of order six,* Ars Combinatoria 19 (1985), pp. 81-94.

[63] S. Ruiz, *Isomorphic decompositions of complete graphs into linear forests,* J. Graph Theory 8 (1985), pp. 189-191.

[64] P. Slater, *Problem 52,* Discrete Math. 47 (1983), 134.

[65] M. Tarsi, *Decompositions of a complete multigraph into simple paths: Non balanced handcuffed designs,* J. Combinatorial Theory A 34 (1983), pp. 60-70.

[66] T. Tillson, *A Hamiltonian decomposition of* $K^*_{2m}, 2m \geq 8$, J. Combinatorial Theory B 29 (1980), pp. 68-74.

[67] M. Truszczyński, *Note on the decomposition of* $\lambda K_{m,n}(\lambda K^*_{m,n})$ *into paths,* Discrete Math. 55 (1985), pp. 89-96.

[68] M.Truszczyński, *Decomposition of graphs into forests of equal size and with bounded maximum degrees,* Ars Combinatoria 21 (1986), pp. 201-215.

[69] K. Ushio, P_3*-factorization of complete bipartite graphs,* Discrete Math. 72 (1988), pp. 361-366.

[70] K. Ushio and R. Tsuruno, *P_3-factorization of complete multipartite graphs,* Graphs and Combinatorics 5 (1989), pp. 385-387.

[71] R.M. Wilson, *Decomposition of complete graphs into subgraphs isomorphic to a given graph,* Proc. 5th British Combinatorial Conf. (1975), pp. 647-659.

[72] M.L. Yu, *On path factorization of complete multipartite graphs,* Discrete Math. (to appear).

[73] M.L. Yu, *Resolvable tree designs,* J. Combinatorial theory A (to appear).

[74] S. Zaks and C.L. Liu, *Decomposition of graphs into trees,* Congressus Numerantum XIX (1977), pp. 643-654.

Department of Mathematics and Statistics
Simon Fraser University
Burnaby BC V5A 1S6