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KINEMATIC STRUCTURES OF GENERALIZED HYPERBOLIC SPACES

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1. Introduction.

Since the genesis of analytic geometry in the 17. century, initiated by R. Descartes and P. De Fermat, there are two general frames in order to introduce a geometry, a synthetic by assuming certain geometric axioms, and an analytic by deriving a geometry from a suitable algebraic structure. Both frames bear possibilities to extend the notion of a certain class of geometries by omitting certain axiomatic assumptions or by taking a more comprehensive class of algebraic structures.
In this century the notion "absolute plane" was heavily extended mainly by grouptheoretical systems of axioms basing on the elementary concept of reflections (cf. e.g. [1], [5], [12], [15]). It is known that the group B of proper motions of any absolute plane $\mathfrak{A}$ can be turned into a *kinematic space*, i.e. B can be provided with an incidence structure $\mathfrak{G}$ such that $(B, \mathfrak{G})$ is a (linear) incidence space with the properties (cf. [13], [5], [6]):

1. $\forall a \in B, \quad \forall X \in \mathfrak{G} : aX,\ Xa \in \mathfrak{G}$
2. $\forall X \in \mathfrak{G}$ with $1 \in X$, X is a subgroup of B.

Then the set $\mathfrak{F} := \{X \in \mathfrak{G} \mid 1 \in X\}$ forms a *kinematic fibration* (= partition) of the group B, i.e. $\mathfrak{F}$ is a set of proper subgroups of B with

(F1) $\quad \cup\mathfrak{F} = B.$
(F2) $\quad \forall X, Y \in \mathfrak{F}$ with $X \neq Y : \ X \cap Y = \{1\},$
(F3) $\quad \forall X \in \mathfrak{F}, \forall b \in B : bXb^{-1} \in \mathfrak{F}.$

(If $\mathfrak{F}$ fulfills (F1), (F3) and the weaker assumption
(F2') There is a subgroup D of B such that: $\forall X, Y \in \mathfrak{F}$ with $X \neq Y : X \cap Y = D$ then $\mathfrak{F}$ is called a *kinematic cover*).

The kinematic space $(B, \mathfrak{G})$ is completely determined by the kinematic fibration $\mathfrak{F}$, since $\mathfrak{G} = \{aF \mid F \in \mathfrak{F},\ a \in B\}$.
In the case of an absolute plane $\mathfrak{U} = (/A, \mathfrak{L}, \equiv)$ (/A resp. $\mathfrak{P}$ denotes the set of points resp. lines and $\equiv$ the congruence relation) the stabilizer B_p of any point $p \in /A$ is a commutative subgroup of B and the set $\mathfrak{F}' := \{B_p \mid p \in /A\}$ of all stabilizers has the properties (F2) and (F3). $\mathfrak{F}'$ can be extended to a kinematic fibration $\mathfrak{F}$ of the motion group B. Each absolute plane $\mathfrak{U}$ can be embedded into its kinematic space by considering the boundle space of a point $a \in B$, in particular of the point $1 \in B$. For this particular case, the map $\alpha : /A \to \mathfrak{F} = \mathfrak{G}(1); p \to B_p$ is an injection where for each line $L \in \mathfrak{L}$, $\alpha(L)$ consists of lines of $\mathfrak{G}$ which contain 1 and are contained in a plane of the kinematic space $(B, \mathfrak{G})$. Then the kinematic space is a very useful tool for the foundation of $\mathfrak{U}$, i.e. for the construction of the coordinate field H of $\mathfrak{U}$, the embedding of $\mathfrak{U}$ into the projective plane over H and the construction of the quadratic form, which describes the congruence in $\mathfrak{U}$ (cf. [5]). Then it turns out that B is a subgroup of one of the following "maximal" kinematic spaces, which are derivable from a *kinematic algebra* (A, H), that is a unitary associative algebra with the property $a^2 \in H + aH$ for all $a \in A$: if U denotes the set of all units of A, then $\mathfrak{C} := \{U \cap H[a] \mid a \in A \setminus H\}$ is a kinematic cover of U with $D = H^{\cdot} := H \setminus \{0\}$, and so the group $G := U/H^{\cdot}$ has the kinematic fibration $\mathfrak{F}_m := \{C/H^{\cdot} \mid C \in \mathfrak{C}\}$. If B is a subgroup of G, then $\mathfrak{F} = \{F \cap B \mid F \in \mathfrak{F}_m\}$.

The kinematic algebras (A, H), which are connected with absolute planes, have all the rank $[A : H] = 4$ and belong to one of the classes:
Let $J := \{x \in A \setminus H^{\cdot} \mid x^2 \in H\}$ if $\operatorname{char} H \neq 2$, $J := \{x \in A \mid x^2 \in H\}$ if $\operatorname{char} H = 2$ and $N := \{x \in A \mid x^2 = 0\}$. Then (cf. [8] p. 475 f):

IIa.β) A/Rad A is a field. Rad A $\neq \{0\}$ and $(\text{Rad A})^2 = \{0\}$ (if $\mathfrak{U}$ is an euclidean plane or more general a rectangular plane).
IIb.β) A/Rad A $\cong H \oplus H$, Rad A $\neq \{0\}$ and $(\text{Rad A})^2 = \{0\}$ (if $\mathfrak{U}$ is a linear Minkowski plane).
IIIa.α) Rad A $= \{0\}$ and (A, H) is a quaternion field (if $\mathfrak{U}$ is an elliptic plane).
IIIb.α) Rad A $= \{0\}$ and $A \cong \mathfrak{M}_{22}(H)$ is the algebra of all 2×2 matrices over H (if $\mathfrak{U}$ is an hyperbolic plane).

For the kinamatic algebras of class IIIb.α) we have $U = GL(2, H)$ and $G = PGL(2, H)$.
If $(K, +, \cdot)$ is any commutative field, then $(PGL(2, K),\ \mathfrak{F}_m)$ is called the hyperbolic *kinematic space* over K and $\mathfrak{F}_m$ the *hyperbolic kinematic fibration*

over K.
All generalized absolute planes $\mathfrak{U}$ split in two classes, in regular ones and in so called "Lotkerngeometries". The corresponding coordinate field H of $\mathfrak{U}$ has in the first case a characteristic $\neq 2$, and in the second case the characteristic $= 2$.
If we denote by I the set of all involutions of B, then for a regular absolute plane $\mathfrak{U} = (/\mathrm{A}, \mathfrak{L}, \equiv)$ we have $|\mathrm{B}_p \cap \mathrm{I}| = 1$ for each point $p \in /\mathrm{A}$. Therefore we observe:
Let $\mathfrak{U} = (/\mathrm{A}, \mathfrak{L}, \equiv)$ be a regular absolute plane and B the group of all proper motions of $\mathfrak{U}$. Then B can be turned in a kinematic space $(\mathrm{B}, \mathfrak{G}, \cdot)$ such that:

a) $/\mathrm{A}$ can be considered as a subset of B, hence $/\mathrm{A} \subset \mathrm{B}$ and the incidence structure $\mathfrak{L}$ of $\mathfrak{U}$ is induced as trace structure by the incidence structure $\mathfrak{G}$ of the kinematic space.

b) If for any $b \in \mathrm{B}$, $\widetilde{b} : \mathrm{B} \to \mathrm{B}; x \to bxb^{-1}$ denotes the inner automorphism, then $\widetilde{b}(/\mathrm{A}) = /\mathrm{A}$, $\widetilde{b}|_{/\mathrm{A}}$ is a motion of $\mathfrak{U}$ and $\{\widetilde{b}|_{/\mathrm{A}} \mid b \in \mathrm{B}\}$ is the motion group B of $\mathfrak{U}$.

These observations lead H. Hotje [4] to the notion of a *kinematic group* $(\mathrm{G}, \mathfrak{G}, \cdot, \mathrm{T})$, i.e.

1. $(\mathrm{G}, \mathfrak{G}, \cdot)$ is a kinematic space and $\mathrm{T} \subset \mathrm{G}$.
2. $\forall g \in \mathrm{G} : \widetilde{g}(\mathrm{T}) = \mathrm{T}$.
3. $\mathrm{G} \cong \widetilde{\mathrm{G}}|_{\mathrm{T}}$.

At present the following question seems to be natural:

(*) Is there also a fine way to connect a spatial absolute geometry with a kinematic space?

In order to tackle this problem it seems necessary to extend Hotjes notion of a kinematic group:
A quintuple $(\mathrm{G}, \mathfrak{G}, \cdot, \square, \mathrm{T})$ is called a *generalized kinematic group*, if:

1. $(\mathrm{G}, \mathfrak{G}, \cdot)$ is a kinematic space and $\mathrm{T} \subset \mathrm{G}$.
2. $\square : \mathrm{G} \to \operatorname{sym} \mathrm{G}; g \to g^{\square}$ is a monomorphism.
3. $\forall g \in \mathrm{G} : g^{\square}(\mathrm{T}) = \mathrm{T}$.
4. If $\mathrm{G}^{\square} := \{g^{\square} \mid g \in \mathrm{G}\}$ then $\mathrm{G} \cong \mathrm{G}^{\square}|_{\mathrm{T}} = \{g^{\square}|_{\mathrm{T}} \mid g^{\square} \in \mathrm{G}^{\square}\}$.

Now we can give (*) a more precise form:

(**) Let $\mathfrak{U} = (/\mathrm{A}, \mathfrak{L}, \equiv)$ be an absolute space and let B be a motion group of $\mathfrak{U}$. Can B be turned into a generalized kinematic group $(\mathrm{B}, \mathfrak{G}, \cdot, \square, /\mathrm{A})$ such that for each $b \in \mathrm{B}$, $b^{\square}|_{/\mathrm{A}}$ is a motion of $\mathfrak{U}$?

Here we will consider this problem for a 3-dimensional hyperbolic space. For this purpose we will present a nice algebraic description of the hyperbolic space which allows a natural generalization of the notion "hyperbolic space". Since

the Minkowski space time world is closely related to the hyperbolic space, we obtain also an elegant representation of this structure and also a possibility to generalize the notion Minkowski-world.
Our analytic description of the classical hyperbolic space starts from the algebra $\mathfrak{M} := \mathfrak{M}_{22}(\mathbb{C})$ of all 2×2-matrices where the coefficients are complex numbers and the set $\mathfrak{H}$ of all Hermitian matrices of $\mathfrak{M}$. Here we have the involutions $^-$, T, $*$ and $\widehat{\ }$ defined by

$$\overline{\mathrm{A}} := (\overline{a}_{ij})\ ,\ \mathrm{A}^T := \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\ ,\ \mathrm{A}^* := (\overline{\mathrm{A}})^T\ ,\ \widehat{\mathrm{A}} := \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

where the conjugation $^-$ is an automorphism, and $T, *$ and $\widehat{\ }$ are antiautomorphisms of $(\mathfrak{M}, +, \cdot)$. The points of the classical hyperbolic space can be identified with the set $\mathfrak{H}_1^{++} := \{\mathrm{X} \in \mathfrak{H} \mid \mathrm{X} + \widehat{\mathrm{X}} > 0\ ,\ \mathrm{X}\widehat{\mathrm{X}} = 1\}$, and then the congruence is given by "$(\mathrm{A},\mathrm{B}) \equiv (\mathrm{C},\mathrm{D}) : \Longleftrightarrow\ \mathrm{A}\widehat{\mathrm{B}} + \mathrm{B}\widehat{\mathrm{A}} = \mathrm{C}\widehat{\mathrm{D}} + \mathrm{D}\widehat{\mathrm{C}}$" for $\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D},\ \in \mathfrak{H}_1^{++}$. The group B_h^+ of all proper resp. B_h of all motions is given by $\mathrm{B}_h^+ = SL(2,\mathbb{C})^\square \cong PSL(2,\mathbb{C}) \cong PGL(2,\mathbb{C})$ resp. $\mathrm{B}_h = \mathrm{B}_h^+ \cup (\widehat{\ } \circ \mathrm{B}_h^+)$, where $SL(2,\mathbb{C})^\square$ consists of all maps $\mathrm{A}^\square : \mathfrak{H}_1^{++} \to \mathfrak{H}_1^{++};\ \mathrm{X} \to \mathrm{AXA}^*$ with $\mathrm{A} \in SL(2,\mathbb{C})$. This shows that B_h^+ becomes via the hyperbolic kinematic fibration $\mathfrak{F}_m$ over $\mathbb{C}$ the hyperbolic kinematic space $(\mathrm{B}_h^+, \mathfrak{G}, \cdot)$ over $\mathbb{C}$ and that $(\mathrm{B}_h^+, \mathfrak{G}, \cdot, \square, \mathfrak{H}_1^{++})$ is a generalized kinematic group.

2. Properties of the algebra of 2×2-matrices.

In this paper let:
K be a commutative field with char $(\mathrm{K}) \neq 2$ such that

1. K has an involutory automorphism $-$
2. $\mathrm{H} := \mathrm{Fix}\,(-)$ has a *half order* P, i.e. P is a subgroup of the multiplicative group $(\mathrm{H}^\cdot, \cdot)$ of index $(\mathrm{H}^\cdot : \mathrm{P}) = 2$ where $\mathrm{H}^\cdot := \mathrm{H} \setminus \{0\}$; P is called an *order* or a *positive-domain* if further more $\mathrm{P} + \mathrm{P} \subset \mathrm{P}$; we set $\mathrm{K}^\cdot := \mathrm{K} \setminus \{0\}$ and $\mathrm{H}' = \{x\overline{x} \mid x \in \mathrm{K}^\cdot\}$.
3. $i \in \mathrm{K} \setminus \mathrm{H}$ such that $i^2 \in \mathrm{H}$ (if $u \in \mathrm{K} \setminus \mathrm{H}$, then $i = u - \overline{u}$ is such an element).

$$\mathfrak{M} := \mathfrak{M}_{22}(\mathrm{K}) := \left\{ \mathrm{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mid x_{ij} \in \mathrm{K} \right\}.$$

$$\mathrm{E}_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ ,\ \mathrm{E}_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ,\ \mathrm{E}_{21} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ ,\ \mathrm{E}_{22} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\ ,$$

$E := E_{11} + E_{22}$. K and KE resp. H and HE shall be identified.
For $X, Y \in \mathfrak{M}$ let:

$$\widehat{X} := \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \quad , \quad X^T := \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}$$

$$\overline{X} := \begin{pmatrix} \overline{x}_{11} & \overline{x}_{12} \\ \overline{x}_{21} & \overline{x}_{12} \end{pmatrix} \quad , \quad X^* := \overline{X}^T = \begin{pmatrix} \overline{x}_{11} & \overline{x}_{21} \\ \overline{x}_{12} & \overline{x}_{22} \end{pmatrix}$$

$$\mathfrak{H} := \text{Fix}\,(*) = \{X \in \mathfrak{M} \mid X^* = X\} = \left\{ \begin{pmatrix} \xi & x \\ \overline{x} & \eta \end{pmatrix} \mid \xi, \eta \in H\,,\ x \in K \right\}.$$

$$\varrho(X, Y) := \frac{(X\widehat{Y} + Y\widehat{X})^2}{X\widehat{X} \cdot Y\widehat{Y}} \quad \text{if} \quad X\widehat{X}, Y\widehat{Y} \neq 0$$

Each half-order P contains the set $H^{(2)} := \{\lambda^2 \mid \lambda \in H \setminus \{0\}\}$ of squares of the field H. We will consider the following particular cases:

P is an order with $-i^2 \in P$

P is an order with $-i^2 \in P$ and $1 + H^{(2)} \subset H^{(2)}$, i.e. H is a *pythagorean field.*

$P = H^{(2)}$ and $1 + H^{(2)} \subset H^{(2)}$, i.e. H is an *euclidean field.*

$H = \mathbb{R}$, $P = \mathbb{R}^{(2)}$, hence $K = \mathbb{C}$.

(2.1) $\widehat{\ }$ is an involutory antiautomorphism of $(\mathfrak{M}, +, \cdot)$ with $\text{Fix}\,\widehat{\ } = K$ and we have:

a) $\det : \mathfrak{M} \to K$; $X \to X\widehat{X}$ is a quadratic form and $f : \mathfrak{M} \times \mathfrak{M} \to K$; $(X, Y) \to \widehat{X}Y + \widehat{Y}X$ the corresponding symmetric bilinear form.

b) For $X \in \mathfrak{M}$, $\text{Tr}\,X = X + \widehat{X}$ is the trace of the matrix X.

c) $J := \text{Fix}\,(-\widehat{\ }) = \{X \in \mathfrak{M} \mid X + \widehat{X} = 0\} = \{X \in \mathfrak{M} \setminus K \mid X^2 \in K\}$ is a 3-dimensional vector subspace of $(\mathfrak{M}, K)$.

d) In the metric vector space $(\mathfrak{M}, K, \det)$ we have $\mathfrak{M} = K \perp J$, hence $\widehat{\ }$ is a reflection with the axis L and the direction J and so $\det(\widehat{\ }) = -1$.

e) For $X, Y, U \in \mathfrak{M}$ with $X\widehat{X}, Y\widehat{Y}, U\widehat{U} \neq 0$, $\varrho\,(UX, UY) = \varrho\,(XU, YU) = \varrho\,(X, Y)$.

(2.2) $(\mathfrak{M})$, H is an eight-dimensional vector space and $*$ an involutory antiautomorphism of $(\mathfrak{M}, +, \cdot)$ and a linear map of $(\mathfrak{M}, H)$ such that:

a) The *Hermitian-matrices* $\mathfrak{H} := \text{Fix}\,* = \left\{ \begin{pmatrix} \xi & x \\ \overline{x} & \eta \end{pmatrix} \mid \xi, \eta \in H,\ x \in K \right\}$ and the *skew-Hermitian*-matrices $\text{Fix}\,(-*) = i \cdot \mathfrak{H}$ are four-dimensional vector subspace of $(\mathfrak{M}, H)$.

b) $\hat{}\circ * = * \circ \hat{}$, hence $\det X = X\widehat{X} \in H$ for all $X \in \mathfrak{H}$, i.e. $\det : \mathfrak{H} \to H$; $X \to X\widehat{X}$ is a quadratic form of the vector space $(\mathfrak{H}, H)$.

c) In the metric vector space $(\mathfrak{H}, H, \det)$, $\mathfrak{H} = H \perp (J \cap \mathfrak{H})$ and $\hat{}$ is a reflection with axis H and direction $(J \cap \mathfrak{H})$. For $X = \xi_1 + \mathfrak{x} \in \mathfrak{H}$ with $\xi_1 \in H$ and

$$\mathfrak{x} = \begin{pmatrix} \xi_2 & x \\ \overline{x} & -\xi_2 \end{pmatrix} \in J \cap G$$

we have $\det X = X\widehat{X} = \xi_1^2 + \mathfrak{x}\widehat{\mathfrak{x}} = \xi_1^2 - \mathfrak{x}^2 = \xi_1^2 - (\xi_2^2 + x\overline{x})$.

d) If P is an order of $(H, +, \cdot)$ such that $-i^2 \in P$, then det is positive definite on H, negative definite on $J \cap \mathfrak{H}$ and for $X, Y \in \mathfrak{H}$ with $\det X$, $\det Y$, $\operatorname{Tr} X$, $\operatorname{Tr} Y \in P$ we have $\det(X + Y) \in P$.

Proof. d) Let $X = \xi_1 + \mathfrak{x}$, $Y = \eta_1 + \mathfrak{y} \in \mathfrak{H}$ with $\xi_1^2 - \mathfrak{x}^2, \eta_1^2 - \mathfrak{y}^2, \xi_1, \eta_1 \in P$, hence $\det(X + Y) = (\xi_1 + \eta_1)^2 - (\mathfrak{x} + \mathfrak{y})^2$. Clearly, if $\mathfrak{x} = 0$ or $\mathfrak{y} = 0$, then $\det(X + Y) \in P$. So let $\mathfrak{x}, \mathfrak{y} \neq 0$, i.e. we may form $\lambda := \dfrac{\mathfrak{x}\mathfrak{y} + \mathfrak{y}\mathfrak{x}}{2 \cdot \mathfrak{y}^2} \in H$.
By c) $(\mathfrak{x} - \lambda\mathfrak{y})^2 = \mathfrak{x}^2 - \lambda(\mathfrak{x}\mathfrak{y} + \mathfrak{y}\mathfrak{x}) + \lambda^2\mathfrak{y}^2 = \mathfrak{x}^2 - 2\mathfrak{y}^2\lambda^2 + \lambda^2\mathfrak{y}^2 = \mathfrak{x}^2 - \lambda^2\mathfrak{y}^2 \in P \cup \{0\}$, hence $4\mathfrak{x}^2\mathfrak{y}^2 - (\mathfrak{x}\mathfrak{y} + \mathfrak{y}\mathfrak{x})^2 \in P \cup \{0\}$, i.e. $(\mathfrak{x}\mathfrak{y} + \mathfrak{y}\mathfrak{x})^2 \leq 4\mathfrak{x}^2\mathfrak{y}^2$ (Schwarz inequality).
By our assumptions $(\mathfrak{x}\mathfrak{y} + \mathfrak{y}\mathfrak{x})^2 \leq 4\mathfrak{x}^2\mathfrak{y}^2 \leq 4\xi_1^2\eta_1^2$ i.e. $|\mathfrak{x}\mathfrak{y} + \mathfrak{y}\mathfrak{x}| \leq 2\xi_1\eta_1$.
Consequently $\det(X + Y) = \xi_1^2 - \mathfrak{x}^2 + \eta_1^2 - \mathfrak{y}^2 + 2\xi_1\eta_1 - (\mathfrak{x}\mathfrak{y} + \mathfrak{y}\mathfrak{x}) \in P$.

In dependence of P we split $\mathfrak{H}$ in the following subsets:

$\mathfrak{H}^\circ := \{X \in \mathfrak{H} \mid \det(X) = 0\}$ (set of *light* like vectors)
$\mathfrak{H}^+ := \{X \in \mathfrak{H} \mid \det(X) \in P\}$ (set of *time* like vectors)
$\mathfrak{H}^- := \{X \in \mathfrak{H} \mid \det X \in H \setminus (P \cup \{0\})\}$ (set of *space* like vectors)

and further $\mathfrak{H}^+$ in

$\mathfrak{H}^{++} := \{X \in \mathfrak{H}^+ \mid \operatorname{Tr}(X) \in P\}$ (*future-cone*)
$\mathfrak{H}^{+-} := \{X \in \mathfrak{H}^+ \mid \operatorname{Tr}(X) \notin P \cup \{0\}\}$ (*past-cone*)
$\mathfrak{H}^{+0} := \{X \in \mathfrak{H}^+ \mid \operatorname{Tr}(X) = 0\}$

(2.3) a) $H^{\cdot} \cdot \mathfrak{H}^\circ = \mathfrak{H}^0$, $H^{\cdot} \cdot \mathfrak{H}^+ = \mathfrak{H}^+$, $H^{\cdot} \cdot \mathfrak{H}^- = \mathfrak{H}^-$

b) $P \cdot \mathfrak{H}^{++} = \mathfrak{H}^{++}$, $P \cdot \mathfrak{H}^{+-} = \mathfrak{H}^{+-}$, $(H^{\cdot} \setminus P) \cdot \mathfrak{H}^{++} = \mathfrak{H}^{+-}$ and $(H^{\cdot} \setminus P) \cdot \mathfrak{H}^{+-} = \mathfrak{H}^{++}$.

c) $\mathfrak{H}^{+\circ} = \emptyset \iff \forall \xi \in H\ \forall x \in K : (\xi^2 + x\overline{x}) \notin -P$

d) If P is an order with $-i^2 \in P$, then $\mathfrak{H}^+ = \mathfrak{H}^{++} \cup \mathfrak{H}^{+-}$, $\mathfrak{H}^{++} + \mathfrak{H}^{++} \subset \mathfrak{H}^{++}$, $P \cdot \mathfrak{H}^{++} = \mathfrak{H}^{++}$, $(-P) \cdot \mathfrak{H}^{++} = \mathfrak{H}^{+-}$ and $J \cap \mathfrak{H} \subset \mathfrak{H}^- \cup \{0\}$.

Proof. d) Let $X, Y \in \mathfrak{H}^{++}$. Then $\operatorname{Tr}(X+Y) = \operatorname{Tr}(X) + \operatorname{Tr}(Y) \in P + P \subset P$ and by (2.2) d), $\det(X+Y) \in P$, then $X + Y \in \mathfrak{H}^{++}$. For $\lambda \in P$, $\operatorname{Tr}(\lambda X) = \lambda \cdot \operatorname{Tr}(X) \in P \cdot P \subset P$ and $\det(\lambda X) = \lambda^2 \cdot \det X \in P \cdot P \subset P$, hence $P \cdot \mathfrak{H}^{++} \subset \mathfrak{H}^{++}$.

A permutation σ of $\mathfrak{M}$ resp. $\mathfrak{H}$ is called a *similarity* if there is a $\lambda \in \mathrm{K}^{\cdot}$ resp. $\lambda \in \mathrm{H}^{\cdot}$ such that $\det(\sigma(\mathrm{X})) = \lambda \cdot \det \mathrm{X}$ for all $\mathrm{X} \in \mathfrak{M}$ resp. $\mathrm{X} \in \mathfrak{H}$, and a similarity is called an *isometry* if $\lambda = 1$.
For $\mathrm{A} \in \mathfrak{M}$ let $\mathrm{A}^{\square} : \mathfrak{M} \to \mathfrak{M} : \mathrm{X} \to \mathrm{AXA}^*$ and let $\mathfrak{M}^{\square} := \{\mathrm{A}^{\square} \mid \mathrm{A} \in \mathfrak{M}\}$.

(2.4) Each map $\sigma \in \mathfrak{M}^{\square} \cup \{\widehat{\ }, T, -\mathrm{id}\}$ is H-linear with $\sigma(\mathfrak{H}) \subset \mathfrak{H}$ and for the restriction $\beta := \sigma|_{\mathfrak{H}}$ onto the metric resp. half-ordered metric vector space $(\mathfrak{H}, \mathrm{H}, \det)$ resp. $(\mathfrak{H}, \mathrm{H}, \det, \mathrm{P})$ we have:

a) $\beta = \widehat{\ } \mid \mathfrak{H}$ is a reflection in the line H with the direction $\mathrm{E}^{\perp} = \mathrm{H}^{\perp} = \mathrm{J} \cap \mathfrak{H}$, $\det \beta = -1$, $\beta(\mathfrak{H}^+) = \mathfrak{H}^+$, $\beta(\mathfrak{H}^-) = \mathfrak{H}^-$, $\beta(\mathfrak{H}^{++}) = \mathfrak{H}^{++}$.

b) $\beta = T|\mathfrak{H} = ^- |\mathfrak{H}$ is a reflection in the hyperplane $\mathrm{Fix}\,\beta = \mathrm{HE}_{11} + \mathrm{HE}_{22} + \mathrm{H}(\mathrm{E}_{12} + \mathrm{E}_{21})$ with the direction $\mathrm{H}i(\mathrm{E}_{12} - \mathrm{E}_{21})$, $\det \beta = -1$ and $\beta(\mathfrak{H}^+) = \mathfrak{H}^+$, $\beta(\mathfrak{H}^-) = \mathfrak{H}^-$, $\beta(\mathfrak{H}^{++}) = \mathfrak{H}^{++}$.

c) $\beta = -\mathrm{id}$ is the reflection in $\{0\}$ with $\det \beta = 1$, $\beta(\mathfrak{H}^+) = \mathfrak{H}^+$, $\beta(\mathfrak{H}^-) = \mathfrak{H}^-$ and $\beta(\mathfrak{H}^{++}) = \mathfrak{H}^{+-} \iff -1 \notin \mathrm{P}$, $\beta(\mathfrak{H}^{++}) = \mathfrak{H}^{++} \iff -1 \in \mathrm{P}$.

d) For $\mathrm{A} \in GL(2, \mathrm{K})$, $\mathrm{A}^{\square} \in GL(4, \mathrm{K})$ and $\beta = \mathrm{A}^{\square}|\mathfrak{H}$ is a similarity of $(\mathfrak{H}, \mathrm{H}, \det)$ with the factor $\lambda := \det \mathrm{A} \cdot \overline{\det \mathrm{A}} \in \mathrm{H}$, i.e. $\det(\beta(\mathrm{X})) = \lambda \cdot \det \mathrm{X}$. Further $\det \beta = \lambda^2$, i.e. β is an isometry if $\lambda = 1$, and $\beta(\mathfrak{H}^+) = \mathfrak{H}^+$, $\beta(\mathfrak{H}^-) = \mathfrak{H}^- \iff \lambda \in \mathrm{P}$, $\beta(\mathfrak{H}^+) = \mathfrak{H}^- \iff \lambda \notin \mathrm{P}$. If P is an order with $-i^2 \in \mathrm{P}$, then $\beta(\mathfrak{H}^{++}) = \mathfrak{H}^{++}$.

e) In the cases a) and b) β is an improper, and in case c) a proper isometry.

f) $\forall \mathrm{A}, \mathrm{B} \in \mathfrak{M} : \mathrm{A}^{\square} \circ \mathrm{B}^{\square} = (\mathrm{AB})^{\square}$, i.e. $\square : \mathfrak{M} \to \mathrm{End}(\mathfrak{H}, \mathrm{H}); A \to \mathrm{A}^{\square}$ is an homomorphism with $GL(2, \mathrm{K})^{\square} \leq GL(\mathfrak{H}, \mathrm{H})$, the restriction $^{\square}|GL(2, \mathrm{K})$ has the kernel $\mathrm{K}_1 := \{z \in \mathrm{K} \mid z\overline{z} = 1\}$, and so $GL(2, \mathrm{K})^{\square} \cong GL(2, \mathrm{K})/\mathrm{K}_1$.

Proof of the last part of d): We have $\mathrm{E} \in \mathfrak{H}^{++}$, $\beta(\mathrm{E}) = \mathrm{AA}^*$ and $\mathrm{Tr}(\mathrm{AA}^*) = a_{11}\overline{a}_{11} + a_{12}\overline{a}_{12} + a_{21}\overline{a}_{21} + a_{22}\overline{a}_{22} \in \mathrm{P}$ since $-i^2 \in \mathrm{P}$, hence $\beta(\mathrm{E}) \in \mathfrak{H}^{++}$ and $\beta(-\mathrm{E}) \in \mathfrak{H}^{+-}$. By $\lambda \in \mathrm{P}$, $\beta(\mathfrak{H}^{++}) \subset \mathfrak{H}^+$. Assume $\mathrm{C} \in \beta(\mathfrak{H}^{++}) \cap \mathfrak{H}^{+-}$. Then there are $\alpha, \gamma \in \mathrm{P}$ such that $\mathrm{Tr}(\alpha\mathrm{AA}^* + \gamma\mathrm{C}) = \alpha\mathrm{Tr}(\mathrm{AA}^*) + \gamma\mathrm{Tr}(\mathrm{C}) = 0$, i.e. by (2.3) d) $(\alpha\mathrm{AA}^* + \gamma\mathrm{C}) \subset \mathfrak{H}^- \cup \{0\}$. But $\beta^{-1}(\mathrm{C}), \beta^{-1}(\mathrm{AA}^*) = \mathrm{E} \in \mathrm{H}^{++}$, hence by (2.3) d) $\gamma(\beta^{-1}(\mathrm{C})) + \alpha\mathrm{E} \in \mathfrak{H}^{++}$, and so $\gamma\mathrm{C} + \alpha\mathrm{AA}^* \subset \beta(\mathfrak{H}^{++}) \subset \mathfrak{H}^+$ a contradiction. Consequently $\beta(\mathfrak{H}^{++}) \subset \mathfrak{H}^{++}$, and also $\beta^{-1}(\mathfrak{H}^{++}) \subset \mathfrak{H}^{++}$, i.e. $\beta(\mathfrak{H}^{++}) = \mathfrak{H}^{++}$.

By [11] (2.9) and [7] (6.4),(6.5) we have:

(2.5) Let Σ resp. $\Sigma_{\mathfrak{H}}$ denote the group of all similarities of $(\mathfrak{M}, \mathrm{K}, \det)$ resp. $(\mathfrak{H}, \mathrm{H}, \det)$ and B resp. $\mathrm{B}_{\mathfrak{H}}$ the group of all isometries of $(\mathfrak{M}, \mathrm{K}, \det)$ resp. $(\mathfrak{H}, \mathrm{H}, \det)$.
Further let $\mathrm{K}^+ := \{z \in \mathrm{K} \mid z\overline{z} \in \mathrm{H}^{(2)}\}$ and $GL(2, \mathrm{K}^+) := \{\mathrm{X} \in GL(2, \mathrm{K}) \mid \det \mathrm{X} \in \mathrm{K}^+\}$. Then

a) $\Sigma = GL(2,K)_\ell \circ GL(2,K)_r \cup (\hat{} \circ GL(2,K)_\ell \circ GL(2,K)_r)$.

b) $\Sigma_{\mathfrak{H}} \cong \Sigma_* := \{\sigma \in \Sigma \mid \sigma \circ * = * \circ \sigma\} = \Sigma'_* \cup (\hat{} \circ \Sigma'_*)$ where $\Sigma'_* := \{\lambda^{\cdot} \circ A^{\square} \mid A \in GL(2,K),\ \lambda \in H^{\cdot}\}$.

c) $B = {}^+\Gamma \ \dot{\cup}\ (\hat{}\circ {}^+\Gamma) \cup \big((J_1)_\ell \circ {}^+\Gamma\big) \ \dot{\cup}\ \big(\hat{}\circ (J_1)_\ell \circ {}^+\Gamma\big)$ where ${}^+\Gamma := \{A_\ell \circ B_r \mid \det AB = 1\}$ and $J_1 := E_{11} - E_{22}$.

d) $B_{\mathfrak{H}} \cong B_* := B \cap \Sigma_* = B_*^+ \ \dot{\cup}\ (\hat{}\circ B_*^+)$ where $B_*^+ := \{\lambda' \circ A^{\square} \mid A \in GL(2,K^+), \lambda \in H^{\cdot} : \lambda^2 \det A \cdot \overline{\det A} = 1\}$.

α) If $-1 \notin P$, then B_*^+ contains the subgroup $B_*^{++} := \{\lambda' \circ A^{\square} \mid A \in GL(2,K^+),\ \lambda \in P : \lambda^2 = (\det A \cdot \overline{\det A}) = 1\}$ of index 2 which is isomorphic with $PGL(2,K^+) := GL(2,K^+)/K^{\cdot}$ and we have $B_*^+ = B_*^{++} \ \dot{\cup}\ \big((-1)^{\cdot} \circ B_*^{++}\big)$.

β) If $-1 \notin P$ and $P = H^{(2)}$ then $B_*^{++} = \{A^{\square} \mid A \in SL(2,K)\}$.

By (2.4) and (2.5) we have:

(2.6) Let $\lambda^{\cdot} \circ A^{\square} \in \Sigma'_*$, $\sigma := \lambda^{\cdot} \circ A^{\square}$ and $\tau := \hat{}\circ \lambda^{\cdot} \circ A^{\square}$. Then:

a) $\sigma(\mathfrak{H}^+) = \mathfrak{H}^+ \iff \sigma(\mathfrak{H}^-) = \mathfrak{H}^- \iff \tau(\mathfrak{H}^+) = \mathfrak{H}^+ \iff \det A \cdot \overline{\det A} \in P$.

b) $\sigma(\mathfrak{H}^+) = \mathfrak{H}^- \iff \tau(\mathfrak{H}^+) = \mathfrak{H}^- \iff \det A \cdot \overline{\det A} \notin P$.

c) $\sigma \in B_* \iff \tau \in B_* \Longrightarrow \sigma(\mathfrak{H}^+) = \tau(\mathfrak{H}^+) = \mathfrak{H}^+$.

Supplement: An analytic description of the hyperbolic plane and its kinematic spaces.

Let H be an euclidean field, let $<$ be the order relation on H defined by: $x < y :\iff y - x \in H^{(2)}$ and let $\mathfrak{M} := \mathfrak{M}_{22}(H)$. To the 4-dimensional vector space $(\mathfrak{M}, H)$ the corresponding 3-dimensional projective space $\Pi(\mathfrak{M}, H)$ is given by:

Let $\mathfrak{M}^\times := \mathfrak{M} \setminus \{0\}$, $\varphi : \mathfrak{M}^\times \to \mathfrak{M}^\times/H^{\cdot};\ X \to H^{\cdot}X$ the canonical map and let $\mathfrak{P}_2$ be the set of all 2-dimensional vector subspaces of $(\mathfrak{M}, H)$, then $\mathfrak{M}^\times/H^{\cdot}$ is the set of points and $\mathfrak{G}_\pi := \{\varphi(L \setminus \{0\}) \mid L \in \mathfrak{L}_2\}$ the set of lines. If

$$\mathfrak{M}^+ := \{X \in M \mid X\widehat{X} > 0\} \quad , \quad \mathfrak{M}^\circ := \{X \in \mathfrak{M}^\times \mid X\widehat{X} = 0\},$$

$$\mathfrak{M}^- := \{X \in \mathfrak{M} \mid X\widehat{X} < 0\} \quad , \quad {}^\circ\mathfrak{M} := \{X \in \mathfrak{M}^\times \mid X + \widehat{X} = 0\},$$

$${}^\circ\mathfrak{M}^+ :={}^\circ \mathfrak{M} \cap \mathfrak{M}^+ \,,\ {}^\circ\mathfrak{M}^\circ :={}^\circ \mathfrak{M} \cap \mathfrak{M}^\circ \,,\ {}^\circ\mathfrak{M}^- :={}^\circ \mathfrak{M} \cap \mathfrak{M}^- \,,$$

then each of these subsets is homogeneneous, $\mathfrak{M}^\times = \mathfrak{M}^+ \ \dot{\cup}\ \mathfrak{M}^\circ \ \dot{\cup}\ \mathfrak{M}^-$ and ${}^\circ\mathfrak{M} ={}^\circ \mathfrak{M}^+ \ \overset{\circ}{\cup}\ \mathfrak{M}^\circ \ \overset{\circ}{\cup}\ \mathfrak{M}^-$ are disjoint unions, $\mathfrak{M}^+ \cup \mathfrak{M}^- = GL(2,H)$ and

$\mathfrak{M}^+$ is a subgroup of index 2 in $GL(2,\mathrm{H})$. Consequently the set $\mathfrak{M}^\times/\mathrm{H}^\cdot$ of points of the projective space splits into the disjoint subsets:

$$\mathfrak{M}^\times/\mathrm{H}^\cdot = \varphi(\mathfrak{M}^+) \,\dot{\cup}\, \varphi(\mathfrak{M}^\circ) \,\dot{\cup}\, \varphi(\mathfrak{M}^-) = PGL(2,\mathrm{H}) \,\dot{\cup}\, \varphi(\mathfrak{M}^\circ)$$

where $\varphi(\mathfrak{M}^\circ)$ is a ruled quadric. $\varphi({}^\circ\mathfrak{M})$ is a projective subplane of $\Pi(\mathfrak{M},\mathrm{H})$ which has the disjoint decomposition:

$$\varphi({}^\circ\mathfrak{M}) = \varphi({}^\circ\mathfrak{M}^+) \,\dot{\cup}\, \varphi({}^\circ\mathfrak{M}^\circ) \,\dot{\cup}\, \varphi({}^\circ\mathfrak{M}^-)$$

where $\varphi({}^\circ\mathfrak{M}^\circ) = \varphi(\mathfrak{M}^\circ) \cap \varphi({}^\circ\mathfrak{M})$, the intersection of the ruled quadric and the projective plane is an ellipse.
The *hyperbolic plane* $\mathfrak{U} = \mathfrak{U}(\mathrm{H}) = (/\mathrm{A}, \mathfrak{L}, \equiv, \alpha)$ over the euclidean field H is given by: $/\mathrm{A} := \varphi({}^\circ\mathfrak{M}^+)$ is the set of *points*, $\mathfrak{L} := \varphi({}^\circ\mathfrak{M}^-)$ is the set of *lines*. A point $\varphi(\mathrm{A}) \in /\mathrm{A}$ and a line $\varphi(\mathrm{B}) \in \mathfrak{L}$ are *incident* iff $\mathrm{A}\widehat{\mathrm{B}} + \mathrm{B}\widehat{\mathrm{A}} = 0$, the *congruence* of two pairs of points $(\varphi(\mathrm{A}),\varphi(\mathrm{B}))$, $(\varphi(\mathrm{C}),\varphi(\mathrm{D})) \in /\mathrm{A} \times /\mathrm{A}$ is defined by $(\varphi(\mathrm{A}),\varphi(\mathrm{B})) \equiv (\varphi(\mathrm{C}),\varphi(\mathrm{D})) :\Longleftrightarrow \varrho(\mathrm{A},\mathrm{B}) = \varrho(\mathrm{C},\mathrm{D})$ and the *order structure* is given by:
Let $\varphi(\mathrm{A}),\varphi(\mathrm{B}),\varphi(\mathrm{C}) \in /\mathrm{A}$ be three distinct collinear points. Then there are $\alpha,\beta \in \mathrm{H}^\cdot$ such that $\mathrm{C} = \alpha\mathrm{A} + \beta\mathrm{B}$.
We set

$$(\varphi(\mathrm{C}) \mid \varphi(\mathrm{A}),\varphi(\mathrm{B})) := \begin{cases} -1 \text{ if } \alpha\beta(\mathrm{A}\widehat{\mathrm{B}} + \mathrm{B}\widehat{\mathrm{A}}) > 0 \\ \ \ 1 \text{ if } \alpha\beta(\mathrm{A}\widehat{\mathrm{B}} + \mathrm{B}\widehat{\mathrm{A}}) < 0 \end{cases}$$

(cf. [12] §29) and say that $\varphi(\mathrm{C})$ lies *between* $\varphi(\mathrm{A}),\varphi(\mathrm{B})$, if $(\varphi(\mathrm{C}) \mid \varphi(\mathrm{A}),\varphi(\mathrm{B})) = -1$.
For $\mathrm{A} \in {}^\circ\mathfrak{M}^+$ and $\mathrm{B} \in GL(2,\mathrm{H})$ we have:

$$\mathrm{BAB}^{-1} + \widehat{\mathrm{BAB}^{-1}} = \mathrm{BAB}^{-1} + \widehat{\mathrm{B}^{-1}}\widehat{\mathrm{A}}\widehat{\mathrm{B}} = \mathrm{B}(\mathrm{A}+\widehat{\mathrm{A}})\mathrm{B}^{-1} = \mathrm{B}O\mathrm{B}^{-1} = 0$$

and

$$\mathrm{BAB}^{-1}\widehat{\mathrm{BAB}^{-1}} = \mathrm{BAB}^{-1}\widehat{\mathrm{B}^{-1}}\widehat{\mathrm{A}}\widehat{\mathrm{B}} = \mathrm{B}(\mathrm{B}^{-1}\widehat{\mathrm{B}^{-1}})\mathrm{A}\widehat{\mathrm{A}}\widehat{\mathrm{B}} = \mathrm{A}\widehat{\mathrm{A}} > 0\,.$$

Consequently ${}^\circ\mathfrak{M}^+$ is an invariant subset of $GL(2,\mathrm{H})$ and so $/\mathrm{A} = \varphi({}^\circ\mathfrak{M}^+)$ is invariant in $PGL(2,\mathrm{H})$. Moreover $/\mathrm{A}$ consists of involutions since $\widehat{\mathrm{A}} + \mathrm{A} = 0$ implies $\widehat{\mathrm{A}} = -\mathrm{A}$ and so $\mathrm{A}^2 = -\mathrm{A}\widehat{\mathrm{A}} \in \mathrm{H}$. The group of proper resp. of all motions of the hyperbolic plane $\mathfrak{U}$ is given by $\mathrm{B} := \varphi(\mathfrak{M}^+)$ resp. $PGL(2,\mathrm{H})$ and the motions are the inner automorphisms of $PGL(2,\mathrm{H})$ restricted onto $/\mathrm{A}$.

Since B and G $:= PGL(2,\mathrm{H})$ are subsets of the point set of the projective space $\Pi(\mathfrak{M},\mathrm{H})$ they can be turned in linear incidence spaces:

$$\mathfrak{G}_{\mathrm{B}} := \{\mathrm{X}\cap\mathrm{B} \mid \mathrm{X}\in\mathfrak{G}_{\pi} : |\mathrm{X}\cap\mathrm{B}| \geq 2\},$$

$$\mathfrak{G}_{\mathrm{G}} := \{\mathrm{X}\cap\mathrm{G} \mid \mathrm{X}\in\dot{\mathfrak{G}}_{\pi} : |\mathrm{X}\cap\mathrm{G}| \leq 2\}$$

Then $(\mathrm{B},\mathfrak{G}_{\mathrm{B}},\cdot)$ and $(PGL(2,\mathrm{H},\mathfrak{G}_{\mathrm{G}},\cdot)$ are kinematic spaces of the hyperbolic plane $\mathfrak{U}$ and $(\mathrm{B},\mathfrak{G}_{\mathrm{B}},\cdot,/\mathrm{A})$ resp. $(PGL(2,\mathrm{H},\mathfrak{G}_{\mathrm{G}},\cdot,/\mathrm{A})$ are kinematic groups.

Remarks.

1. For each euclidean field $\mathrm{H},\mathfrak{U}(\mathrm{H})$ is a hyperbolic plane in the sense of [12] §26, and conversely, to each such hyperbolic plane $\mathfrak{U}$ there is an uniquely determined euclidean field H such that $\mathfrak{U}(\mathrm{H}) = \mathfrak{U}$. For $\mathrm{H} = \mathbb{R}$ we obtain the classical hyperbolic plane (cf. [12] §30).
2. The notion "hyperbolic plane" can be generalized in many steps by taking for H fields of larger classes of fileds.
3. With the function ϱ the congruence relation $\equiv$ of $\mathfrak{U}$ can be extended onto the kinematic spaces $(\mathrm{B},\mathfrak{G}_{\mathrm{B}},\cdot)$ and $(PGL(2,\mathrm{H}),\mathfrak{G}_{\mathrm{G}},\cdot)$.

3. Generalized Minkowski-world, their Lorentz-groups and their kinematic structures.

The 4-dimensional half-ordered metric vector space $(\mathfrak{H},\mathrm{H},\det\mathrm{P})$ shall be provided with the following structures:

1. The elements of $\mathfrak{H}$ are called *events*.
2. On $\mathfrak{H}$ we define:

a) the *causality relation*: $\mathrm{A}\twoheadrightarrow\mathrm{B} :\Longleftrightarrow \mathrm{B}-\mathrm{A}\in\mathfrak{H}^{++}$,
b) the *signal relation*: $\mathrm{A}\dashrightarrow\mathrm{B} :\Longleftrightarrow \mathrm{B}-\mathrm{A}\in\mathfrak{H}^{\circ+} := \{\mathrm{X}\in\mathfrak{H}^{\circ} \mid \mathrm{Tr}\,\mathrm{X}\in\mathrm{P}\}$,
c) the *congruence relation*: $(\mathrm{A},\mathrm{B}) \equiv (\mathrm{C},\mathrm{D}) :\Longleftrightarrow \det(\mathrm{A}-\mathrm{B}) = \det(\mathrm{C}-\mathrm{D})$.

3. The elements of $\mathfrak{G} := \{\mathrm{A}+\mathrm{HB} \mid \mathrm{A},\mathrm{B}\in\mathfrak{H},\ \mathrm{B}\neq 0\}$ are called *lines*. They split in the three classes $\mathfrak{G}^{\circ}$, $\mathfrak{G}^{+}$, $\mathfrak{G}^{-}$ whether $\mathrm{B}\in\mathfrak{H}^{\circ}$, $\mathrm{B}\in\mathfrak{H}^{+}$ or $\mathrm{B}\in\mathfrak{H}^{-}$.

Then $(\mathfrak{H},\mathfrak{G},\twoheadrightarrow,\dashrightarrow,\equiv)$ is called a *generalized Minkowski-world* and we have:

(3.1) If $\mathrm{H}=\mathbb{R}$, hence $\mathrm{K}=\mathbb{C}$, then $(\mathfrak{H},\mathfrak{G},\twoheadrightarrow,\dashrightarrow,\equiv)$ is the classical Minkowski space time world.

(3.2) All line preserving permutations π of $\mathfrak{H}$ with $(\mathrm{A},\mathrm{B}) \equiv (\pi(\mathrm{A}),\pi(\mathrm{B}))$ for all $\mathrm{A},\mathrm{B}\in\mathfrak{H}$ form the *generalized Lorentz-group* $\mathrm{M} := \mathrm{M}(\mathfrak{H},\mathfrak{G},\equiv)$ and

M is the semidirect product $M = T \rtimes L$, where T is the normal subgroup consisting of all translations $A^+ : \mathfrak{H} \to \mathfrak{H}; X \to A + X$ with $A \in \mathfrak{H}$ and $L := \{\sigma \in M \mid \sigma(0) = 0\} = B_{\mathfrak{H}} \cong B_* = B_*^+ \dot\cup (\frown \circ B_*^+)$ (cf. (2.5) b)).

By (2.5) $L = B_{\mathfrak{H}}$ consists of linear maps and for each $\sigma \in B_{\mathfrak{H}}$ either $\det\sigma = 1$ or $\det\sigma = -1$ by (2.4), $\sigma(\mathfrak{H}^\circ) = \mathfrak{H}^\circ$, $\sigma(\mathfrak{H}^+) = \mathfrak{H}^+$ and $\sigma(\mathfrak{H}^-) = \mathfrak{H}^-$.
In the Lorentz-group M we will consider the following subgroups:
$L^+ := \{\sigma \in L \mid \det\sigma = 1\} = B_*^+$ and $M^+ := T \rtimes L^+$.
(We have $(M : M^+) = (L : L^+) = 2$ and $L^+ = B_*^+$ by (2.4) a) and (2.5))
${}^+L := \{\sigma \in L \mid \sigma(\mathfrak{H}^{++}) = \mathfrak{H}^{++}\}$, and ${}^+M := T \rtimes {}^+L$ the group of all *orthochronous* Lorentz-transformations which are also characterized by preserving the causality relation "$\to$".
${}'L :=^+ L \dot\cup (-id) \circ^+ L$ and ${}'M := T \rtimes {}'L$ the group of all maps which either preserve or reserve the causality relation.

(3.3) If ${}^+L^+ :=^+ L \cap L^-$, then:

a) ${}^+L =^+ L^+ \dot\cup (\frown \circ^+ L^+)$ (by (2.4) a), and ${}'L :=^+ L^+ \dot\cup (\frown \circ^+ L^+) \dot\cup ((\text{-id}) \circ^+ L^+) \dot\cup (\frown \circ (\text{-id}) \circ^+ L^+)$.

b) If P is an order with $-i^2 \in P$, then ${}^+L^+ = B_*^{++}$ by (2.3) d), (2.4) d) and (2.5) d), hence ${}'L = L$.

c) if $(H, +, \cdot)$ is a euclidean field, then ${}^+L^+ = \{A^\square \mid A \in SL(2, K)\} \cong SL(2, K)$.

d) For $H = \mathbb{R}$, ${}^+L^+ \cong SL(2, \mathbb{C})$ is the classical group of all homogeneous, proper and orthochronous Lorentz-transformations.

Problems: Determine ${}^+L^+$ for the cases:

i) P is an order but $-i^2 \notin P$.

ii) P is a halforder but not an order for $-i^2 \in P^2$ and for $-i^2 \notin P^2$.

(3.4) The following statements are equivalent:

a) The causality relation "$\to$" is antisymmetric and transitive

b) $\mathfrak{H}^{++} + \mathfrak{H}^{++} \subset \mathfrak{H}^{++}$

c) P is an order with $-i^2 \in P$.

Proof. a) $\Longleftrightarrow$ b): Since there are $A, B \in \mathfrak{H}$ with $A \to B$, $B - A \in \mathfrak{H}^{++}$, i.e. $\mathfrak{H}^{++} \neq \phi$. Let $A, B \in \mathfrak{H}^{++}$. Then $0 \to A$ and $A \to A+B$, and by the transitivity of "$\to$" we have $0 \to A + B$, i.e. $A + B \in \mathfrak{H}^{++}$, i.e. $\mathfrak{H}^{++} + \mathfrak{H}^{++} \subset \mathfrak{H}^{++}$.
b) $\Rightarrow$ c): Let

$$A = \begin{pmatrix} \alpha & a \\ \bar{a} & \beta \end{pmatrix} \in \mathfrak{H}^{++},$$

hence $\alpha+\beta\in \mathrm{P}$. Since by assumption $\mathrm{A}+\mathrm{A}\in\mathfrak{H}^{++}$ we have $2(\alpha+\beta)\in\mathrm{P}$, and so $2\in\mathrm{P}$ since $(\mathrm{P},\cdot)$ is a group. But $2\in\mathrm{P}$ implies

$$\mathrm{P}=\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} \mid \lambda\in\mathrm{P}\right\}\in\mathfrak{H}^{++},$$

consequently $\mathrm{P}+\mathrm{P}\subset\mathfrak{H}^{++}+\mathfrak{H}^{++}\subset\mathfrak{H}^{++}$. For $\lambda,\mu\in\mathrm{P}$ we obtain

$$\begin{pmatrix}\lambda+\mu & 0\\ 0 & \lambda+\mu\end{pmatrix}\in\mathrm{H}^{++},$$

hence $2(\lambda+\mu)\in\mathrm{P}$, thus $\lambda+\mu\in\mathrm{P}$, and so P is an order of $(\mathrm{H},+,\cdot)$. Suppose $i^2\in\mathrm{P}$. Let $\alpha,\beta\in\mathrm{H}$ such that $\beta^2>\alpha>i^2$. Then

$$\mathrm{A}:=\begin{pmatrix}\alpha & \beta i\\ -\beta i & -i^2\end{pmatrix}\in\mathfrak{H}^{++}.$$

For $\mu\in\mathrm{P}$ with $\mu<\dfrac{\alpha i^2}{\alpha+\beta^2}$ and $\mathrm{B}:=\begin{pmatrix}\beta^2 & 0\\ 0 & \mu\end{pmatrix}$ we have $\mathrm{B}\in\mathfrak{H}^{++}$ but $\det(\mathrm{A}+\mathrm{B})=\mu(\alpha+\beta^2)-\alpha i^2<\alpha i^2-\alpha i^2=0$, hence $\mathrm{A}+\mathrm{B}\notin\mathfrak{H}^{++}$.
c) $\Rightarrow$ a). Since $-1\notin\mathrm{P}$, "$\rightarrow$" is antisymmetric and since $2\in\mathrm{P}$, $0\rightarrow\mathrm{E}$. Now let $\mathrm{A}\rightarrow\mathrm{B}$ and $\mathrm{B}\rightarrow\mathrm{C}$, i.e. $\mathrm{X}:=\mathrm{B}-\mathrm{A}$, $\mathrm{Y}:=\mathrm{C}-\mathrm{B}\in\mathfrak{H}^{++}$, i.e. $\det\mathrm{X},\det\mathrm{Y},\mathrm{Tr}\,\mathrm{X},\mathrm{Tr}\,\mathrm{Y}\in\mathrm{P}$. By (2.2) d) $\det(\mathrm{X}+\mathrm{Y})\in\mathrm{P}$ and $\mathrm{Tr}(\mathrm{X}+\mathrm{Y})=\mathrm{Tr}\,\mathrm{X}+\mathrm{Tr}\,\mathrm{Y}\in\mathrm{P}+\mathrm{P}\subset\mathrm{P}$, i.e. $\mathrm{X}+\mathrm{Y}=\mathrm{C}-\mathrm{A}\in\mathfrak{H}^{++}$, hence $\mathrm{A}\rightarrow\mathrm{C}$.

By $\psi:\mathrm{L}^+\rightarrow PGL(2,\mathrm{K}^+)$; $\lambda^{\cdot}\circ\mathrm{A}^{\square}\rightarrow\mathrm{K}^{\cdot}\mathrm{A}$ a map is defined (for $\lambda^{\cdot}\circ\mathrm{A}^{\square}=\mu^{\cdot}\circ\mathrm{B}^{\square}\iff(\lambda\mu^{-1})^{\cdot}=\mathrm{B}^{\square}\mathrm{A}^{\square-1}=(\mathrm{BA}^{-1})^{\square}\iff\mathrm{KA}=\mathrm{KB}$) which is a homomorphism with the kernel $\{\mathrm{id.}-\mathrm{id}\}$. Since the group $PGL(2,\mathrm{K}^+)$ is a subgroup of $PGL(2,\mathrm{K})$ and since $PGL(2,\mathrm{K})$ is the hyperbolic kinematic space over $(\mathrm{K},+,\cdot)$ with respect to the fibration $\mathfrak{F}=\{\varphi((\mathrm{KE}+\mathrm{KA})\cap GL(2,\mathrm{K}))\mid\mathrm{A}\in\mathrm{J}^{\times}\}$ where $\varphi:\mathfrak{M}^{\times}\rightarrow\mathfrak{M}^{\times}/\mathrm{K}^{\cdot}$; $\mathrm{X}\rightarrow\mathrm{K}^{\cdot}\mathrm{X}$, $\mathfrak{F}$ induces a kinematic fibration $\mathfrak{F}':=\{\mathrm{F}\cap PGL(2,\mathrm{K}^+)\mid\mathrm{F}\in\mathfrak{F}\}$ for $PGL(2,\mathrm{K}^+)$ and therefore we can provide the Lorentz-group L^+ with the kinematic cover $\mathfrak{C}:=\{\psi^{-1}(\mathrm{F})\mid\mathrm{F}\in\mathfrak{F}'\}$. If $-1\notin\mathrm{P}$, then B_*^{++} is a subgroup of L^+ of index 2 and the restriction $\psi|_{\mathrm{B}_*^{++}}$ is an isomorphism between B_*^{++} and $PGL(2,\mathrm{K}^*)$, i.e. B_*^{++} can be turned into a kinematic space.

This gives us the result:
(3.5) The proper homogeneus Lorentz-group $\mathrm{L}^+=\mathrm{B}_*^+$ has a *kinematic cover* $\mathfrak{C}$ with $\mathrm{D}=\{\mathrm{id},-\mathrm{id}\}$ and we have:

a) $L^+/D \cong PGL(2, K^+)$ has a kinematic fibration.
b) If $-1 \notin P$, then $L^+/D \cong B_*^{++}$ and ${}^+L^+ \leq B_*^{++}$.
c) If P an order with $-i^2 \in P$, hence ${}^+L^+ = B_*^{++}$, then the homogeneus orthochronous Lorentz group ${}^+L^+$ has the kinematic fibration $\mathfrak{F}'$.
d) If P is an order with $-i^2 \in H^{(2)}$ and $1 + H^{(2)} \subset H^{(2)}$, then ${}^+L^+ \cong PGL(2, K$ and ${}^+L^+$ can be turned into the hyperbolic kinematic space over $(K, +, \cdot)$.
e) For the classical case $H = \mathbb{R}$, the orthochronous Lorentz group ${}^+L^+ \cong PGL(2, \mathbb{C}) \cong SL(2, \mathbb{C})$ has the hyperbolic kinematic fibration $\mathfrak{F}_m$ of $\mathbb{C}$.

4. Hyperbolic and generalized hyperbolic spaces.

In this section we use the same notations as in §2 and we will assume that H has a halforder P.
To the 4-dimensional vector space $(\mathfrak{H}, H)$ we consider the corresponding 3-dimensional projective space $\Pi(\mathfrak{H}, H)$ (cf. e.g. [12] p. 61):
Let $\mathfrak{H}^\times := \mathfrak{H} \setminus \{0\}$ and $\varphi : \mathfrak{H}^\times \to \mathfrak{H}^\times/H^{\cdot}; X \to H^{\cdot} \cdot X$ the canonical map; then $\mathfrak{H}^\times/H^{\cdot}$ is the set of points and if $\mathfrak{P}_2$ denotes the set of all 2-dimensional vector subspaces of $(\mathfrak{H}, H)$, then $\{\varphi(L^\times) \mid L \in \mathfrak{P}_2\}$ is the set of lines of $\Pi(\mathfrak{H}, H)$ and the incidence is the inclusion. $\varphi(\mathfrak{H}^{0\times})$ is not a ruled quadric. The quadratic form "det" turns $\Pi(\mathfrak{H}, H)$ in a *projective metric space* by defining on the set of points which do not lay on the quadratic $\varphi(\mathfrak{H}^{0\times})$, the following congruence relation:

$$(H^{\cdot}A, H^{\cdot}B) \equiv (H^{\cdot}C, H^{\cdot}D) :\iff \varrho(A, B) = \varrho(C, D).$$

From now on if $A \in \mathfrak{H}^\times$ we denote by A also the point $H^{\cdot}A$ of $\Pi(\mathfrak{H}, H)$.
The halforder P of $(H, +, \cdot)$ induces a *halforder* α of the projective space $\Pi(\mathfrak{H}, H)$ by (cf. [14] or [10] p.130):

1. For $\lambda \in H^*$ let $\operatorname{sgn}\lambda := 1$ if $\lambda \in P$ and $\operatorname{sgn}\lambda = -1$ if $\lambda \notin P$.
2. Let $\mathfrak{H}^{(4)} := \{(A, B, C, D) \in \mathfrak{H}^{\times 4} \mid HA, HB, HC, HD$ are distinct and $HA + HB = HC + HD\}$. Then to each $(A, B, C, D) \in \mathfrak{H}^{(4)}$ there are $\gamma_1, \gamma_2, \delta_1, \delta_2 \in H^{\cdot}$ with $C = \gamma_1 A \in \gamma_2 B$, $D = \delta_1 A + \delta_2 B$, and $DV(A, B \mid C, D) := \gamma_1 \delta_1^{-1} \delta_2 \gamma_2^{-1}$ is the crossratio.
3. The *separation function* $\alpha : \mathfrak{H}^{(4)} \to \{1, -1\}; A, B, C, D) \to [A, B \mid C, D] := \operatorname{sgn} DV(A, B \mid C, D)$ is the halforder of $\Pi(\mathfrak{H}, H)$. In this way we obtain a *halfordered projective metric space* $(\Pi(\mathfrak{H}, H), \equiv, \alpha)$.

By the *hyperbolic-derivation* $\eta(\mathfrak{H}, H, \det, P)$ we understand the trace structure on $\mathfrak{V} := \varphi(\mathfrak{H}^+)$, i.e.: $\mathfrak{V}$ is the set of *points*. $\mathfrak{G} := \{\varphi(L \cap \mathfrak{H}^+) \mid L \in \mathfrak{L}_2 : |L \cap \mathfrak{H}^+| \geq 2\}$ is the set of *lines*, $\equiv$ and α are restricted on $\mathfrak{V}$. The triple

$(\mathfrak{V}, \mathfrak{G}, \equiv, \alpha)$ is also called a *generalized hyperbolic space* and the points of the quadratic $\varphi(\mathfrak{H}^{0\times})$ the *ends* of the hyperbolic space. A line $\varphi(\mathrm{L} \cap \mathfrak{H}^+) \in \mathfrak{G}$ is said to be incident with an end $\varphi(\mathrm{A}) \in \varphi(\mathfrak{H}^{0\times})$, if $\mathrm{A} \in \mathrm{L}$. This gives us the following subdivision of $\mathfrak{G}$:

A line $\varphi(\mathrm{A}) \in \mathfrak{G}$ belongs to $\mathfrak{G}_0$ resp. $\mathfrak{G}_1$ resp. $\mathfrak{G}_2$ if $\varphi(\mathrm{A})$ is incident with no resp. one resp. two ends. For each $\mathrm{A} \in \mathfrak{H}^\times$, $\mathrm{A}^\perp := \{\mathrm{H}^\cdot\mathrm{X} \mid \mathrm{X} \in \mathfrak{H}^\times : \mathrm{A}\widehat{\mathrm{X}} = -\mathrm{X}\widehat{\mathrm{A}}\}$ is a plane of $\Pi(\mathfrak{H}, \mathrm{H})$ and to each plane ε of $\Pi(\mathfrak{H}, \mathrm{H})$ there is an $\mathrm{A} \in \mathfrak{H}^\times$ with $\varepsilon = \mathrm{A}^\perp$. Therefore there are the three classes of planes $A^\perp$ of $\Pi(\mathfrak{H}, \mathrm{H})$ depending whether $\mathrm{A} \in \mathfrak{H}^+, \mathrm{A} \in \mathfrak{H}^0$ or $A \in \mathfrak{H}^-$, and for the set $\mathfrak{E}$ of *planes* of the hyperbolic space we have $\mathfrak{E} = \{\varphi(\mathrm{A}^\perp \cap \mathfrak{H}^+) \mid \mathrm{A} \in \mathfrak{H}^\times : \mathrm{A}^\perp \cap \mathfrak{H}^+ \neq \emptyset\}$. Also the set $\mathfrak{E}$ splits into the subsets $\mathfrak{E}^+, \mathfrak{E}^0, \mathfrak{E}^-$.

The function $\chi : \mathfrak{H} \to \mathrm{H}/\mathrm{H}^{(2)} : \mathrm{X} \to \mathrm{H}^{(2)}\mathrm{X}\widehat{\mathrm{X}}$ gives us a further subdivision of the sets of points and of planes of the projective space $\Pi(\mathfrak{H}, \mathrm{H})$ by: $\mathrm{A} \sim \mathrm{B} \iff \chi(\mathrm{A}) = \chi(\mathrm{B})$. For each $\mathrm{A} \in \mathfrak{H}^\times$ we have:

A is an end $\iff \chi(\mathrm{A}) = 0 \iff A^\perp$ is a tangent plane of $\varphi(\mathfrak{H}^{0\times})$ in A.

The pair $(\varphi(\mathfrak{H}^+), \mathfrak{G})$ is an incidence space (cf. [12] p.11) if each line of $\mathfrak{G}$ contains at least two points.

We have:

(4.1) a) $(\mathfrak{P} := \varphi(\mathfrak{H}^+), \mathfrak{G})$ is an incidence space if $\mathrm{H} \neq \mathbb{Z}_3$.

b) $\mathfrak{G} = \mathfrak{G}_0 \cup \mathfrak{G}_1 \cup \mathfrak{G}_2$.

c) If $G := \varphi(\mathrm{L} \cap \mathfrak{H}^+) \in \mathfrak{G}_1$, then $\mathrm{L} \subset \mathfrak{H}^+ \cup \mathfrak{H}^\circ$, i.e. $\varphi(\mathrm{L} \cap \mathfrak{H}^+)$ contains up to the end $\varphi(\mathrm{L}^\times \cap \mathfrak{H}^\circ)$ all points of the projective line $\varphi(\mathrm{L}^\times)$; hence $|\mathrm{G}| = 2|\mathrm{P}|$.

d) Any two distinct ends are incident with exactly one line.

e) If $\mathrm{G} \in \mathfrak{G}_2$, then $|\mathrm{G}| = |\mathrm{P}|$.

f) If $\mathrm{G} \in \mathfrak{G}_0$, then $|\mathrm{G}| = |\mathrm{P}| + 1$.

Proof. Let $\mathrm{L} \in \mathfrak{L}_2$ such that $\mathrm{L} \cap \mathfrak{H}^+ \neq \emptyset$, $\mathrm{G} := \varphi(\mathrm{L} \cap \mathfrak{H}^+)$ and let $\mathrm{A} \in \mathrm{L} \cap \mathfrak{H}^+$. Then $\mathrm{A} \notin \mathrm{A}^\perp$ and $\mathrm{L} \cap \mathrm{A}^\perp \neq \{0\}$. For $\mathrm{B} \in \mathrm{L} \cap \mathrm{A}^\perp$ with $\mathrm{B} \neq 0$, $\mathrm{L} = \mathrm{HA} + \mathrm{HB}$. We set $a := \mathrm{A}\widehat{\mathrm{A}}$, $b := \mathrm{B}\widehat{\mathrm{B}}$ and remark $a \in \mathrm{P}$. Then $\varphi(\mathrm{L}^\times) \setminus \varphi(\mathrm{A}) = \{\varphi(\lambda\mathrm{A} + \mathrm{B}) \mid \lambda \in \mathrm{H}\}$ and

$$(*) \qquad \det(\lambda\mathrm{A} + \mathrm{B}) = \lambda^2\mathrm{A}\widehat{\mathrm{A}} + \mathrm{B}\widehat{B} = \lambda^2 a + b$$

hence

$$(**) \qquad |\varphi(\mathrm{L} \cap \mathfrak{H}^+)| = 1 + |\{\lambda \in \mathrm{H} | \lambda^2 a + b \in \mathrm{P}\}|.$$

If $b = 0$ then $\mathrm{G} \in \mathfrak{G}_1$ and $\mathrm{L} \subset \mathfrak{H}^+ \cup \mathfrak{H}^0$ which proves c).

For $b \neq 0$, G is incident with no ends if $-ab \notin \mathrm{H}^{(2)}$, and with two ends if $-ab \in \mathrm{H}^{(2)}$. With this b) is proved.

Now let us assume $|G| = 1$. This implies $b \neq 0$; $b \notin P$ and $\{\lambda \in H \mid \lambda^2 a + b \in P\} = \emptyset$, hence $H^{(2)}a + b \subset b \cdot P \cup \{0\}$ which is equivalent with $H^{(2)}ab^{-1} + 1 \subset P \cup \{0\}$. Since $|H| > 3$, $|H^{(2)}| \geq 1$, and therefore we may assume $b \neq -a$. Then $a^{-2}b^2ab^{-1} + 1 \in P$ hence $b + a \subset aP = P$ which is a contradiction. This shows a).
In order to prove d) and e) let $A, B \in \mathfrak{H}^{0\times}$ with $\varphi(A) \neq \varphi(B)$ and $L := HA+HB$. Then $\det(\lambda A + B) = \lambda(A\widehat{B} + B\widehat{A})$, i.e. $|L \cap \mathfrak{H}^+| \neq \emptyset$ if $A\widehat{B} + B\widehat{A} \neq 0$. But $A\widehat{B} + B\widehat{A} = 0$ would imply the contradiction $L \subset \mathfrak{H}^0$. Since $A\widehat{B} + B\widehat{A} \neq 0$ we may assume $A\widehat{B} + B\widehat{A} = 1$ and then we see directly: $\varphi(\lambda A + B) \in \varphi(\mathfrak{H}^+) \iff \lambda \in P$.
f) Here $a \in P$ and $-ab \notin H^{(2)}$, hence $\lambda^2 a + b \in P \iff \lambda^2 a^2 + ab \in P$. Since $H^{(2)} \subset P$ we have for each $\lambda \in H$ of the form $\lambda = \dfrac{ab - \mu^2}{2\mu a}$ with $\mu \in H^{\cdot}$, $\lambda^2 a^2 + ab \in H^{(2)} \subset P$. If $\dfrac{ab - \mu^2}{2\mu a} = \dfrac{ab - \nu^2}{2\nu a}$ with $\nu \neq \mu$, then $\nu = -\dfrac{ab}{\mu}$. Therefore by $(**)$

$$1 + |P| = 1 + \frac{1}{2}|H^{\cdot}| \leq |G| \leq 1 + |H|,$$

and if $|H| \notin \mathbb{N}$, then $|P| = |H|$, and so $|G| = 1 + |P|$. If $q := |H| \in \mathbb{N}$, then $P = H^{(2)}$. Hence

$$|G| = 1 + \frac{1}{2}(q-1) = \frac{1}{2}(q+1) = 1 + |P|.$$

Remarks: 1. Let $L_2^\times := \{L \in \mathfrak{L}_2 \mid L \not\subset \mathfrak{H}^0\}$ and let $x : \mathfrak{L}_2^\times \to H^{\cdot}/H^{(2)} \cup \{0\}$ be the following map: For $L \in \mathfrak{L}_2^\times$ let $A \in L \setminus \mathfrak{H}^0$, $B \in A^\perp \cap L^\times$ and $x(L) = A\widehat{A}B\widehat{B} \cdot H^{(2)}$. Then x is independent of the choice of $A \in L \setminus \mathfrak{H}^0$ and $B \in A^\perp \cap L^\times$. For $G := \varphi(L \cap \mathfrak{H}^+) \in \mathfrak{H}$ we have:

$$\begin{aligned} G \in \mathfrak{G}_0 &\iff x(L) \neq (-1)H^{(2)} \\ G \in \mathfrak{G}_1 &\iff x(L) = 0 \\ G \in \mathfrak{G}_2 &\iff x(L) = (-1)H^{(2)} \end{aligned}$$

2. $\mathfrak{G}_2 \neq \emptyset : \mathfrak{G}_1 \neq \emptyset \iff H' \cap (-P) \neq \emptyset$ (in the finite case $H' = H$, and so $\mathfrak{G}_1 \neq \emptyset$; for $H = \mathbb{Q}$ there are examples with $\mathfrak{G}_1 = \emptyset$ and such with $\mathfrak{G}_1 \neq \emptyset$:

(i) Let $K = \mathbb{Q}(i)$, and P be a subgroup of $(\mathbb{Q}^{\cdot}, \cdot)$ of index 2 such that $-1 \notin P$ resp. $-1 \in P$. Then in the first case $\mathfrak{G}_1 = \emptyset$, in the second $\mathfrak{G}_1 \neq \emptyset$.

(ii) For $K = \mathbb{Q}(\sqrt{2})$, $H' = \{x^2 - 2y^2 \mid x, y \in \mathbb{Q}\}$, and so $-1 \in H'$, hence $-1 \in H' \cap (-P)$.

Consequently by any choice of P we have here $\mathfrak{G}_1 \neq \emptyset$).

3. $\mathfrak{G}_0 \neq \emptyset \iff \exists L \in \mathfrak{L}_2 : L \cap \mathfrak{H}^+ \neq \emptyset$ and $x(L) \notin (-1)H^{(2)}$. Since $E \in \mathfrak{H}^+$ and

$$E^\perp = \{X \in \mathfrak{H} \mid X + \widehat{X} = 0\} = \left\{ \begin{pmatrix} \xi & x \\ \overline{x} & -\xi \end{pmatrix} \mid \xi \in H, x \in K \right\}$$

we have for each $L = HE = HX \in \mathfrak{P}_2$ with $X \in E^\perp$, $L \cap \mathfrak{H}^+ \neq \emptyset$ and $x(L) = X\widehat{X} \cdot H^{(2)} = -(\xi^2 + x\overline{x})H^{(2)}$. This shows:

$$\mathfrak{G}_0(1) \neq \emptyset \iff \exists \xi \in H,\ x \in K : \xi^2 + x\overline{x} \notin H^{(2)}.$$

If H is finite, hence $H' = H$, then $\mathfrak{G}_0 \neq \emptyset$. For $H = \mathbb{R}$, hence $K = \mathbb{C}$, and $P = \mathbb{R}^{(2)}$ we have always $\xi^2 + x\overline{x} \in \mathbb{R}^{(2)}$, and so $\mathfrak{G}_0 = \emptyset$.

By a *hiperbolic space* we understand a quadrupel $(\mathfrak{P}, \mathfrak{G}, \alpha, \equiv)$ such that:

1. $(\mathfrak{P}, \mathfrak{G}, \alpha, \equiv)$ is an absolute space in the sense of [9] §1.
2. Each plane E of $(P, \mathfrak{G})$ is with respect to the trace structures of $\mathfrak{G}, \alpha$ and $\equiv$ a hyperbolic plane in the sense of [12] §26.

We have the following representation theorem:

(4.2) a) If $(H, +, \cdot)$ is an euclidean field, then $\eta(\mathfrak{H}, H, \det, P)$ is a 3-dimendsional hyperbolic space.

b) If $(\mathfrak{P}, \mathfrak{G}, \alpha, \equiv)$ is a 3-dimensional hyperbolic space, then there is exactly one euclidean field $(H, +, \cdot)$ such that $(\mathfrak{P}, \mathfrak{G}, \alpha, \equiv) = \eta(\mathfrak{H}, H, \det, P)$.

Remarks. 1. Each euclidean field $(H, +, \cdot)$ has besides the uniquely determined order $P = H^{(2)}$, exactly one quadratic field extension $(K, +, \cdot)$, so that the involutory H- automorphism of $(K, +, \cdot)$ is fixed and with this the set of Hermitian matrices $\mathfrak{H}$.

2. $\eta(\mathfrak{H}H, \det, P)$ is a continuous hyperbolic space (cf. [12] §30) if and only if $H = \mathbb{R}$, thus $K = \mathbb{C}$ and $P = \mathbb{R}^{(2)}$.

5. The kinematic structure of a hyperbolic space.

Let B_h be the *motion group* of the generalized hyperbolic space $\eta(\mathfrak{H}, H, \det, P)$, i.e. the set of all permutations π of the point set $\mathfrak{P} = \mathfrak{H}^+/H^{\cdot}$ which preserve the incidence sturcture $\mathfrak{G}$, the halforder α and for which $(\pi(X), \pi(Y)) = (X, Y)$ is valid for all $X, Y \in \mathfrak{H}^+$. Each $\sigma = \lambda^{\cdot} \circ A^{\square} \in H^{\cdot} \circ GL(2, K)^{\square} = \Sigma'_*$ or $\sigma = \frown$ induces by $\sigma' : \mathfrak{H}^{\times}/H^{\cdot} \to \mathfrak{H}^{\times}/H^{\cdot};\ H^{\cdot}X \to H^{\cdot}\sigma(X)$ a collineation of the projective space $\Pi(\mathfrak{H}, H)$ which preserves ϱ (for by (2.1) e) $\varrho(H^{\cdot}\sigma(X), H^{\cdot}\sigma(Y)) = \varrho(AXA^*, AYA^*) = \sigma(X, Y) = \varrho(H^{\cdot}X, H^{\cdot}Y)$), the crossatio (since ϱ is linear!) and so the separation function α. For $\varrho = \lambda^{\cdot} \circ A^{\square}$ we have $\sigma' = (A^{\square})'$ and by (2.6) $\sigma'(\mathfrak{P}) = \mathfrak{P}$ if $\det A \cdot \overline{\det A} \in P$. Hence:

(5.1) Let $GL(2, K, P) := \{A \in GL(2, K) \mid \det A \cdot \overline{\det A} \in P\}$. Then

a) $$\Phi : \begin{cases} GL(2, K, P) \to (GL(2, K, P))^{\square} \to B_h \\ A \to A^{\square} \to (A^{\square})' \end{cases}$$

is a homomorphism with the kernel $K^{\cdot}$, i.e;

$$B_h^+ := \Phi(GL(2, K, P)) \cong PGL(2, K, P) := GL(2, K, P)/K^{\cdot}E^{\cdot}.$$

(The elements of B_h^+ are induced by linear maps σ of $(\mathfrak{H}, H)$ with $\det \sigma \in H^{(2)}$ (cf. (2.4) d) and are called *proper motions*).

b) $\frown' \in B_h$ with $\det \frown|_{\mathfrak{H}} = -1$ (cf.(2.4) e) is a reflection in the point $\varphi(E)$ and if $E^{\perp} \cap \mathfrak{H}^+ \neq \emptyset$ then $\frown'$ fixes the plane $\varphi(E^{\perp} \cap \mathfrak{H}^+)$ pointwise and $\frown \notin B_h^+$.

c) $B_h = B_h^+ \dot\cup (\frown' \circ B_h^+)$.

This theorem tells us that the proper motion group B_h^+ of the generalized hyperbolic space $\eta(\mathfrak{H}, H, \det, P)$ can also be turned into a kinematic space which is isomorphic to the kinematic space $(PGL(2, KP), \mathfrak{F}'')$ where the fibration $\mathfrak{F}'' := \{F \cap PGL(2, K, P) \mid F \in \mathfrak{F}_m\}$ is also induced by the hyperbolic fibration $\mathfrak{F}_m$ over $(K, +, \cdot)$.

By our definitions $GL(2, K^+) = GL(2, K, H^{(2)}) := \{X \in GL(2, K) \mid \det X \cdot \overline{\det X} \in H^{(2)}\}$ and since for each half-order P of $(H, +, \cdot), H^{(2)} \subset P$ we have $GL(2, K^+) \leq GL(2, K, P)$. Further $GL(2, K, P) = GL(2, K)$ if and only if $z\bar{z} \in P$ for all $z \in K^{\cdot}$. This gives us the results:

(5.2) For the group B_h^+ of all proper motions of the generalized hyperbolic space $\eta(\mathfrak{H}, H, \det, P)$ and the Lorentz group ${}^+L^+$ of $(\mathfrak{H}, H, \det, P)$ we have :

a) $B_h^+ \cong PGL(2, K) \iff \forall z \in K^{\cdot} : z\bar{z} \in P$; in this case B_h^+ has the hyperbolic kinematic fibration $\mathfrak{F}_m$ over K.

b) If P is an order with $-i^2 \in$ P, then $\forall z \in$ K$^{\cdot}$, $z\bar{z} \in$ P, and so $\mathrm{B}_h^+ \cong PGL(2,\mathrm{K})$ and $^+\mathrm{L}^+ \cong PGL(2,\mathrm{K},\mathrm{H}^{(2)})$.

c) If P is an order with $-i^2 \in$ P and $\forall z \in \mathrm{K}^{\cdot}, z\bar{z} \in \mathrm{H}^{(2)}$, then $^+\mathrm{L}^+ \cong \mathrm{B}_h^+ \cong PGL(2,\mathrm{K})$.

d) if $\eta(\mathfrak{H}, \mathrm{H}, \det, \mathrm{P})$ is a hyperbolic space, i.e. $\mathrm{H}^{(2)}$ is an order, then $^+\mathrm{L}^+ \cong \mathrm{B}_h^+ \cong PGL(2,\mathrm{K})$.

Embedding: the hyperbolic space as a generalized kinematic group.

For the last part we assume $-1 \in \mathrm{K}^{(2)}$. We consider the following maps:

$$\begin{aligned}
&\Psi : \mathfrak{M}^\times \to \mathfrak{M}^\times/\mathrm{K}^{\cdot}; \quad \mathrm{X} \to \mathrm{K}^{\cdot}\mathrm{X}\\
&\varphi : \mathfrak{H}^\times \to \mathfrak{H}^\times/\mathrm{H}^{\cdot}; \quad \mathrm{X} \to \mathrm{H}^{\cdot}\mathrm{X}\\
&\tau : \mathfrak{H}^\times/\mathrm{H}^{\cdot} \to \mathfrak{M}^\times/\mathrm{K}^{\cdot}; \quad \mathrm{H}^{\cdot}\mathrm{X} \to \mathrm{K}^{\cdot}\mathrm{X}\\
&\square : \mathfrak{M}^\times \to \mathrm{Map}\,(\mathfrak{M}); \quad \mathrm{A} \to \mathrm{A}^\square = A_\ell \circ (\mathrm{A}^*)_r\\
&\Theta : \mathfrak{M}^\times/\mathrm{K}^\times \to \mathfrak{M}/\mathrm{K}^{\cdot}; \quad \mathrm{x} = \mathrm{K}^{\cdot}\mathrm{X} \to \mathrm{x}^\Theta = \mathrm{K}^{\cdot}\mathrm{X}^*\\
&\boxtimes : \mathfrak{M}^\times/\mathrm{K}^{\cdot} \to \mathrm{Map}\,(\mathfrak{M}^\times/\mathrm{K}^{\cdot}); \quad a \to a^\boxtimes := a_\ell \circ (a^\Theta)_r
\end{aligned}$$

and remark that with respect to the multiplication the set $\mathfrak{M}^\times$ and so the set $\mathfrak{M}^\times/\mathrm{K}^{\cdot}$ are semigroups and also the sets Map$\,(\mathfrak{M})$ resp. Map$(\mathfrak{M}^x/\mathrm{K}^{\cdot})$ of all maps from $\mathfrak{M}$ into $\mathfrak{M}$ resp. $\mathfrak{M}^\times/\mathrm{K}^{\cdot}$ into $\mathfrak{M}^\times/\mathrm{K}^{\cdot}$ form semigroups under composition of maps.

(5.3) Θ is an involutory antiautomorphism, Ψ and $\square|_{GL(2,\mathrm{K})}$ are homomorphisms with the kernels $\mathrm{K}^{\cdot}$ and K_1 respectively and $\boxtimes|_{PGL(2,\mathrm{K})}$ is a monomorphism. Further:

a) $\Psi^{-1}\Psi(\mathfrak{H}^\times) = \{\mathrm{X} \in \mathfrak{M}^\times \mid \mathrm{X}^* \in \mathrm{K}_1\mathrm{X}\}$.

b) $\Psi^{-1}\Psi(\mathfrak{H}^+) = \{\mathrm{X} \in \mathfrak{M}^\times \mid \exists a \in \mathrm{K}^{\cdot} : \mathrm{X}^* = a\bar{a}^{-1}\mathrm{X},\ a^2\mathrm{X}\widehat{\mathrm{X}} \in \mathrm{P}\} \subset GLGL(2,\mathrm{K})$.

c) τ is an injection and $/\mathrm{A} := \tau \circ \varphi(\mathfrak{H}^+) \subset PGL(2,\mathrm{K})$.

d) $\forall \mathrm{A} \in GL(2,\mathrm{K}) : \mathrm{A}^\square(\Psi^{-1}\Psi(\mathfrak{H}^\times)) = \Psi^{-1}\Psi(\mathfrak{H}^\times)$ and if $\mathrm{H}^{\cdot} \subset \mathrm{P}$ then $\mathrm{A}^\square(\Psi^{-1}\Psi(\mathfrak{H}^+)) = \Psi^{-1}\Psi(\mathfrak{H}^+)$.

e) If $\mathrm{H}^{\cdot} \subset \mathrm{P}$ then $\forall a \in PGL(2,\mathrm{K}) : a^\boxtimes(/\mathrm{A}) = /\mathrm{A}$.

Proof. With $* : \mathfrak{M} \to \mathfrak{M}$ also $\Theta : \mathfrak{M}^\times/\mathrm{K}^{\cdot} \to \mathfrak{M}^\times/\mathrm{K}^{\cdot}$ is an involutory antiautomorphism. Since $\mathrm{K}^{\cdot}$ is the center of $(\mathfrak{M}^\times, \cdot)$, Ψ is a homomorphism with the kernel $\mathrm{K}^{\cdot}$. For $\mathrm{A}, \mathrm{B} \in \mathfrak{M}^\times$, $(\mathrm{AB})^\square = (\mathrm{AB})_\ell \circ ((\mathrm{AB})^*)_r = \mathrm{A}_\ell \circ \mathrm{B}_\ell \circ (\mathrm{B}^*\mathrm{A}^*)_r = \mathrm{A}^\square \circ \mathrm{B}^\square$, $\mathrm{A} \in GL(2,\mathrm{K})$ with $\mathrm{A}^\square = \mathrm{A}_\ell \circ \mathrm{A}_r^* = \mathrm{id}$ implies $\mathrm{AEA}^* = \mathrm{E}$, hence $\mathrm{A}^* = \mathrm{A}^{-1}$, and so $\mathrm{AXA}^{-1} = \mathrm{X}$ for all $\mathrm{X} \in \mathfrak{M}$.

Consequently $A \in K$ and $\overline{A}A = A^*A = A^{-1}A = E$ implies $A \in K_1$ is the kernel of $\Box|_{GL(2,K)}$ and this shows that $\boxtimes|_{PGL(2,K)}$ is a monomorphism.
a) For each $X \in \Psi^{-1}\Psi(\mathfrak{H}^\times)$ there is an $a \in K^\cdot$ such that $aX \in \mathfrak{H}$, hence $aX = (aX)^* = \overline{a}X^*$, i.e; $X^* = a\overline{a}^1 X \in K_1 \cdot X$. Now let $X \in \mathfrak{M}^\times$, such that $X^* = \epsilon X$ for some $\epsilon \in K_1$. Then by $\epsilon\overline{\epsilon} = 1$, $((1+\epsilon)X)^* = (1+\overline{\epsilon})X)^* = (1+\overline{\epsilon})\epsilon X = (1+\epsilon)X$, hence $(1+\epsilon)X \in \mathfrak{H}$ and $X \in \Psi^{-1}\Psi(\mathfrak{H}^\times)$ if $\epsilon \neq -1$. If $\epsilon = -1$, then by $-1 \in K^{(2)}$ there is an $i \in K$ with $i^2 = -1$ and this implies $\overline{i} = -i$. Consequently $(iX)^* = \overline{i}X^* = \overline{i}X^* = (-i)(-X) = iX \in \mathfrak{H}$, thus in both cases $X \in \Psi^{(-1)}\Psi(\mathfrak{H}^\times)$.
b) If $X \in \Psi^{-1}\Psi(\mathfrak{H}^+)$, then $aX \in \mathfrak{H}^+$ for some $a \in K^\cdot$ hence $aX \cdot a\widehat{X} = a^2 X\widehat{X} \in P$, and so $X\widehat{X} \neq 0$, i.e. $\Psi^{-1}\Psi(\mathfrak{H}^+) \subset GL(2,K)$.
c) is a onsequence of b).
d) Let $X = bB \in \Psi^{-1}\Psi(\mathfrak{H}^\times)$ with $b \in K^\cdot, B \in \mathfrak{H}^\times$. Then $A^\Box(X) = AbBA^* = bABA^* \in \Psi^{-1}\Psi(\mathfrak{H}^\times)$ since $(ABA^*)^* = AB^*A^*$ by $B^* = B$, and $ABA^*\widehat{(ABA^*)} = ABA^* \cdot \widehat{A}^*\widehat{B}\widehat{A} = A\widehat{A} \cdot \overline{A\widehat{A}} \cdot B\widehat{B} \in H^\cdot \cdot B\widehat{B}$. Hence, if $H^\cdot \subset P$, then: $X \in \Psi^{-1}\Psi(\mathfrak{H}^+) \iff B \in \mathfrak{H}^+ \iff B\widehat{B} \in P \iff A^\Box(B) \in \mathfrak{H}^+ \iff A^\Box(X) \in \Psi^{-1}\Psi(\mathfrak{H}^+)$.
e) Since $/A = \Psi(\mathfrak{H}^+)$, e) is a consequence of d).

The hyperbolic kinematic space $(PGL(2,K), \mathfrak{F}_m)$ can be provided with the *congruence structure* $(K^\cdot A, K^\cdot B) \equiv (K^\cdot C, K^\cdot D) :\iff \varrho(A,B) = \varrho(C,D)$ and the *restricted halforder* α, which is defined on all collinear quadruples (a,b,c,d) of $PGL(2,K)$ where the crossatio $DV(a,b \mid c,d)$ is an element of H; then $[a,b \mid c,d] := \operatorname{sgn} DV(a,b \mid c,d)$.

Now we have the result:

(5.4) Let $-1 \in K^{(2)}$ and $H^\cdot \subset P$. Then $\big(PGL(2,K), \mathfrak{F}_m, \cdot, \boxtimes, \Psi(H^+)\big)$ is a generalized kinematic group and $/A := \Psi(\mathfrak{H}^+)$ is with respect to the trace-structures of incidence, congruence and halforder of the hyperbolic kinematic space $(PGL(2,K), \mathfrak{F}_m, \equiv, \alpha)$ a generalized hyperbolic space,which is isomorphic with $\eta(\mathfrak{H}, H, \det, P)$. If the field H is euclidean then we have the additional properties:

a) $PGL(2,K) = PSL(2,K) \cong SL(2,K)^\Box \cong SL(2,K) \setminus \{E, -E\}$
b) $\varphi(\mathfrak{H}^+) = \varphi(\mathfrak{H}^{++}) = \varphi(\mathfrak{H}_1^{++})$ and $\Psi(\mathfrak{H}^+) = \Psi(\mathfrak{H}^{++}) = \Psi(\mathfrak{H}_1^{++}) \subset PGL(2,K)$ where $\mathfrak{H}_1^{++} := \{X \in \mathfrak{H} \mid X + \widehat{X} > 0,\ X\widehat{X} = 1\} \subset SL(2,K)$.
c) $\varphi|\mathfrak{H}_1^{++}$, $\Psi|\mathfrak{H}_1^{++}$ and $\Box|\mathfrak{H}_1^{++}$ are injective, i.e. $\mathfrak{H}_1^{++}$ can be considered as the point set of the hyperbolic space.
d) $SL(2,K)$ can be provided (as in the papers of H. Hotie [2],[3]) with a *spherical structure*: $(-1)^\cdot SL(2,K) \to SL(2,K)$; $X \to -X$ is the *antipodal*

map. If $\mathfrak{L}_2$ denotes the set of all 2-dimensional vector subspaces of $(\mathfrak{M}, \mathrm{K})$, then $\mathfrak{K} := \{\mathrm{L} \cap SL(2,\mathrm{K}) \mid \mathrm{L} \in \mathfrak{L}_2\}$ is the set of all *great circles*. Then $(SL(2,\mathrm{K}), \mathfrak{K}, (-1)^{\cdot}, \cdot)$ is a two-sided incidence group and $\mathfrak{C} := \{SL(2,\mathrm{K}) \cap (\mathrm{K}+\mathrm{KA}) \mid \mathrm{A} \in SL(2,\mathrm{K}) \setminus \{\mathrm{E}, -\mathrm{E}\}\}$ is a kinematic cover with $\mathrm{D} = \{\mathrm{E}, -\mathrm{E}\}$ and $\mathfrak{K} = \{\mathrm{AC} \mid \mathrm{A} \in SL(2,\mathrm{K}),\ \mathrm{C} \in \mathfrak{C}\}$.
Further, if we provide $SL(2,\mathrm{K})$ with a congruence $\equiv$ by

$$(\mathrm{A},\mathrm{B}) \equiv (\mathrm{C},\mathrm{D}) :\Longleftrightarrow \mathrm{A}\widehat{\mathrm{B}} + \mathrm{B}\widehat{\mathrm{A}} = \mathrm{C}\widehat{\mathrm{D}} + \mathrm{D}\widehat{\mathrm{C}}$$

and an order structure α by
Let $\mathrm{A},\mathrm{B},\mathrm{C} \in SL(2,\mathrm{K})$ such that $\mathrm{A} \neq \mathrm{B}, -\mathrm{B}$, then we say: C lies between A and B if and only if there are $\lambda, \mu \in \mathrm{H}^{(2)}$ such that $\mathrm{C} = \lambda\mathrm{A} + \mu\mathrm{B}$,
then the left-and right-translations of $SL(2,\mathrm{K})$ are motions of $(SL(2,\mathrm{K}), \equiv)$ which preserve α.

e) The structure $(SL(2,\mathrm{K}), \mathfrak{K}, (-1)^{\cdot}, \cdot, \square, \mathfrak{H}_1^{++})$ has the properties: If we restrict the structures $\mathfrak{K}, \square, \alpha$ onto $\mathfrak{H}_1^{++}$ then $\mathfrak{H}_1^{++}$ becomes the 3-dimensional hyperbolic space over H and $SL(2,\mathrm{K})^{\square}$ is the group of all proper motions.

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