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PROPERTIES OF INFINITE HARMONIC FUNCTIONS RELATIVE TO RIEMANNIAN VECTOR FIELDS

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We employ Riemannian jets which are adapted to the Riemannian geometry to obtain the existence-uniqueness of infinite harmonic functions in Riemannian spaces. We then show such functions are equivalent to those that enjoy comparison with Riemannian cones. Using comparison with cones, we show that the Riemannian distance is a supersolution to the infinite Laplace equation, but is not necessarily a solution. We find some geometric conditions under which the Riemannian distance is infinite harmonic and under which it fails to be infinite harmonic.

1. Introduction

In this article, we present an extension of the Euclidean results of Jensen [11] and Juutinen [12] that viscosity infinite harmonic functions exist and are unique. Our approach to the existence-uniqueness of viscosity infinite harmonic functions differs from that of [12] in that we use Riemannian vector fields, which have the added complexity that the second order derivative matrix of a smooth function is not necessarily symmetric. We also use Riemannian jets and the Riemannian maximum principle as detailed in [2]. These extend the Euclidean jets and maximum principle from [9] into Riemannian spaces and employs the natural Riemannian geometry.

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To create a Riemannian space, we begin with $\mathbb{R}^n$ and replace the Euclidean vector fields $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$ with an arbitrary collection of vector fields or *frame*

$$\mathfrak{X} = \{X_1, X_2, \ldots, X_n\}$$

consisting of n linearly independent smooth vector fields

$$X_i(x) = \sum_{j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_j}$$

for some choice of smooth functions $a_{ij}(x)$. Denote by $\mathbb{A}(x)$ the matrix whose (i, j)-entry is $a_{ij}(x)$. We always assume that $\det(\mathbb{A}(x)) \neq 0$ in $\mathbb{R}^n$ and we note that when $\mathbb{A}$ is the identity matrix, we have the Euclidean environment. We fix an inner product $\langle \cdot, \cdot \rangle$ and related norm $\| \cdot \|$ so that this frame is orthonormal. The Riemannian metric $d(x,y)$ is defined by considering this frame to be an orthonormal basis of the tangent space at each point.

The natural gradient is the vector

$$D_{\mathfrak{X}} u = (X_1(u), X_2(u), \ldots, X_n(u))$$

and the natural second derivative is the $n \times n$ *not necessarily symmetric* matrix with entries $X_i(X_j(u))$. Because of the lack of symmetry, we introduce the symmetrized second-order derivative matrix with respect to this frame, given by

$$(D^2_{\mathfrak{X}} u)^\star = \frac{1}{2}(D^2_{\mathfrak{X}} u + (D^2_{\mathfrak{X}} u)^t).$$

With this frame, we may define the infinite Laplace operator

$$\Delta_{\mathfrak{X},\infty} u = \langle (D^2_{\mathfrak{X}} u)^\star D_{\mathfrak{X}} u, D_{\mathfrak{X}} u \rangle$$

and for $2 \leq p < \infty$, we may define the p-Laplace operator

$$\Delta_{\mathfrak{X},p} u = \left[\|D_{\mathfrak{X}} u\|^{p-2} \Delta_{\mathfrak{X}} u + (p-2) \|D_{\mathfrak{X}} u\|^{p-4} \Delta_{\mathfrak{X},\infty} u \right] = \operatorname{div}\, (\|D_{\mathfrak{X}} u\|^{p-2} D_{\mathfrak{X}} u).$$

In addition, we define the Sobolev spaces $W^{1,p}$ and $W^{1,p}_0$ for $1 \leq p \leq \infty$ with respect to this frame in the usual way.

Our results concern viscosity solutions to the infinite Laplace equation. We recall the definition of the Riemannian jets $J^{2,+}_{\mathfrak{X}}$ and $J^{2,-}_{\mathfrak{X}}$. (See [2] for a more complete analysis of such jets.)

Definition 1. Let u be an upper semi-continuous function. Consider the set

$$K^{2,+}_{\mathfrak{X}} u(x) = \Big\{ \varphi \in C^2 \text{ in a neighborhood of } x,\ \varphi(x) = u(x),$$
$$\varphi(y) \geq u(y),\ y \neq x \text{ in a neighborhood of } x \Big\}.$$

Each function $\varphi \in K^{2,+}_{\mathfrak{X}} u(x)$ determines a vector-matrix pair (η, X) via the relations

$$\begin{array}{rcl} \eta & = & \big(X_1\varphi(x), X_2\varphi(x), \ldots, X_n\varphi(x)\big) \\ X_{ij} & = & \frac{1}{2}\big(X_i(X_j(\varphi))(x) + X_j(X_i(\varphi))(x)\big). \end{array} \tag{1}$$

We then define the *second order superjet of u at x* by

$$J^{2,+}_{\mathfrak{X}} u(x) = \{(\eta, X) : \varphi \in K^{2,+} u(x)\},$$

the *second order subjet of u at x* by

$$J^{2,-}_{\mathfrak{X}} u(x) = -J^{2,+}_{\mathfrak{X}}(-u)(x)$$

and the set-theoretic closure

$$\begin{aligned} \overline{J}^{2,+}_{\mathfrak{X}} u(x) = \{(\eta, X) : & \exists \{x_n, \eta_n, X_n\}_{n\in\mathbb{N}} \text{ with } (\eta_n, X_n) \in J^{2,+}_{\mathfrak{X}} u(x_n) \\ & \text{and } (x_n, u(x_n), \eta_n, X_n) \to (x, u(x), \eta, X)\}. \end{aligned}$$

We then use these Riemannian jets to define viscosity infinite harmonic functions as follows:

Definition 2. A lower semi-continuous function v is **viscosity infinite superharmonic** in a bounded domain Ω if $v \not\equiv \infty$ in each component of Ω and for all $x_0 \in \Omega$, whenever $(\eta, \mathscr{Y}) \in \overline{J}^{2,-}_{\mathfrak{X}} v(x_0)$, we have

$$-\langle \mathscr{Y}\eta, \eta\rangle \geq 0.$$

An upper semi-continuous function u is **viscosity infinite subharmonic** in a bounded domain Ω if $u \not\equiv -\infty$ in each component of Ω and for all $x_0 \in \Omega$, whenever $(\eta, \mathscr{X}) \in \overline{J}^{2,+}_{\mathfrak{X}} u(x_0)$, we have

$$-\langle \mathscr{X}\eta, \eta\rangle \leq 0.$$

A **viscosity infinite harmonic** function is both a viscosity infinite subharmonic and a viscosity infinite superharmonic function.

Our main result is the following theorem. (See Definitions 3, 4, and 6 for appropriate definitions.)

Main Theorem. Given a domain Ω and a function u, the following are equivalent.

1. u is a gradient minimizer.

2. u is viscosity infinite harmonic.

3. u is potential infinite harmonic.

4. u enjoys comparison with cones.

In addition, the corresponding "one-sided" statements hold. Namely, the following are equivalent.

(I) u is a gradient sub(super)-minimizer.

(II) u is viscosity infinite sub(super)-harmonic.

(III) u is potential infinite sub(super)-harmonic.

(IV) u enjoys comparison with cones from above(below).

2. Existence-Uniqueness of Infinite Harmonic Functions

In this section, we present an extension of the Euclidean results of Jensen [11] and Juutinen [12] that viscosity infinite harmonic functions exist and are unique in the Euclidean setting. Because of the Riemannian structure, we must use the Riemannian maximum principle as detailed in [2], which extends the Euclidean maximum principle from [9] into Riemannian spaces and employs the natural Riemannian geometry. We note that the results of [2] do not directly apply to our setting, because the infinite Laplace operator is degenerate elliptic but not uniformly elliptic. However, we shall use those results as a key first step in our proof. We combine those results with the Jensen auxiliary functions introduced in [11] in order to prove uniqueness. The existence proof, which constructs a (Riemannian) Lipschitz solution, follows proofs found in [11], [12], [3], and [4] and is omitted.

In order to prove uniqueness, we shall use the Riemannian Maximum Principle proved in [2]. For our purposes, we will need the following corollary to the Maximum Principle.

Corollary 2.1. *Let u be an upper semi-continuous function in a bounded domain $\Omega \subset \mathbb{R}^n$ and let v be a lower semi-continuous function in Ω. Let one of u or v be locally Lipschitz. Then for the vectors η_τ^+ and η_τ^- and the matrices $\mathscr{X}_\tau$ and $\mathscr{Y}_\tau$ in the Riemannian maximum principle, we have*

$$\begin{aligned} \|\eta_\tau^+\|^2 - \|\eta_\tau^-\|^2 &= o(1) \\ \textit{and} \quad \langle \mathscr{X}_\tau \eta_\tau^+, \eta_\tau^+ \rangle - \langle \mathscr{Y}_\tau \eta_\tau^-, \eta_\tau^- \rangle &= o(1). \end{aligned}$$

Proof. We may assume without loss of generality that u is locally Lipschitz. We then obtain

$$u(x) - v(y) - \tau\psi(x,y) \leq u(x_\tau) - v(y_\tau) - \tau\psi(x_\tau, y_\tau)$$

where $\psi(x,y) = |x-y|^2$, the square of the Euclidean distance between points x and y. Setting $x = y = y_\tau$, we have

$$\tau\psi(x_\tau, y_\tau) \leq u(x_\tau) - u(y_\tau) \leq Cd(x_\tau, y_\tau) \leq C(\psi(x_\tau, y_\tau))^{\frac{1}{2}}. \tag{2}$$

Before continuing with the proof, we recall from [9] that there exist $n \times n$ symmetric matrices X_τ, Y_τ so that

$$(\tau D_x(\psi(x_\tau, y_\tau)), X_\tau) \in \overline{J}^{2,+}_{\text{eucl}}\, u(x_\tau)$$

and

$$(-\tau D_y(\psi(x_\tau, y_\tau)), Y_\tau) \in \overline{J}^{2,-}_{\text{eucl}}\, v(y_\tau)$$

with the property

$$\langle X_\tau \gamma, \gamma\rangle - \langle Y_\tau \chi, \chi\rangle \leq \langle C\gamma \oplus \chi,\ \gamma \oplus \chi\rangle \tag{3}$$

where the vectors $\gamma, \chi \in \mathbb{R}^n$, and

$$C = \tau(A^2 + A) \ \text{ and } \ A = D^2_{x,y}(\psi(x_\tau, y_\tau))$$

are $2n \times 2n$ matrices. These Euclidean jets are related to the Riemannian jets via the twisting lemma given by

Lemma 2.2. *[2, Lemma 3] For smooth functions u we have*

$$D_{\mathfrak{X}} u(x) = \mathbb{A}(x) \cdot \nabla_{eucl} u(x),$$

and for all $t \in \mathbb{R}^n$

$$\begin{aligned}\langle \left(D^2_{\mathfrak{X}} u(x)\right)^{*} \cdot t, t\rangle = \\ \langle \mathbb{A}(x) \cdot D^2 u(x) \cdot \mathbb{A}^t(x) \cdot t, t\rangle + \sum_{k=1}^{n} \langle \mathbb{A}^t(x) \cdot t, D\left(\mathbb{A}^t(x) \cdot t\right)_k\rangle \frac{\partial u}{\partial x_k}(x)\end{aligned}$$

where D^2u is the Euclidean second derivative matrix and D represents Euclidean differentiation.

In the computations that follow, we shall denote the evaluations of the summand derivatives $D(\mathbb{A}^t(x) \cdot t)_k$ at the point x_0 and for the vector $\eta \in \mathbb{R}^n$ by $D(\mathbb{A}^t(x) \cdot t)_k[x_0, \eta]$.

Proceeding with the proof and using the notation of [2], we define the vector $\xi_\tau = \tau D_x(\psi(x_\tau, y_\tau))$ where D_x denotes (Euclidean) differentiation with respect to the components of the point x. For the vectors, because Ω is a bounded domain and $\mathbb{A}$ is a matrix with smooth entries (as defined in Section 1), we have

$$\begin{aligned}
\|\eta_\tau^+\|^2 - \|\eta_\tau^-\|^2 &= (\|\eta_\tau^+\| - \|\eta_\tau^-\|)(\|\eta_\tau^+\| + \|\eta_\tau^-\|) \\
&\le \|\eta_\tau^+ - \eta_\tau^-\|(\|\mathbb{A}(x_\tau)\xi_\tau\| + \|\mathbb{A}(y_\tau)\xi_\tau\|) \\
&\le (C_1\tau\psi(x_\tau,y_\tau))(\|\mathbb{A}(x_\tau)\| + \|\mathbb{A}(y_\tau)\|)\|\xi_\tau\| \\
&\le C_2(\psi(x_\tau,y_\tau))^{\frac{1}{2}}\tau(\psi(x_\tau,y_\tau))^{\frac{1}{2}}
\end{aligned}$$

and the result follows from the fact that $\tau(\psi(x_\tau,y_\tau)) \to 0$ as $\tau \to \infty$ [2].

We then obtain

$$\begin{aligned}
&\langle \mathscr{X}_\tau\eta_\tau^+, \eta_\tau^+\rangle - \langle \mathscr{Y}_\tau\eta_\tau^-, \eta_\tau^-\rangle = \\
&\quad \langle X_\tau\mathbb{A}^t(x_\tau)\eta_\tau^+, \mathbb{A}^t(x_\tau)\eta_\tau^+\rangle - \langle Y_\tau\mathbb{A}^t(y_\tau)\eta_\tau^-, \mathbb{A}^t(y_\tau)\eta_\tau^-\rangle \\
&\quad + \sum_{k=1}^n \langle \mathbb{A}^t(x_\tau)\eta_\tau^+, D(\mathbb{A}^t(p)\cdot t)_k[x_\tau,\eta_\tau^+]\rangle(\xi_\tau)_k \\
&\quad + \sum_{k=1}^n \langle \mathbb{A}^t(y_\tau)\eta_\tau^-, D(\mathbb{A}^t(q)\cdot t)_k[y_\tau,\eta_\tau^-]\rangle(\xi_\tau)_k
\end{aligned}$$

so that we obtain for the matrix $\mathscr{C}$ as in [9]

$$\begin{aligned}
&\langle \mathscr{X}_\tau\eta_\tau^+, \eta_\tau^+\rangle - \langle \mathscr{Y}_\tau\eta_\tau^-, \eta_\tau^-\rangle \le \\
&\quad \langle \mathscr{C}(\mathbb{A}^t(x_\tau)\eta_\tau^+ \oplus \mathbb{A}^t(y_\tau)\eta_\tau^-), \mathbb{A}^t(x_\tau)\eta_\tau^+ \oplus \mathbb{A}^t(y_\tau)\eta_\tau^-\rangle \\
&\quad + \tau\sum_{k=1}^n \langle \mathbb{A}^t(x_\tau)\eta_\tau^+, D(\mathbb{A}^t(x)\cdot t)_k[x_\tau,\eta_\tau^+]\rangle \frac{\partial\psi}{\partial x_k}(x_\tau,y_\tau) \\
&\quad - \tau\sum_{k=1}^n \langle \mathbb{A}^t(y_\tau)\eta_\tau^-, D(\mathbb{A}^t(y)\cdot t)_k[y_\tau,\eta_\tau^-]\rangle \frac{\partial\psi}{\partial x_k}(x_\tau,y_\tau) \\
&= I + II + III.
\end{aligned}$$

We first observe that term I is controlled by

$$C\tau\|\mathbb{A}^t(x_\tau)\eta_\tau^+ - \mathbb{A}^t(y_\tau)\eta_\tau^-\|^2.$$

Using the definition of η_τ^+ and η_τ^- and the fact that

$$\frac{\partial\psi}{\partial x_k}(x_\tau,y_\tau) \le C(\psi(x_\tau,y_\tau))^{\frac{1}{2}}$$

we have that term I is controlled by

$$C\tau^3\psi(x_\tau,y_\tau)\|\mathbb{A}^t(x_\tau)\mathbb{A}(x_\tau) - \mathbb{A}^t(y_\tau)\mathbb{A}^t(y_\tau)\|^2.$$

By the smoothness of the entries of the matrix $\mathbb{A}$, we conclude that term I is dominated by $C\tau^3\psi(x_\tau,y_\tau)^2$. From Equation (2), we conclude term I is $o(1)$.

Similarly, the difference of the last two terms is controlled by

$$C\tau(\psi(x_\tau,y_\tau))^{\frac{1}{2}}\sum_{k=1}^{n}\tau^2(\psi(x_\tau,y_\tau))|x_\tau-y_\tau|$$

which is $o(1)$ by Equation (2). The corollary follows. □

Using the Jensen auxiliary functions [11] and following the techniques of [11], [12] and [4], we have the following theorem, giving uniqueness of viscosity infinite harmonic functions.

Theorem 2.3. *Let u be a viscosity infinite subharmonic function and v be a viscosity infinite superharmonic function in a domain Ω such that if $x\in\partial\Omega$,*

$$\limsup_{y\to x} u(y)\le\limsup_{y\to x} v(y)$$

where both sides are not $-\infty$ or $+\infty$ simultaneously. Then $u\le v$ in Ω.

We now present two corollaries to the comparison principle which will be useful in the sequel. We first begin with a definition.

Definition 3. A lower semi-continuous function $u:\Omega\to\mathbb{R}\cup\{\infty\}$ that is not identically infinity in each component of Ω is **potential infinite superharmonic** if for each open set $U\subset\subset\Omega$ and each viscosity infinite harmonic function f defined on U,

$$u\ge f \text{ on } \quad \partial U\Rightarrow u\ge f \text{ in } \quad U.$$

A function u is **potential infinite subharmonic** if $-u$ is potential infinite superharmonic. A function u is **potential infinite harmonic** if it is both potential infinite subharmonic and superharmonic.

We then have the first corollary, whose proof follows that of Theorem 5.8 of [12] and is omitted.

Corollary 2.4. *A function u is viscosity infinite subharmonic if and only if it is potential infinite subharmonic. A function u is viscosity infinite superharmonic if and only if it is potential infinite superharmonic. A function is viscosity infinite harmonic if and only if it is potential infinite harmonic.*

Next, we recall the definition of gradient minimizers.

Definition 4. The function $u \in \text{Lip}(\overline{\Omega})$ is an *gradient minimizer* if for every $V \subset \Omega$ and $v \in \text{Lip}(V)$, such that $u = v$ on ∂V, then

$$\|D_{\mathfrak{X}}u\|_{L^\infty(V)} \leq \|D_{\mathfrak{X}}v\|_{L^\infty(V)}.$$

The function $u \in \text{Lip}(\overline{\Omega})$ is an *gradient super-minimizer* if the above holds for $v \geq u$. The function $u \in \text{Lip}(\overline{\Omega})$ is an *gradient sub-minimizer* if the above holds for $v \leq u$.

It is clear from the definition that a function is a gradient minimizer means it is both a gradient sub-minimizer and gradient super-minimizer. The converse implication can be found in Section 4.3 of [1]. Also, a function u is a sub-minimizer exactly when $-u$ is a super-minimizer. It was shown in [6] (for C^1) and [16] (for arbitrary) that gradient minimizers in Riemannian spaces are viscosity infinite harmonic functions. In particular, the proofs show that a gradient sub(super)-minimizer is viscosity infinite sub(super)-harmonic.

We also have the following corollary, whose proof is similar to that in [14] and omitted.

Corollary 2.5. *Given a domain Ω, let u be viscosity infinite harmonic in Ω. Then $u \in W^{1,\infty}_{loc}$ and it is a gradient minimizer with respect to its trace.*

We summarize our results of this section, which is most of the Main Theorem:

Theorem 2.6. *Given a domain Ω and a function u, the following are equivalent.*

1. *u is a gradient minimizer.*

2. *u is viscosity infinite harmonic.*

3. *u is potential infinite harmonic.*

We also have what [1] refers to as "one-sided results". Namely,

Theorem 2.7. *Given a domain Ω and a gradient sub(super)-minimizer u, then u is viscosity sub(super)-harmonic. In addition, u is viscosity sub(super)-harmonic if and only if it is potential infinite sub(super)-harmonic.*

3. Riemannian Cones

In this section, we discuss Riemannian cones and extend results found in [14] and [1] including the important property of comparison with cones. In the Euclidean case, functions enjoying comparison with cones were shown to be exactly those that were viscosity infinite harmonic [8] and we extend this result to Riemannian spaces. This will complete the proof of the Main Theorem.

We begin with the definition of Riemannian cones, recalling that the Riemannian distance between points x and y is denoted $d(x,y)$.

Definition 5. Let $a,b \in \mathbb{R}$. Given a point x and an open set U, we define the function $\theta : \partial(U \setminus \{x\}) \to \mathbb{R}$ by

$$\theta(y) = a + b\, d(x,y).$$

The Riemannian cone based on (U,x) is the unique viscosity infinite harmonic function $\omega_{U,x}^{a,b}$ in $U \setminus \{x\}$ such that

$$\omega_{U,x}^{a,b} = \theta \ \text{ on } \ \partial(U \setminus \{x\}).$$

Note that for sufficiently large p, a point has positive p-capacity. In addition, the previous sections show the existence-uniqueness of Riemannian cones. Thus, they are well-defined.

To obtain bounds on cones, we state a result of Monti and Serra-Cassano [15].

Theorem 3.1. *Given $y \in \mathbb{R}^n$, for almost every $x \in \mathbb{R}^n$, we have*

$$\|D_{\mathfrak{X}} d(x,y)\| \leq 1.$$

We then prove the following.

Proposition 3.2. *Given a pair (U,x) and $a,b \in \mathbb{R}$, the cone $\omega_{U,x}^{a,b}$ satisfies*

$$a - |b|\, d(x,y) \leq \omega_{U,x}^{a,b}(y) \leq a + |b|\, d(x,y)$$

for $y \in U$.

Proof. If $x \in \overline{U}$, we compute

$$\omega_{U,x}^{a,b}(y) - a = \omega_{U,x}^{a,b}(y) - \omega_{U,x}^{a,b}(x) \leq \|D_{\mathfrak{X}} \omega_{U,x}^{a,b}\|_{L^\infty(U)} d(x,y).$$

The cones, as viscosity infinite harmonic functions, are gradient minimizers. Thus,

$$\|D_{\mathfrak{X}} \omega_{U,p}^{a,b}\|_{L^\infty(U)} \leq \|b D_{\mathfrak{X}} d(x,y)\,\|_{L^\infty(U)} \leq |b|.$$

We also note that we have

$$\omega_{U,x}^{a,b}(y) - a = \omega_{U,x}^{a,b}(y) - \omega_{U,x}^{a,b}(x) \geq -\|D_{\mathfrak{X}} \omega_{U,x}^{a,b}\|_{L^\infty(U)} d(x,y).$$

Similarly, we obtain

$$\omega_{U,x}^{a,b}(y) - a \geq -|b| d(x,y).$$

If $x \notin \overline{U}$, let γ be the geodesic between x and y and observe that there is a point $y' \in \partial U \cap \gamma$. Thus,

$$d(x,y) = d(x,y') + d(y',y).$$

In particular, we have

$$\omega_{U,x}^{a,b}(y) - a - bd(x,y') = \omega_{U,x}^{a,b}(y) - \omega_{U,x}^{a,b}(y') \leq \|D_{\mathfrak{X}}\omega_{U,x}^{a,b}\|_{L^\infty(U)} d(y,y').$$

As above, we then obtain

$$\omega_{U,x}^{a,b}(y) \leq a + bd(x,y') + |b|d(y,y') \leq a + |b|d(x,y') + |b|d(y,y').$$

We also then have

$$\omega_{U,x}^{a,b}(y) - a - bd(x,y') = \omega_{U,x}^{a,b}(y) - \omega_{U,x}^{a,b}(y') \geq -\|D_{\mathfrak{X}}\omega_{U,x}^{a,b}\|_{L^\infty(U)} d(y,y').$$

We then conclude

$$\omega_{U,x}^{a,b}(y) \geq a - |b|d(x,y') - |b|d(y,y').$$

The results then follow from our choice of y'. $\square$

Having analyzed the Riemannian cones we now define the concept of comparison with cones, analogous to the Euclidean case [8].

Definition 6. Let $U \subset \mathbb{R}^n$ be an open set, and let $u : U \to \mathbb{R}$. Then u enjoys comparison with cones from above in U if for every open $V \subset U$ and $a,b \in \mathbb{R}$ for which $u \leq \omega_{U,x}^{a,b}$ holds on $\partial(V \setminus \{x\})$, then we have $u \leq \omega_{U,x}^{a,b}$ in V. A similar definition holds for the function u enjoying comparison with cones from below in U. The function u enjoys comparison with cones in U exactly when it enjoys comparison with cones from above and below.

With these definitions, we obtain the following implication.

Lemma 3.3. *A viscosity infinite superharmonic function in U enjoys comparison with cones from below in U. Similarly, a viscosity infinite subharmonic function enjoys comparison with cones from above in U and a viscosity infinite harmonic function enjoys comparison with cones in U.*

In order to relate the property of comparison with cones with the previous properties, we must proceed through a series of propositions. We again focus on "one-sided results" and so we shall focus our attention on comparison with cones from above and follow the general scheme of [1]. The two main technical issues in extending these results are the differences between Euclidean and Riemannian geometry and the lack of an explicit formula for the Riemannian cones. The proofs below highlight where these issues appear.

Proposition 3.4. *Let u be an upper semi-continuous function in a domain U that enjoys comparison with cones from above. For a point $y \in U$, define the function $S(y,r)$ by*

$$S(y,r) \equiv \max\{\frac{u(w)-u(y)}{r} : d(w,y) = r\}.$$

Then, we have

1. $$\max\{u(x) : d(x,y) = r\} = \max\{u(x) : d(x,y) \leq r\}.$$

2. $$u(x) \leq u(y) + S(y,r)d(x,y)$$

   for points $x \in U$ so that $d(x,y) \leq r$ with $0 \leq r < d(y, \partial U)$

3. *$S(y,r)$ is monotonic and non-negative*

4. *$u \in W^{1,\infty}_{loc}(U)$*

Proof. To prove Equation (1), we note that the cone with boundary data $M \equiv \max\{u(x) : d(x,y) = r\}$ on $B_r(y)$ is the constant M itself. Since $u \leq M$ on $\partial B_r(y)$, comparison with cones from above implies

$$u \leq M \text{ in } B_r(y).$$

Equation (1) follows. To prove Equation (2), we notice that it holds when $d(x,y) = 0$ and when $d(x,y) = r$. Using $u(y) + S(y,r)d(x,y)$ as the boundary data for the cone ω, we have by comparison with cones from above that

$$u(x) \leq \omega(x).$$

Equation (2) then follows from Proposition 3.2. The proof of statement 3 is identical to Lemma 2.4 of [8] and the proof of statement 4 is identical to Lemma 2.5 of [8] and therefore omitted. □

In light of the previous proposition, it is reasonable to define the function $S(y)$ by

$$S(y) \equiv \lim_{r \to 0} S(y,r) = \inf\{S(y,r) : 0 < r < d(y, \partial U)\}.$$

In addition, if we let $L_u(U)$ denote the smallest constant L so that

$$|u(x) - u(y)| \leq L\, d(x,y)$$

for $x,y \in U$, then we may define the similar function $T_u(x)$ by

$$T_u(x) \equiv \lim_{r\to 0} L_u(B_r(x)) = \inf\{L_u(B_r(x)) : 0 < r < d(x,\partial U)\}.$$

We note that this is well-defined on the extended reals and if u is Lipschitz, we have

$$\sup_{x\in U} T_u(x) \equiv \|D_{\mathfrak{X}} u\|_{L^\infty}.$$

We then have the following proposition.

Proposition 3.5. *Let u be as in the previous proposition. Then, we have*

1. *$T_u(x)$ is upper semi-continuous.*
2. *Let $x,y \in U$ so that $\gamma_{xy} \subset U$ where γ_{xy} is the geodesic between x and y with $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$. Then,*

$$|u(x) - u(y)| \leq (\max\{S(w) : w \in \gamma_{xy}\})d(x,y).$$

3. *$T_u(y) = S(y)$*

Proof. The proof of the first statement can be found in Section 1.5 of [1]. The next two statements follow from Lemma 2.15 in [1] and the fact that along a Riemannian geodesic, we have

$$d(x,\gamma_{xy}(t)) = td(x,y).$$

□

Lastly, we state the key technical lemma, given as Proposition 4.7 in [1]. The proof is omitted.

Lemma 3.6. *Let U be bounded and let $u \in C(\overline{U})$ enjoy comparison with cones from above. Let $x_0 \in U$ so that $S(x_0) > 0$ and let $\delta > 0$. Then, there is a sequence of points $\{x_j\} \subset U$ and a point $x_\infty \in \partial U$ so that*

1. *$d(x_j,x_{j-1}) \leq \delta$*
2. *γ_j, the geodesic between x_j and x_{j-1} is contained in U*
3. *$S(x_j) \geq S(x_{j-1})$*
4. *$\lim_{j\to\infty} x_j = x_\infty$*
5. *$u(x_\infty) - u(x_0) \geq S(x_0)\sum_{j=1}^{\infty} d(x_j,x_{j-1})$.*

We now have the technical information needed to prove the following Theorem.

Theorem 3.7. *A function that enjoys comparison with cones from above is a gradient sub-minimizer. A function that enjoys comparison with cones from below is a gradient super-minimizer. A function that enjoys comparison with cones is a gradient minimizer.*

Proof. Again, the last statement follows from the first two, and the first two are symmetric, so we shall prove only the first statement.

We suppose that u enjoys comparison with cones from above and is not a gradient sub-minimizer in the domain U. Let v be a Lipschitz function so that $v \leq u$ in U and $u = v$ on ∂U. Suppose that

$$\|D_{\mathfrak{X}} u\|_{L^\infty(V)} > \|D_{\mathfrak{X}} v\|_{L^\infty(V)}.$$

This is equivalent to

$$\sup\{T_u(x) : x \in U\} > \sup\{T_v(x) : x \in U\}.$$

Thus, there is a $x_0 \in U$ so that

$$T_u(x_0) > \sup\{T_v(x) : x \in U\}.$$

Using the Lemmas above, we then have

$$\begin{aligned}
u(x_\infty) - u(x_0) &\geq S(x_0) \sum_{j=1}^{\infty} d(x_j, x_{j-1}) = T_u(x_0) \sum_{j=1}^{\infty} d(x_j, x_{j-1}) \\
&> \sup\{T_v(x) : x \in U\} \sum_{j=1}^{\infty} d(x_j, x_{j-1}) \geq \sum_{j=1}^{\infty} |v(x_j) - v(x_{j-1})| \\
&\geq \sum_{j=1}^{\infty} (v(x_j) - v(x_{j-1})) = v(x_\infty) - v(x_0).
\end{aligned}$$

Since $x_\infty \in \partial U$ and $v = u$ on ∂U, we then obtain

$$u(x_0) < v(x_0)$$

contradicting the fact that $v \leq u$ in U. Our supposition is therefore false, and u is a gradient sub-minimizer. □

We summarize the equivalence results with the following theorem.

Main Theorem. Given a domain Ω and a function u, the following are equivalent.

1. u is a gradient minimizer
2. u is viscosity infinite harmonic
3. u is potential infinite harmonic
4. u enjoys comparison with cones.

In addition, the corresponding "one-sided" statements hold. Namely, the following are equivalent.

(I) u is a gradient sub(super)-minimizer.

(II) u is viscosity infinite sub(super)-harmonic.

(III) u is potential infinite sub(super)-harmonic.

(IV) u enjoys comparison with cones from above(below).

Corollary 3.8. *Let $a,b \in \mathbb{R}$ with $b \geq 0$. Let U be a domain and x an arbitrary point. Define the function $D : U \to \mathbb{R}$ by $d(y) = a + b\, d(x,y)$. Then $D(y)$ is a viscosity infinite superharmonic function. In particular, the distance function is a viscosity infinite superharmonic function. By symmetry, $D_{-}(y) \equiv a - b\, d(x,y)$ is a viscosity infinite subharmonic function.*

Proof. We will show the function $D(y)$ enjoys comparison with cones from below. Let U, x, a, b and $D(y)$ be as above. Let ω_D be the Riemannian cone equal to $D(y)$ on $\partial(U \setminus \{x\})$. Suppose the cone ω has the property that $\omega \leq \omega_D$ on ∂U. Then, by the comparison principle and Proposition 3.2,

$$\omega \leq \omega_D \leq a + b\, d(x,y) = D(y)$$

in U. □

4. Geometry of Riemannian Cones

In the Euclidean environment, it is well-known ([8], [1]) that the functions $D(y)$ are also viscosity infinite subharmonic functions and thus the Euclidean cones are exactly those functions $D(y)$. By symmetry, the restriction that $b \geq 0$ can be removed, and the result holds for all $D(y)$. Due to the richness of the geometry, the analogous result in the Riemannian environment does not necessarily hold. Before beginning our study of this phenomenon, we will explore two examples concerning the Riemann sphere.

Example 4.1. Consider the Riemann sphere with the spherical metric so that the geodesics are arcs of great circles. Let the domain U be the southern hemisphere and fix the point x as the north pole. Then, on the boundary of U, $D(y)$ equals a fixed constant D. Having constant boundary data, the corresponding cone is the constant D. Clearly, we have $D < D(y)$ in U and $D = D(y)$ on ∂U. We note that the interior points are farther away from the north pole than the boundary points. We also note that geodesics from the north pole to any boundary point do not intersect the interior.

Example 4.2. Again consider the Riemann sphere as in Example 4.1. Let $\mathscr{U}$ be the southern hemisphere joined with a semicircle lying in the northern hemisphere whose diameter is on the equator. We make the radius of the semicircle one eighth the distance from the equator to the north pole, so that it does not intersect the north pole and it is contained entirely within the western hemisphere. (For example, $\mathscr{U}$ is the southern hemisphere of the Earth, plus South America.) The point x is again the north pole. We see that geodesics from the north pole to $\partial\mathscr{U}$ do not intersect $\mathscr{U}$. We also note that the northern hemisphere points of $\mathscr{U}$ are closer to x than some boundary points of $\mathscr{U}$ while the southern hemisphere points of $\mathscr{U}$ are further away from the north pole than any boundary point of $\mathscr{U}$.

These examples motivate the following definitions concerning points in a domain U.

Definition 7. 1 Let U be a bounded domain, and x an arbitrary point. A point $y \in U$ is **geodesically near with respect to the point x** if

$$y \in \Lambda = \{ \bigcup_{z \in \partial U} \gamma : \gamma \text{ is a geodesic between } x \text{ and } z \}.$$

2 A point $y \in U$ that is not geodesically near is **geodesically far with respect to the point x**. That is, $y \notin \Lambda$.

3 A point $y \in U$ is **boundary near with respect to the point x** if there exists $z \in \partial U$ so that

$$d(x,y) < d(x,z).$$

4 A point $y \in U$ that is not boundary near is **boundary far with respect to the point x**. That is, for all $z \in \partial U$, we have

$$d(x,y) \geq d(x,z).$$

We drop the phrase "with respect to x" in these definitions when the point x is understood.

We first note that because geodesics need not be unique, the set Λ actually includes *all* geodesics between points x and z. Points that are geodesically near with respect to x lie on *some* geodesic from x to the boundary point z. Additionally, it is clear that geodesically near implies boundary near, or equivalently, boundary far implies geodesically far.

We next note that in Example 4.1, it is clear that no interior point is boundary near and no interior point is geodesically near. However, in Example 4.2, the northern hemisphere points (lying in the semicircle) are geodesically far, but not boundary far. This differs from the Euclidean environment, in which all interior points are boundary near and geodesically near. (See [1] for a more complete discussion.)

In Example 4.1, the cone boundary data is constant, which leads us to first consider cones with constant boundary data. Here again, we stray from the Euclidean environment, in which bounded domains with $b \neq 0$ produce non-constant cone boundary data. In particular, when in the Euclidean environment, given $b \neq 0$, a vertex point x, and a bounded domain Ω with $D(y)$ constant on $\partial\Omega$, we conclude that Ω is an n-dimensional ball with x as the center. Using the definition of cones from Definition 5, the boundary used to construct the cone is $\partial(\Omega \setminus \{x\}) = \partial\Omega \cup \{x\}$. Since $D(x) = a \neq D(y)$ for any $y \in \partial\Omega$, the cone boundary data is non-constant. This differs from Example 4.1, where we have constant cone boundary data when $b \neq 0$ and we have $\partial(\Omega \setminus \{x\}) = \partial\Omega$, because $x \notin \overline{\Omega}$.

Returning to the Riemannian environment, when $b = 0$ we have $\omega_D(y) = D(y) = a$ for all points y in any bounded domain U. In the case when $b > 0$, the constant boundary data and uniqueness of the cones produces the constant cone ω_D. We have the following theorem concerning constant Riemannian cones when $b > 0$.

Theorem 4.3. *Let U be a bounded domain, $x \in \mathbb{R}^n$, and $a, b \in \mathbb{R}$ with $b > 0$. Define $D(y) = a + b\, d(x,y)$ as above. Suppose $D(z) = K$ for $z \in \partial(U \setminus \{x\})$ for some constant K. Let ω_D be the (constant) Riemannian cone with boundary data K. Then the point $y \in U$ is boundary far with respect to x exactly when $\omega_D(y) < D(y)$.*

Proof. Suppose that y is boundary far with respect to x. Because y is an interior point to $U \setminus \{x\}$, there is an $r > 0$ so that the ball $B_r(y) \subset (U \setminus \{x\})$. Let γ be a geodesic from x to y. Then, there is a point $x^\star \in (B_r(y) \setminus \{y\}) \cap \gamma$ with the property

$$d(x,y) = d(x,x^\star) + d(x^\star,y).$$

Using this property, we see that $D(y) > D(x^\star)$. We would then have

$$\omega_D(y) = K = \omega_D(x^\star) \leq D(x^\star) < D(y).$$

We note that the penultimate inequality is a consequence of Proposition 3.2.

Suppose next that $\omega_D(y) < D(y)$. Then by Proposition 3.2, we have

$$K = \omega_D(y) < D(y).$$

That is, for any $z \in \partial(U \setminus \{x\})$,

$$a + b\, d(x,z) < a + b\, d(x,y).$$

Because $b > 0$, we conclude that y is boundary far with respect to x. □

The case of non-constant cones is more involved. We have the following partial result that parallels the constant case.

Theorem 4.4. *Let U, x, a, b be as in Theorem 4.3. Suppose that $D(z)$ is non-constant on $\partial(U \setminus \{x\})$ and let ω_D be the (non-constant) Riemannian cone with boundary data $D(z)$. Then we have the implications*

$$\begin{gathered} y \text{ is boundary far with respect to } x \Rightarrow \\ \omega_D(y) < D(y) \Rightarrow y \text{ is geodesically far with respect to } x. \end{gathered}$$

Proof. We first observe that as a non-constant (continuous) infinite harmonic function on a compact set, we have that ω_D achieves its maximum on $\overline{U}$. A consequence of the Harnack inequality [14] is the strong maximum principle which states that this maximum occurs only on the boundary.

Now assume that y is boundary far. Suppose $\omega_D(y) = D(y)$. Because y is boundary far and $b > 0$, for all $z \in \partial(U \setminus \{x\})$ we have $D(y) \geq D(z)$. That is,

$$\omega_D(y) \geq \omega_D(z)$$

for all $z \in \partial(U \setminus \{x\})$. This contradicts the fact that the maximum of ω_D occurs only on the boundary. We conclude that $\omega_D(y) < D(y)$.

Next, suppose that there is a point y so that $\omega_D(y) < D(y)$ and let γ be a geodesic from x to $z \in \partial(U \setminus \{x\})$ with $\gamma(0) = x, \gamma(1) = z$, and for some $t_0 \in (0,1)$, $\gamma(t_0) = y$. Then

$$d(x,z) = d(x,y) + d(y,z).$$

We then have

$$\begin{aligned} \omega_D(z) - \omega_D(y) &= D(z) - \omega_D(y) > D(z) - D(y) \\ &= b\big(d(x,z) - d(x,y)\big) = bd(y,z) \end{aligned}$$

and so the Lipschitz constant of ω_D in $\overline{U}$ is strictly larger than b. Since ω_D is a gradient minimizer of its trace, which equals $D(\cdot)$, we know

$$\|D_{\mathfrak{X}}\omega_D\|_{L^\infty(\overline{U})} \leq b.$$

We then conclude that t_0 does not exist. □

Ideally, we would like to prove both converse implications of the above theorem. This, however, is not possible, since if both converse statements are true, we would have proved that geodesically far points are boundary far, which is shown to be false by Example 4.2. We conclude that the converse statements can not both be true. We then have the following Lemma motivated by Example 4.2 showing that boundary near need not imply equality.

Lemma 4.5. *Let $U, x, a, b, d(y)$ and $\omega_D(y)$ be as in Theorem 4.4. Additionally, suppose U has points that are boundary far with respect to x and points that are boundary near with respect to x. Suppose that at least one boundary far point in U is the limit point of a sequence of boundary near points in U. Then there exists a point $y \in U$ that is boundary near with $\omega_D(y) < D(y)$. Thus, $\omega_D(y) < D(y)$ does not necessarily imply that y is boundary far.*

Proof. Suppose that $\omega_D(y) < D(y)$ implies y is boundary far. Then the logically equivalent implication that y is boundary near implies $\omega_D(y) = D(y)$ would be true. We will show, however, that the latter implication is false.

By assumption, we may construct a sequence $\{y_n\}_{n\in\mathbb{N}}$ of points in U that are boundary near with respect to x and converge to the point $y \in U$ that is boundary far with respect to x. By our assumption, we have $\omega_D(y_n) = D(y_n)$. By continuity of the cone function, this implies $\omega_D(y) = D(y)$. However, y is boundary far, and so Theorem 4.4, which showed that $\omega_d(y) < d(y)$, is contradicted. $\square$

It is an open problem to explore the last geometric condition further in order to determine precisely the conditions for equality. Unlike the Euclidean case, we are hampered by a lack of explicit formulas for both the Riemannian cones and the Riemannian geodesics.

REFERENCES

[1] G. Aronsson - M- Crandall - P. Juutinen. *A Tour of the Theory of gradiently Minimizing Functions*, Bull. AMS. *41* (4) (2004), 439–505.

[2] F. Beatrous - T. Bieske - J. Manfredi, *The Maximum Principle for Vector Fields. The p-harmonic equation and recent advances in analysis*, Contemp. Math., 370, Amer. Math. Soc., Providence, RI, 2005, 1–9.

[3] T. Bhatthacharya - E. DiBenedetto - J. Manfredi, *Limits as $p \to \infty$ of $\Delta_p u_p = f$ and Related Extremal Problems*, Rend. Sem. Mat. Univ. Politec. Torino 1989 Special Issue (1991), 15–68.

[4] T. Bieske, *On Infinite Harmonic Functions on the Heisenberg Group*, Comm. in PDE. *27* (3&4) (2002), 727–762.

[5] T. Bieske, *Lipschitz extensions on Grushin-type spaces*, Mich Math J. *53* (1) (2005), 3–31.

[6] T. Bieske - L. Capogna, *The Aronsson-Euler Equation for gradiently Minimizing Lipschitz Extensions with respect to Carnot-Carathéodory Metrics*, Trans. Amer. Math. Soc. *357* (2) (2005), 795–823.

[7] M. Crandall, *Viscosity Solutions: A Primer*; Lecture Notes in Mathematics 1660; Springer-Verlag: Berlin, 1997.

[8] M. Crandall - L. C. Evans - R. Gariepy, *Optimal Lipschitz Extensions and the Infinity Laplacian*, Calc. Var. *13* (2001), 123–139.

[9] M. Crandall - H. Ishii - P. Lions, *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations*, Bull. of Amer. Math. Soc. *27* (1) (1992), 1–67.

[10] L. Hörmander, *Hypoelliptic second order differential operators* Acta. Math. *119* (1967), 637–665.

[11] R. Jensen, *Uniqueness of Lipschitz Extensions: Minimizing the Sup Norm of the Gradient*, Arch. Ration. Mech. Anal. *123* (1993), 51–74.

[12] P. Juutinen, *Minimization Problems for Lipschitz Functions via Viscosity Solutions*, Ann. Acad. Sci. Fenn. Math. Diss. No. 115 (1998).

[13] P. Juutinen - P. Lindqvist - J. Manfredi, *The ∞-eigenvalue Problem*, Arch. Rational Mech. Anal. 148 no. 2 (1999), 89–105.

[14] G. Lu - J. Manfredi - B. Stroffolini, *Convex Functions on the Heisenberg Group*, Calc. Var. PDE *19* (1) (2004), 1–22.

[15] R. Monti - F. Serra Cassano, *Surface measures in Carnot–Carathéodory spaces*, Calc. Var. PDE. *13* (2001), 339–376.

[16] C. Y. Wang, *The Aronsson equation for gradient minimizers of L^∞-functionals associated with vector fields satisfying Hörmander's condition*, Trans. Amer. Math. Soc. **359** (2007), 91–113.

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