LE MATEMATICHE
Vol. XLVII (1992) – Fasc. II, pp. 295-311

RECENT RESULTS ON THE EMBEDDING OF LATIN SQUARES AND RELATED STRUCTURES, CYCLE SYSTEMS AND GRAPH DESIGNS

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1. Introduction.

The purpose of this paper is to give a survey of recent results on the embedding of latin squares and related structures, cycle systems and other graph designs, and to give some understanding of the techniques that have led to these developments.

A *(partial) latin square* of *order* n on the symbols $\{1, \dots, t\}$ is an $n \times n$ array in which each cell contains (at most) one symbol and each symbol occurs (at most) once in each row and column.A partial latin square L of order n is *embedded* in a latin square T of order v if for each cell (i, j) of L that contains a symbol k, cell (i, j) of T also contains k. An $r \times s$ *incomplete latin rectangle* is an $r \times s$ array in which each cell contains one symbol and each symbol occurs at most once in each row and column.

The embedding of partial latin squares has a long and celebrated history, attracting the attention of such mathematicians as Marshall Hall, Herb Ryser, Trevor Evans, Allan Cruse, Anthony Hilton and Lars Andersen. In most cases, these efforts have resulted in best possible theorems. For example, Hall [11] found that every $r \times n$ incomplete latin rectangle on the symbols $\{1, \dots, n\}$ can be embedded in a latin square of order n, and Evans [10] showed that every partial latin square of order n on the symbols $\{1, 2, \dots, n\}$ can be embedded in

This research was supported by NSF grant DMS-9024645.

a latin square of order t, for any $t \geq 2n$ (and this is "best possible" in the sense that if $1 < n < t < 2n$, there exists a partial latin square of order n that cannot be embedded in a latin square od order t). Both of these results follow quickly from the general result of Ryser [29]:

Theorem 1.1. *The $r \times s$ incomplete latin rectangle R on the symbols $\{1, 2, \ldots, t\}$ can be embedded in a latin square of order t iff*

$$R(i) \geq r + s - t \quad \textit{for} \quad 1 \leq i \leq t,$$

where $R(i)$ is the number of cells in R containing symbol i.

Cruse went on to consider symmetric and idempotent latin squares. A partial latin square L is *symmetric* if whenever cell (i, j) contains symbol k, so does cell (j, i), for all cells (i, j) in L, and is idempotent if cell (i, i) contains symbol i for all i. Cruse [8] found necessary and sufficient conditions for the embedding of a symmetric incomplete latin rectangle in a symmetric latin square, providing a beautiful companion theorem to that of Ryser. As a corollary, he proved the following.

Theorem 1.2. *Every partial symmetric idempotent latin square of order n on the symbols $\{1, \ldots, t\}$ can be embedded in a symmetric idempotent latin square of order t, for all odd integers t, $t \geq 2n + 1$.*

Again, this is a best possible theorem in the same sense as the theorem of Evans. Notice that since each symbol occurs once on the diagonal (idempotent) and an even number of times off the diagonal (symmetric), t must be odd. (When we consider the embedding of symmetric idempotent groupoids in the next section, having t odd will not be necessary.)

The last theorem in this sequence of embeddings of latin squares obtained a companion to Theorem 1.1 for idempotent latin squares when Andersen, Hilton and Rogers [3] proved the following result.

Theorem 1.3. *Every partial idempotent latin square L of order n on the symbols $\{1, \ldots, n\}$ can be embedded in an idempotent latin square of order t, for all $t \geq 2n + 1$.*

This result was subsequently improved by Rodger [24] to allow L to be defined on the symbols $\{1, \ldots, t\}$. However, the idempotent version of Ryser's Theorem is yet to be proved. Indeed, proving such a theorem is likely to be very difficult because it has been shown [4] that the arrangement of the symbols within the incomplete latin rectangle can determine whether or not it can be embedded (so conditions on the number of times each symbol occurs are not sufficient).

In Section 2, recent generalizations of Theorems 1.1, 1.2 and 1.3 to embeddings of partial groupoids are described. (A *(partial) groupoid* of *order* n on the symbols $\{1, 2, \ldots, t\}$ is an $n \times n$ array in which each cell contains (at most) one symbol. A groupoid is row *(column) latin* if each symbol appears at most once in each row (column).) In Section 3, these generalizations are used to obtain smallest known embeddings for partial cycle systems of odd lenght, a problem which itself has a long history, but we leave the description of such results until then. Section 4 considers the embedding of (complete) cycle systems, and finally Section 5 uses the results of Section 2 to find small embedding for a class of partial graph designs.

2. Embedding partial groupoids.

For the applications described in Section 3 and 5, it is crucial to be able to find a partial latin square L of order t in which the following are all specified:

(a) the cells which are to be empty;
(b) the symbols that are to be missing from each row;
and
(c) the symbols that are to be missing from each column.

Fortunately, the cells required to be empty occur in a subsquare of order n in L. But filling these cells with symbols that are to be missing from the corresponding rows and columns of L and thereby forming a partial *latin square* S is often impossible. (If it were possible, S could then be embedded in a latin square using the classical embedding results described in Section 1 and the entries in S removed to produce the desired objective, L.) Apart from being a pleasing generalization in its own right, it is with this application in mind that the following definition was made by Lindner and Rodger [16].

An $r \times s$ *patterned hole* on the symbols $\{1, 2, \ldots, t\}$ is an ordered triple (H, R, C) where H is a subset of the cells of an $r \times s$ array, and R and C are partial $r \times s$ groupoids on the symbols $\{1, 2, \ldots, t\}$ that are row latin and column latin respectively, $R(i) = C(i)$ for $1 \leq i \leq t$, and in which the occupied cells are precisely the cells of H. A patterned hole is *idempotent* if for $1 \leq i \leq n$, cell $(i, i) \in H$ and contains symbol i in both R and C. A patterned hole of *order* n is an $n \times n$ patterned hole.

Example 2.1.

$R=$

1	2	3	5
1	2	3	5
	5	3	6
2	7	8	4

and $C=$

1	1	2	3
2	2	5	6
	3	3	8
5	5	7	4

(H, R, C) is an idempotent patterned hole of order 4, where H is the set of cells occupied by symbols in R(or in C).

Let $P_x(H)$ be a partial latin square of order x in which the empty cells are precisely the cells in H, and let $P_x(R)$ or $P_x(C)$ be the groupoid obtained from $P_x(H)$ by filling the empty cells of $P_x(H)$ with the symbols in the corresponding cells in R or C respectively. We say that the patterned hole (H, R, C) is *embedded* in $P_x(H)$ if $P_x(R)$ is row latin and $P_x(C)$ is column latin.

Example 2.2.

				8	7	6	4
				7	4	8	6
4				2	8	1	7
				6	1	5	3
8	7	6	1	5	3	4	2
3	8	4	7	1	6	2	5
6	4	8	5	3	2	7	1
7	6	1	2	4	5	3	8

The patterned hole (H, R, C) of Example 2.1 is embedded in this partial latin square.

Clearly this definition gives us the structure we need to find a latin square L satisfyng properties (a), (b) and (c). All that remains is to be able to embed (H, R, C) in a partial latin square L. It turns out that we can obtain the analogues of Theorems 1.1, 1.2 and 1.3, as the following results show.

Theorem 2.3. [15] *The $r \times s$ patterned hole (H, R, C) with $|H| = rs$ can be embedded in a partial latin square of order t iff $R(i) = C(i) \geq r + s - t$ for $1 \leq i \leq n$.*

Just as Evan's Theorem can be deduced from Theorem 1.1, the following follows immediately from Theorem 2.3.

Corollary 2.4. *The patterned hole (H, R, C) of order n can be embedded in a partial latin square $P_t(H)$ of order t for all $t \geq 2n$.*

A patterned hole (H, R, C) is *symmetric* if $C = R^T$. The generalization of Cruse's Theorem 1.2 now allows for the size of the containing latin square to be even. (See also [1], where Lars Andersen defined the less general notion of an externally symmetric embedding.)

Theorem 2.5. [16] *Every symmetric idempotent patterned hole (H, R, R^T) of order n on the symbols $\{1, 2, \dots, t\}$ can be embedded in a partial symmetric idempotent latin square $P_t(H)$ of order t, for any $t \geq 2n + 1$, providing*

$$R(i) \equiv \begin{cases} t & (mod\ 2) \quad for \quad 1 \leq i \leq n \\ t+1 & (mod\ 2) \quad for \quad n+1 \leq i \leq t. \end{cases}$$

(**Remark.** In the case of an idempotent embedding, instead of requiring that the empty cells of $P_t(H)$ be precisely the cells in H, we modify the definition of *embedding* (H, R, C) so that the diagonal cells are precisely the cells in H that contain symbols in $P_t(H)$.)

Example 2.6. The idempotent symmetric patterned hole with

$R =$

1	2		8
1	2	3	5
	2	3	4
7	6	2	4

is embedded in the partial symmetric idempotent latin square

$L =$

1		6		7	4	3	5
	2			8	7	4	6
6		3		1	8	5	7
			4	3	5	8	1
7	8	1	3	5	2	6	4
4	7	8	5	2	6	1	3
3	4	5	8	6	1	7	2
5	6	7	1	4	3	2	8

Note that it is impossible to complete L to a symmetric idempotent latin square, so it is not possible to use Cruse's theorems to obtain L.

Finally, we obtain the analogue of the Andersen, Hilton and Rodger theorem, Theorem 1.3 (see also [25], where Rodger obtained a slightly more general result).

Theorem 2.7. [17] *An idempotent patterned hole (H, R, C) of order n on the symbols $\{1, 2, \ldots, n\}$ can be embedded in a partial idempotent latin square of order t, for all $t \geq 2n + 1$.*

3. Embedding partial odd cycle systems.

An *m-cycle* is a graph with vertex set $(v_0, v_1, \ldots, v_{m-1})$ and edge set $\{\{v_i, v_{i+1}\} | \, i \in \mathbf{Z}_m\}$, reducing the subscript modulo m. An *m-cycle system* of a graph G is an ordered pair $(V(G), C)$, where C is a set of m-cycles defined on vertices in $V(G)$ that form a partition of the edges of G. An *m-cycle system* of order n is an m-cycle system of K_n. A *partial* m-cycle system is an m-cycle system of a subgraph of K_n. A partial m-cycle system (V, P) is *embedded* in an m-cycle system (W, C) if $V \subseteq W$ and $P \subseteq C$.

The embedding of partial m-cycle systems has a long history in the case when $m = 3$ (3-cycle systems are Steiner triple systems). After several earlier results [13, 32, 33], the best embedding to date is by Andersen, Hilton and Mendelsohn [2] who showed that any partial Steiner triple system of order n can be embedded in a Steiner triple system of order v, for any $v \geq 4n + 1, v \equiv 1$ or 3 (mod 6).

More recently, the embedding for partial m-cycle systems has been considered in general. Some remarks about embedding partial m-cycle systems when

m is even are made in the last section of this survey; but the results from Section 2 are used when m is odd, so we focus on that here. Therefore, assume that m is odd. The embedding rely on the following construction for m-cycle systems (we give the construction formally at first, then proceed with an informal discussion of it).

The Construction (of odd-cycle systems).

Construct an m-cycle system $(\mathbf{Z}_{2s+1} \times \mathbf{z}_m, C)$ as follows:

(1) For each $a \in \mathbf{Z}_{2s+1}$, let $(\{a\} \times \mathbf{Z}_m, C_a)$ be an m-cycle system of order m(this is just a hamiltonian decomposition of K_m) and let $C_a \subseteq C$;

and

(2) let $(\mathbf{Z}_{2s+1}, \circ)$ be a symmetric idempotent quasigroup, and for each $\{a,b\} \subseteq \mathbf{Z}_{2s+1}, a \neq b$ let $c_{a,b}$ be the m-cycle defined by

$$c_{a,b} = \begin{cases} ((a,0),(b,1),(a,-1),(b,2),\ldots,(b,(m-1)/4), \\ \quad (a \circ b, -(m-1)/4), (a,(m-1)/4), (b, -(m-1)/4+1), \\ \quad \ldots,(b,0)) \quad \text{if} \quad m \equiv 1 \pmod 4 \\ ((a,0),(b,1),(a,-1),(b,2),\ldots,(a,-(m-3)/4), \\ \quad (a \circ b, (m+1)/4), (b,-(m-3)/4), (a,(m+1)/4-1), \\ \quad \ldots,(b,0)) \quad \text{if} \quad m \equiv 3 \pmod 4), \end{cases}$$

where each second component is reduced modulo m. Let $c_{a,b,i}$ be defined by adding i (modulo m) to the second component of each vertex in $c_{a,b}$. Then let $\{c_{a,b,i} \mid i \in \mathbb{Z}_m, \{a,b\} \subseteq \mathbb{Z}_{2s+1}\} \subseteq C$.

Clearly the complicated part of this construction is part (2) (see Figure 1). If we think of the vertex (a,j) as "being in column a and on level j", then two vertices can be at most $\lfloor m/2 \rfloor$ levels apart, modulo m. The set of m-cycles $\{c_{a,b,i} \mid i \in \mathbb{Z}_m\}$ uses each edge joining vertices in columns a and b that are less than $\lfloor m/2 \rfloor$ levels apart, and uses each edge joining vertices in columns a and b to vertices in column $a \circ b$ that are exactly $\lfloor m/2 \rfloor$ levels apart. So as we let a and b range over all 2-element subsets of $\mathbb{Z}_{2s+1}$, we consider each pair of columns in turn, so every edge joining vertices in different columns is used in an m-cycle. Therefore, together with the m-cycles in (1) that contain the edges joining vertices that are in the same column, an m-cycle system of order $m\,(2s+1)$ is produced. The first small embedding for partial m-cycle systems [21] used this construction by attempting to define a partial symmetric idempotent latin square to represent the partial m-cycle system (V,P) as follows: for each m-cycle $(v_0, v_1, \ldots, v_{m-1}) \in P$, define cells (v_i, v_{i+1}) and (v_{i+1}, v_i) to contain symbol $v_{i+\lceil m/2 \rceil}$ ($v_{i+\lceil m/2 \rceil}$ is the vertex on the "opposite side" of the m-cycle).

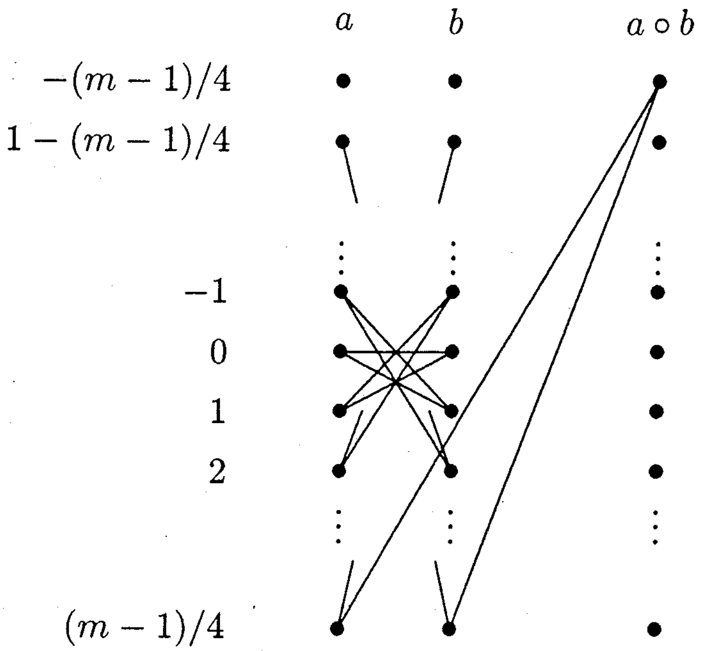

Figure 1: The m-cycle $c_{a,b} = c_{a,b,0}$

Example 3.1. Let $m = 5$, and $(V, P) = (\mathbb{Z}_8, \{(0,1,2,3,4), (0,5,6,3,7)\})$. Then $(\mathbb{Z}_8, P)$ is represented by the following partial symmetric idempotent groupoid.

0	3			2	3		6
3	1	4					
	4	2	0			0	5
		0	3	1			
2			1	4			
3					5	7	
		0			7	6	
6		5					7

As can be seen from this example, it is not a partial latin square that is defined in this way, but is a partial groupoid (since here symbol 3-occurs twice in row and column 1), so the construction appears to be of no use. However for certain types of cycle systems, known as $\lfloor m/2 \rfloor$-*perfect* m-cycle systems, defining a partial groupoid in this way does produce a partial latin square, and then using The Construction, an embedding of a partial $\lfloor m/2 \rfloor$-perfect m-cycle system of order n in an m-cycle system of order $(2n+1)\,m$ is produced. It turns out [21] that by adding many m-cycles and new vertices to the given partial m-cycle system (V, P), and then using a mutually balanced set of m-cycles (see [26] for further discussion on this), (V, P) can be "turned into" an $\lfloor m/2 \rfloor$-perfect partial m-cycle system, resulting in an embedding of an m-cycle system of order $m\,((m-2)\,n\,(n-1)+2n+1)$.

The problem with this method is that it strives to represent (V, P) with a partial *latin square*. However, with the results from Section 2 in hand, a closer look at The Construction shows that using a groupoid to represent (V, P) works exceptionally well, as we shall now see.

Given a partial m-cycle system $(\mathbb{Z}_n, P)$ of order n, define a partial idempotent groupoid R' as follows:

(a) let cell (i, i) to contain symbol i for all $i \in V$, and
(b) for each m-cycle $(v_0, v_1, \ldots, v_{m-1}) \in P$ let cells (v_i, v_{i+1}) and (v_{i+1}, v_i) contain symbols v_{i+1} and v_i respectively. (See Example 3.2.)

Clearly R is a row latin partial groupoid, and since each vertex has even degree in G (where (V, P) is an m-cycle system of G), each symbol in V occurs in an odd number of cells of R. We can now take a symmetric latin square L of order n on the symbols $n, n+1, \ldots, 2n-1$ and then fill each empty cell (i, j) of R' with the symbol in cell (i, j) of L to form a row latin groupoid R' in which $R'(i)$ is odd if $i \in \mathbb{Z}_n$ and is even if $n \le i \le 2n-1$.

Example 3.2. Using $(\mathbb{Z}_8, P) = (\mathbb{Z}_8, \{(0,1,2,3,4), (0,5,6,3,7)\})$ as defined in Example 3.1, R is defined using (a) and (b) by

$R =$

0	1			4	5		7
0	1	2					
	1	2	3				
		2	3	4		6	7
0			3	4			
0					5	6	
			3		5	6	
0			3				7

(compare R with the partial groupoid defined in Example 3.1). Then with L

defined below, we get R' from R and L (the entries from R in R' are in bold typer).

$L =$

8	9	10	11	12	13	14	15
9	10	11	12	13	14	15	8
10	11	12	13	14	15	8	9
11	12	13	14	15	8	9	10
12	13	14	15	8	9	10	11
13	14	15	8	9	10	11	12
14	15	8	9	10	11	12	13
15	8	9	10	11	12	13	14

$R' =$

0	1	10	11	4	5	14	7
0	1	2	12	13	14	15	8
10	1	2	3	14	15	8	9
11	12	2	3	4	8	6	7
0	13	14	3	4	9	10	11
0	14	15	8	9	5	6	12
14	15	8	3	10	5	6	13
0	8	9	3	11	12	13	7

Now the patterned hole (H', R', R'^T) can be embedded in a partial symmetric idempotent latin square of order $2n+1$ using Theorem 2.5. Therefore we have an embedding of the patterned hole (H, R, R^T) in a partial symmetric idempotent latin square T. Notice that

(a) the only empty cells in T are the off-diagonal cells in R that contain symbols, and

(b) symbol i is missing from row j of T
iff it occurs in row j of R
iff $\{i, j\}$ is an edge in an m-cycle in P.

So now suppose that we apply The Construction using T, ignoring in step (2) the pairs $\{a, b\} \subseteq \mathbb{Z}_{2n+1}$ for which cell (a, b) of T is empty. This produces a partial m-cycle system $(\mathbb{Z}_{2n+1}, P')$. But which edges occur in m-cycles in P'?

As described in the informal discussion following the justification of The Construction, all edges joining vertices in different columns a and b that are less than $\lfloor m/2 \rfloor$ levels apart will occur in an m-cycle in P' EXCEPT if cell (a, b) is empty. And all edges joining vertices in columns a and $b = a \circ z (a \circ z$ is the symbol in cell (a, z) of $T)$ that are $\lfloor m/2 \rfloor$ levels apart will occur in an m-cycle in P' EXCEPT if b does not occur in row a of T (so there is no z such that b occurs in cell (a, z) of T). But by (b) above, b is missing from row a of T iff $\{a, b\}$ is an edge in an m-cycle in P iff cell (a, b) of T is empty. Therefore the ONLY edges that are not contained in m-cycles in P' are the edges joining vertices in columns a and b for all pairs $\{a, b\}$ for which the edge $\{a, b\}$ occurs in an m-cycle of P. But these edges are easily placed in m-cycles by defining:

(3) For each m-cycle $(v_0, v_1, \ldots, v_{m-1})$ in P, let $(\mathbb{Z}_m, \otimes)$ be an idempotent quasigroup and for each $i, j \in \mathbb{Z}_m$ (including $i = j$) place the m-cycle $((v_0, i), (v_1, j), (v_2, i), (v_3, j), \ldots, (v_{m-2}, j), (v_{m-1}, i \otimes j))$ in C.

So we obtain an m-cycle system $(\mathbb{Z}_{m(2n+1)}, C)$ by applying The Construction modified by using (2) only when cell (a, b) of T contains a symbol, and adding (3).

This modification has the great advantage that if $(v_0, v_1, v_2, \ldots, v_{m-1}) \in P$ then since $(\mathbb{Z}_m, \otimes)$ is idempotent, $((v_0, 0), (v_1, 0), \ldots, (v_{m-1}, 0)) \in C$, so we have the required embedding of $(\mathbb{Z}_n, P)$ in $(\mathbb{Z}_{m(2n+1)}, C)$. Therefore we have the following result.

Theorem 3.3. [16] *Any partial m-cycle system of order n can be embedded in an m-cycle system of order $(2n + 1)m$.*

While this embedding is very satisfying, it is a long way from a best possible embedding. It is likely to be the case that any partial m-cycle system of order n can be embedded in an m-cycle system of any admissible (admissible means satisfies obvious numerical necessary conditions) order $v \geq n\,(m + 1)/(m - 1) + 1$ (see [26] and Section 4).

4. Embedding complete odd cycle systems.

It is worth briefly mentioning the problem of taking a (complete) m-cycle system (V, P) of order n and showing that for all admissible values of $v \geq n\,(m + 1)/(m - 1) + 1$, (V, P) can be embedded in an m-cycle system (W, C) of order v (v is admissible if v is odd, $v \geq m$ and $2m$ divides $v\,(v - 1)$). The fact that v cannot be less that $n\,(m + 1)/(m - 1) + 1$ is easy to see [26] by observing that since m is odd, every m-cycle containing an edge joining a vertex in V to a vertex in $W \setminus V$ must contain an edge joining two vertices in $W \setminus V$; as there are $n\,(v - n)$ edges joining vertices in V to vertices in $W \setminus V$, and as there are $(v - n)(v - n - 1)/2$ edges joining two vertices in $W \setminus V$, we get $n\,(v - n)/(m - 1) \leq (v - n)(v - n - 1)/2$.

Doyen and Wilson [9] have obtained such a result for 3-cycle systems.

Theorem 4.1. *Any 3-cycle system of order n can be embedded in a 3-cycle system of order $v > n$ for any $v \geq 2n + 1, v \equiv 1$ or 3 (mod 6).*

Subsequently, another proof in the case where $m = 3$ was obtained by Stern and Lenz [31]. Their proof made great use of a very useful lemma they proved that showed that the graph $G\,(D)$ with vertex set $\mathbb{Z}_v$ and edge ser $\{\{i, i+d\} \mid d \in D\}$, reducing sums modulo v, has a 1-factorization if their exists a $d \in D$ such taht $v/gcd\{v, g\}$ is even. For a survey of this technique, see [23].

Very recently, Bryant and Rodger [5] have extended the Doyen-Wilson Theorem to 5-cycle systems.

Theorem 4.2. *Any 5-cycle system of order n can be embedded in a 5-cycle of order $v > n$, for any $v > 3n/2$ with $v \equiv 1$ or 5 (mod 10).*

It is likely that the technique used in this paper will generalize to larger values of m. However, since it is not even known for what values of v m-cycle systems exist, complete solutions of this problem seem very unlikely to be found in the near future.

5. Embedding partial $K_m \setminus K_{m-2}$ designs.

In this section we use the results of Section 2 to find small embedding of another family of graphs. A graph is *simple* if it has no loops, and each pair of vertices is joined by at most one edge. Let $K_m \setminus K_{m-2}$ be the simple graph, denoted by $(a_1, a_2, \ldots, a_m)$, with vertex set $\{a_1, a_2, \ldots, a_m\}$ in which a_1 and a_2 have degre $m-1$, and $\{a_3, \ldots, a_m\}$ are an independent set of vertices (or, remove the edges joining vertices in $\{a_3, a_4, \ldots, a_m\}$ from the edges in K_m). For any simple graph G, let λG denote the multigraph in which two vertices are joined by λ edges if $\{u, v\} \in E(G)$, and otherwise u and v are not adjacent.

A $K_m \setminus K_{m-2}$ *design* (P, B) of a multigraph G is a collection B of copies of $K_m \setminus K_{m-2}$, each defined on a subset of the vertices in P, such that each edge of G occurs in exactly one copy of $K_m \setminus K_{m-2}$. A (partial) $K_m \setminus K_{m-2}$ design of *order* n and of *index* λ is a $K_m \setminus K_{m-2}$ design of (a subgraph of) λK_n. The *embedding* problem is again to show that any partial $K_m \setminus K_{m-2}$ design od order n and index λ can be embedded in a $K_m \setminus K_{m-2}$ design of order v and index λ, where v is as small as possible. (Of course, Richard Wilson has shown [33] that any partial graph design can be finitely embedded, but finding a small embedding remains a difficult problem in general.) As we will now see, we can use the techinque described in Section 3 to obtain a small embedding.

The first step is to take the partial $K_m \setminus K_{m-2}$ design (P, B) of order n and of index λ, say (P, B) is a $K_m \setminus K_{m-2}$ design of the multigraph G, and to sort the edges of G out into λ sets $S_1, S_2, \ldots, S_\lambda$, such that for $1 \le i \le \lambda$,

(a) the subgraph G_i induced by the edges in S_i is a simple graph,
and
(b) each vertex in G, has even degree.

Obtaining property (a) is simple: let $\{u, v\} \in E(G'_i)$ iff u and v are joined by at least i edges in G. Certainly G'_i satisfies (a), but almost certainly doesn't satisfy (b)! However, it is not hard to add some extra vertices to P and some extra copies of $K_m \setminus K_{m-2}$ to G_i' (these copies are added to B too) to make sure that the resulting graph G_i satisfies (a) and (b) (see [19] for details, where $2m - 4 + \lfloor n/2 \rfloor$ vertices are added in this step). Let (P', B') be the resulting $K_m \setminus K_{m-2}$ design of the multigraph $\bigcup_{i=1}^{\lambda} G_i$.

The second step is to form λ partial idempotent groupoids $(P', \circ_i)$ for $1 \le i \le \lambda$ as follows:

(1) $x \circ_i x = x$ for all $x \in P'$,

and

(2) if $\{x, y\} \in E(G_i)$ then $x \circ_i y = y$ and $y \circ_1 x = x$, and no other cells of the groupoids contain symbols (compare this to the formation of R in Section 3).

Clearly $(P', \circ_i) = R_i$ is a row latin partial groupoid. By (b) and (1) above, each symbol in P' occurs in an odd number of cells of $(P', \circ_i)$, so the empty cells of $(P', \circ_i)$ can be filled with $|P'|$ new symbols to form the row latin groupoid R_i' (as R' was formed from R in Section 3), and then Theorem 2.5 provides an embedding of the patterned hole (H, R_i, R_i^T) into a partial symmetric idempotent latin square L on the symbols $0, 1, \dots, t = 2|P'|$. Note that

$$(*) \qquad \begin{array}{c} \text{cell } \{a, b\} \text{ of } L \text{ is empty} \\ \text{iff row } a \text{ of } L \text{ is missing symbol } b \\ \text{iff } \{a, b\} \text{ occurs in } E(G_i). \end{array}$$

Thirdly, we can now define the containing $K_m \setminus K_{m-2}$ design (W, C). If m is even or odd then let $W = \{\infty\} \cup (\mathbb{Z}_t \times \mathbb{Z}_{m-3}$ or $\mathbb{Z}_t \times \mathbb{Z}_{m-3}$ respectively. Define

(1) For each $a \in \mathbb{Z}_t$, let $(\{\infty\} \cup (\{a\} \times \mathbb{Z}_{m-3}), g(a))$ or $(\{a\} \times \mathbb{Z}_{m-3}, g(a))$ if m is even or odd respectively be a $K_m \setminus K_{m-2}$ design of index λ (such designs do exist [19]), and let $g(a) \subseteq C$.

(2) For $1 \le i \le \lambda$, if $\{a, b\}$ does not occur in $E(G_i)$ then place $((a, j), (b, j), (a \circ_i b, j + 1), (a \circ_i b, j + 2), \dots, (a \circ_i b, j + m - 2))$ in C.

As in Section 3 when (3) was defined, the main point to notice here is that by $(*)$, any two vertices in columns a and b are joined by exactly $\lambda - m_{a,b}$ edges in copies of $K_m \setminus K_{m-2}$ defined in (2), where $m_{a,b}$ is the number of graphs G_i that contain the edge $\{a, b\}$. So the edge $\{a, j), (b, k))$ must still occur in $m_{a,b}$ further copies of $K_m \setminus K_{m-2}$. This is easily accomplished because $m_{a,b}$ is the number of edges joining vertices a and b in $\bigcup_{i=1}^{\lambda} G_i$, and we know that (P', B')

is a $K_m \setminus K_{m-2}$ design of $\bigcup_{i=1}^{\lambda} G_i$; so we also define

(3) for each copy $(a_1, a_2, \ldots, a_m)$ of $K_m \setminus K_{m-2}$ in B', let $((a_1, j), (a_2, k), (a_3, j \otimes k), (a_4, j \otimes k), \ldots, (a_m, j \otimes k)) \in C$ for each j, k in $\mathbb{Z}_{m-3}$ (possibly $j = k$) and where $(\mathbb{Z}_{2m-3}, \otimes)$ is an idempotent quasigroup.

The embedding has now been achieved, since $(\mathbb{Z}_{2m-3}, \otimes)$ in (3) is idempotent and so for each $(a_1, a_2, \ldots, a_m) \in B$, $((a_1, 0), (a_2, 0), \ldots, (a_m, 0)) \in C$.

This gives the following result.

Theorem 5.1. [19] *Any partial $K_m \setminus K_{m-2}$ design of order n and of index λ can be embedded in a $K_m \setminus K_{m-2}$ design of order v and index λ, where $v \leq (3n + 4m - 7)(2m - 3) + 1$, and if m is odd and $\lambda = 1$ then $v = (2n + 1)(2m - 3)$.*

(The improvement in the case where m is odd and $\lambda = 1$ in Theorem 5.1 is because in G_1, every vertex already has even degree, and so already satisfies property (b).)

One of the interesting features of this embedding technique is that the way the edges in G are distributed among the graphs $G_1, \ldots, G_\lambda$ is unrestricted by the actual copies of $K_m \setminus K_{m-2}$ that define G. One might hope that this freedom can eventually be exploited to obtain an embedding of a partial K_4 design of index λ_1, (block design of block size 4 and index λ_1) in a K_4 design of index λ_2, where λ_2 is not too big compared to λ_1.

Concluding remarks.

So far in this survey, we have focused on the new results on embedding patterned holes, and shown how one of these results, namely Theorem 2.5, has led to vastly improved embedding for m-cycle systems when m is odd. The same principles can be applied to obtain small embeddings of partial directed m-cycle systems when m is odd [17] by using Theorem 2.7, on embedding idempotent patterned holes, instead of Theorem 2.5. The embeddings of partial m-cycle systems when m is even [20] are much simpler because of a result of Dominique Sotteau [30] who found necessary and sufficient conditions for the existence of an m-cycle system of $K_{a,b}$ (the complete bipartite graph).

Finally we should mention that the embedding of partial cycle systems of index $\lambda > 1$ have also been considered. Again the case where $m = 3$ has quite a history [6, 7, 22], with the best result to date being that a partial 3-cycle system of order n and index λ can be embedded in a 3-cycle system of order v and index λ for any admissible $v \geq 2n+1$ if 4 divides λ [12] and any admissible $v \geq 4n+1$

otherwise [28]. For $m > 3$, the techniques described in Section 2,3 and 4 can be used to obtain an embedding, where as in Section 4, the edges in the given partial m-cycle system are divided up among λ simple graphs in which each vertex has even degree (so the graphs satisfy conditions (a) and (b) of Section 4). Again, when using this approach, the edges can be distributed among the λ graphs without ever worring about which copy of $K_m \setminus K_{m-2}$ they belong to. The embedding of a partial m-cycle system of order n and index λ in an m-cycle system of order $(4n + 17)m$ and index λ in [18] uses a generalization of the techiniques used in [2], and has also been used to obtain small embeddings of linear spaces into linear spaces with no lines of size 2 [27]. But that technique is another story, not to be described here!

We conclude with a summary of the latest results concerning embeddings of m-cycle systems, as indicated in the following table [18].

Parity of m	K_n or D_n	Best Embedding	Conditions	Reference
Odd	Undirected	$(4n + 17)m$	$\lambda \geq 2$ and $m > 3$	[18]
		$4n + 1$	$\lambda \not\equiv 0$ (mod4) and $m = 3$	[2, 28]
		$2n + 1$	$\lambda \equiv 0$ (mod 4) and $m = 3$	[12]
		$(2n + 1)m$	$\lambda = 1$ and $m > 3$	[16]
Odd	Direct	$(2n + 1)m$	$\lambda \geq 1$ or $m > 3$	[18]
		$4n + 1$	$\lambda = 1$ and $m = 3$	[25]
Even	Undirected	nm	λ even	[18]
		$2nm + 1$	$\lambda \geq 3$ and odd, or $\lambda = 1$ and $m > 4$	[18]
		$\approx 2n + \sqrt{2n}$	$\lambda = 1$ and $m = 4$	[14]
Even	Directed	nm	$\lambda \geq 1$ and $m \geq 8$	[18]
		$nm + 1$	$\lambda \geq 1$ and $m = 6$ or $\lambda \geq 2$ and $m = 4$	[18]
		$\approx 2n + \sqrt{2n}$	$\lambda = 1$ and $m = 4$	[17]

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