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THE KINETIC AND DYNAMIC BEHAVIOUR OF A SIMPLE GAS MODEL

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The so called Lebowitz stick model of a gas is studied. We discuss the particle level, as well as the gas kinetic, and gas dynamic levels of the model, and consider how the three levels are connected. In particular attention is given to the validation of the kinetic level from a stochastic Liouville equation, and to the asymptotic behaviour of the kinetic level.

1. Introduction.

The modelling of rarefied gases can be done at various levels, the most popular ones being by particle dynamics, by gas dynamics and by gas kinetics. However, the mathematical understanding of how to connect those three levels is not very complete, at least not outside of in some sense small data. In such a situation it is often interesting to study special model systems to gain insight and hopefully also to develop some fresh mathematical machinery applicable also to the full problem. One such model with a certain interest for the present context is the so called Lebowitz stick model. In comparison to other (velocity continuous) gas models in current use, the stick model is amenable to more explicit and more transparent constructions, and so is useful for a first concrete pencil test of mathematical ideas and hypotheses. At the same time it gives an idea of the minimal level of complexity to expect for the general problem. On the other hand, in the case to be treated it is far from interesting physics, due to the here trivial collision mechanism.

We will first give a short introduction to the stick model, and then discuss some properties on the kinetic level including the connection to the hydrodynamic level. Towards the end the validation of the kinetic equations from a stochastic Liouville description will be treated.

The Lebowitz model treats a gas of sticks in the (x, y)-plane R^2 [5]. Each stick is oriented along the y-axis, and moves in a straight line with uniform velocity until it collides with another stick side to side. Then the two sticks exchange their velocity component v_x in the x-direction. On the particle level for a gas of N sticks, the evolution can e.g. be given by the time evolution of a density f^N through the Liouville equation in the $4N$ dimensional phase space with phase variable $z^N = (z_1, z_2, \ldots, z_N)$, where $z_j = (x_j, v_j, v_{x_j}, v_{y_j})$.

We will use a stochastic version of the Liouville equation and take f^N as a probability density, symmetric under particle exchange. From the Liouville equation the BBGKY hierarchy is obtained by integrating away all but the first j particles. At a cruder scaling the evolution can be described by gas kinetics, and in still more compressed terms by gas dynamics.

At the particle level, the last equation in the BBGKY hierarchy is

$$(\partial_t + v_x \partial_x + v_y \partial_y) f_1^N = \text{constant } Q_2(f_2^N),$$

where

$$Q_2(f_2^N)(z^1) = \int |v_{x_1} - v_{x_2}| \{ f_2^N(x_1, y_1, v_{x_2}, v_{y_1}, x_1, y_1 - \delta r, v_{x_1}, v_{y_2})$$
$$- f_2^N(z_1, x_1, y_1 + \delta r, v_{x_2}, v_{y_2})\} dr dv_{y_2} dv_{x_2}, -\frac{1}{2} \leq r \leq \frac{1}{2}.$$

Here δ is the length of the stick. If f_2^N is independent of the y-variables, then by the symmetry and change of variables,

$$\int Q_2(f_2^N) dv_{y_1} = 0, \; (\partial_t + v_x \partial_x) \int f_1^N dv_{y_1} = 0.$$

This is the *simplifying property* of the stick model on the particle level, that it partly decouples the equations of the hierarchy. On the kinetic level the Boltzmann equation for the sticks can similarly be decoupled into two linear equations. We shall rely on the simplifying property from now on and so drop the y-dependence, only considering phase points $z^N = (z_1, z_2, \dots, z_n)$ with $z_j = (x_j, v_{x_j}, v_{y_j})$.

2. The kinetic level.

This section discusses the Boltzmann equation for the sticks, in particular its asymptotic behaviour, and towards the end the problem of the hydrodynamic limit.

For simplicity let us carry out the discussion in the following setting,

$$(\partial_t + v_x \partial_x) f = Q(f, f), -1 \le x \le 1,\ (v_x, v_y) \in \mathbb{R}^2,\ t > 0.$$

Here $Q(f,f) = Q_2(f \otimes f)$, as defined in Section 1. As initial value we take $f|_{t=0} = f_0$, where $(1 + |v_x|) f_0 \in L^1_+$, and assume diffusive reflexion at the boundary,

$$f(-1, v_x, v_y, t) = \phi_-(v_x)\psi_-(v_y) \int_{v_x<0} |v_x| f(-1, v_x', v_y', t) dv_x' dv_y',\ v_x > 0,$$

$$f(1, v_x, v_y, t) = \phi_+(v_x)\psi_+(v_y) \int_{v_x>0} v_x f(1, v_x', v_y', t) dv_x' dv_y',\ v_x < 0.$$

The inflow equals the outflow if

$$\int_R \psi_\pm(v_y) dv_y = 1,\ \int_{v_x \lessgtr 0} |v_x| \phi_\pm(v_x) dv_x = 1.$$

As already mentioned an integration $\int dv_y$ decouples the problem into one linear problem for $g = \int f dv_y$, and then with g inserted into Q, another linear problem for f. If we write the equations in suitably integrated forms, then both the stationary and time dependent g and f cases can be solved by iteration, which produces monotone sequences with uniformly bounded masses, hence convergence to solutions. With the masses of the stationary solutions f_∞ and g_∞ equal that of f_0, it follows from the iteration procedure that $g_t \le C g_\infty$, if $g_0 \leq Cg_\infty$ and $t > 0$. Using this and making an asymptotic analysis when $t \to \infty$, in a joint work with N. Ianiro and L. Triolo we have obtained the following result [3].

THEOREM *If $g_0 \leq Cg_\infty$, and supp $\psi_\pm$ is bounded, then the solution f of the present initial boundary value problem satisfies*

$$f_t \to f_\infty \text{ in the weak * topology of measures, when } t \to \infty.$$

Remark With boundary conditions of this generality, such results are still an open question for the full Boltzman equation.

Our proof is based on measure theoretic arguments, and there nonstandard arguments are often quite helpful both in making the proofs more transparent and in discovering them in the first place. Let me illustrate the first point on the step: $g_t \to g_\infty$ strongly in L^1. For this it is enough to prove convergence in measure, since mass is conserved. Let us first in a routine way discuss the inequality behind the convergence. The condition $g_t \leq Cg_\infty$ gives $(g_t - g_\infty)^2/g_\infty \in L^1$. So by the equation we obtain

$$(I) \qquad \begin{aligned} &\partial_t \int (g_t - g_\infty)^2 g_\infty^{-1} dx dv + \int v_x (g_t - g_\infty)^2 g_\infty^{-1} dv_{x|x=1} - \\ &\int v_x (g_t - g_\infty)^2 g_\infty^{-1} dv_{x|x=-1} = 0. \end{aligned}$$

Now for $x = 1$ the boundary condition is

$$(g_t - g_\infty)/g_{\infty|x=1} = \phi_+ g_\infty^{-1} \int_{v'_x > 0} v'_x (g(1, v'_x, t) - g_\infty(v'_x)) dv'_x, v_x < 0.$$

Using Jensen's inequality we get

$$\begin{aligned} &|v_x| g_\infty(v_x)(g_t - g_\infty)^2 g_\infty^{-2}(v_x)_{|x=1} \leq \\ &|v_x|\phi_+(v_x) \int_{v'_x > 0} v'_x g_\infty(v'_x)(g_t - g_\infty)^2 g_\infty^{-2}(v'_x)_{|x=1} dv'_x. \end{aligned}$$

Thus by an integration with respect to $v_x < 0$,

$$\int v_x g_\infty (g_t - g_\infty)^2 g_\infty^{-2}|_{x=1} dv_x \geq 0,$$

with equality if and only if $(g_t - g_\infty)/g_\infty$ =constant= $c_1(t)$. This is a well known boundary entropy inequality for diffuse reflexion, and it holds also at $x = -1$. From here integrating (I) with respect to time we get

$$(II) \qquad \int_0^\infty dt \int v_x g_\infty (g_t - g_\infty)^2 g_\infty^{-2}|_{x=1} dv_x < \infty.$$

The nonstandard argument starts from the above computations but using, instead of the reals R, an extended hyperreal line $^*R$, containing besides the reals also infinitesimals and their inverses, infinite hyperreal numbers. We can go back and forth between the two structures using Leibniz' principle (transfer) meaning that whatever is proved in one context holds true also in the other. In particular the integral in (II) is finite in the nonstandard context, and so for $t_1 < t_2$, both infinite positive hyperreals, $\int_{t_1}^{t_2} dt \int v_x g_\infty (g_t - g_\infty)^2 g_\infty^{-2}|_{x=1} dv_x \approx 0$ ($\approx$ infinitesimally close to 0). Since the inner integral is positive, it is infinitesimal (Loeb dt) almost everywhere. The previous argument using Jensen's inequality implies that for some time dependent constant $c_1(t)$,

$$(g_t(v_x) - g_\infty(v_x))g_\infty^{-1}(v_x)|_{x=1} \approx c_1(t).$$

Analogously

$$(g_t(v_x) - g_\infty(v_x))g_\infty^{-1}(v_x)|_{x=-1} \approx c_{-1}(t).$$

By the equation g is constant along the characteristics, and so (Loeb dt) a.e.

$$c_1(t) \approx \text{ constant } c \text{ independent of } t \approx c_{-1}(t).$$

It follows that $g_t \approx (c+1)g_\infty$, which together with mass conservation implies $g_t \approx g_\infty$, and this for arbitrary infinite t. The last result in the standard context means that g_t converges in measure to g_∞ when $t \to \infty$.

Before turning to the validation problem, let me also mention just one hydrodynamic limit result due to N. Ianiro and L. Triolo [4]. They consider the stationary equation in the interval $\left[\frac{-1}{\varepsilon}, \frac{1}{\varepsilon}\right]$

with the previous diffusive reflexion at the end points. If we write $f_\infty = g_\infty \cdot h$, and set $q = \varepsilon x$, then the equation for h in the q-variable becomes

$$v_1\partial_q h = \varepsilon^{-1} \int |v_{x_1} - v_{x_2}| g(v_{x_2})(h(q, v_{x_2}, v_{y_1}) - h(q, v_{x_1}, v_{y_1}))dv_{x_2},$$

with boundary values

$$h(-1, v_x, v_y) = \psi_-(v_y), v_x > 0,$$

$$h(1, v_x, v_y) = \psi_+(v_y), v_x < 0.$$

Since v_y is only a kind of parameter, whereas v_x behaves as a real velocity in this equation, we should expect h to become independent at least of v_x in the hydrodynamic limit $\varepsilon \to 0$. That is also what happens at least when $g_\infty \in L^\infty$.

THEOREM (Triolo, Ianiro [4]) $h_\varepsilon(q, v_x, v_y) = h_0(q, v_y) + \varepsilon R_\varepsilon(q, v_x, v_y)$, *where* R_ε *is a bounded family in* L^∞, *and* h_0 *is linear,*

$$h_0(q, v_y) = (\psi_+(v_y) - \psi_-(v_y))q/2 + (\psi_+(v_y) + \psi_-(v_y))/2.$$

3. The validation problem.

The final result to be discussed concerns the validation problem for the gas of sticks. We shall start from a stochastic version of the Liouville equation without y-dependence, and prove that the j-particle contractions in the BBGKY hierarchy converge to solutions of the Boltzmann hierarchy. And the hierarchy solutions will be seen to factorize when the initial values do. In particular the Boltzmann equation is obtained in the limit of the 1-particle BBGKY equation. This is a report on joint work with S. Caprino and M. Pulvirenti [2], and shows on the stick model some ideas from our study of a more general one space dimensional validation situation.

As a first step let us discuss a suitable family of solutions for the Liouville equation. Introduce the notation

$$f_t^{N\sharp}(z^N) = f_t^N(x_1 + tv_{x_1}, v_{x_1}, v_{y_1}, \ldots, x_N + tv_{x_N}, v_{x_N}, v_{y_N}),$$

and

$$Lf_t^n(z^N) = \sum_{1\le i\le k\le N} \{f^N(z_1,\dots,z_{i-1},x_i,v_{x_k},v_{y_i},\dots,z_{k-1},x_k,$$
$$v_{x_i},v_{y_k},\dots,z_N) - f_t^N(z^N)\}|v_{x_i}-v_{x_k}|\delta(x_i-x_k).$$

We shall consider the stochastic Liouville equation given by

$$D_t f^{N\sharp} = \varepsilon (Lf_t^N)^\sharp \quad (\varepsilon = \lambda/N), \quad \text{with} \quad f_{t=0}^N = f_0^N \in L_+^1.$$

Here $f_0^N(Pz^N) = f_0^N(z^N)$ (symmetry) and $\int f_0^N dz^N = 1$ (normalization), and P belongs to $\mathcal{P}$=the set of permutations of $1,\dots,N$; The problem is considered with $x \in R$ (and not as previously $[-1,1]$). Integrating out all but the first j particles, we get

$$D_t f_j^{N\sharp} = \varepsilon (Lf_j^N)^\sharp + \varepsilon (N-j)(Q_{j+1}f_{j+1}^N)^\sharp,$$

the j-th equation in the BBGKY hierarchy. Here L, the Liouville collision operator only treats collisions between the j first particles, and Q of Boltzmann type, collision between one of the particles $1,\dots,j$ and one with a higher index. The solution of interest for $N=2$ is

$$f_t^\sharp(z^2) = f_0(z^2) \text{ for } t>0, \text{ if } (x_1-x_2)/(v_{x_2}-v_{x_1}) \le 0$$

(since $\delta(x_1+tv_{x_1}-x_2-tv_{x_2}) = 0$ for $t>0$ in that case). If

$$t_0 = (x_1-x_2)/(v_{x_2}-v_{x_1}) > 0,$$

then

$$f_t^\sharp(z^2) = f_0(z^2),\ t<t_0,\ f_t^\sharp(z^2) = (1-\varepsilon)f_0(z^2)+$$
$$\varepsilon f_0(x_1,v_{x_1},v_{y_1},x_2,v_{x_2},v_{y_1}), \quad t>t_0,$$

and we define $f_{t_0}^\sharp(z^2) = f_{t_0-}^\sharp(z^2) = f_0(z^2)$. It follows that for $0<\varepsilon\le 1$, positivity is conserved, L^1 norm, i.e. mass, is conserved, and L^∞ norm is non-increasing.

For $N>2$ and z^N in a set of full measure in phase space, at any time t with collision for z^N, there is exactly one two-particle collision, so existence of f^N for $N>2$ can be reduced to the $N=2$ case. This means that we can solve the Liouville equation, hence the BBGKY hierarchy for any N.

In a second step we discuss the convergence of the BBGKY hierarchy to the Boltzmann hierarchy when $N \to \infty$. It follows from the above solution discussion that

$$f_t^{N\sharp} = \sum_{P \in \mathcal{P}} \lambda_P^t f_0^N(x^N, v_x^N, Pv_y^N), \sum \lambda_P = 1,\ t > 0.$$

Here the coefficients λ_P^t only depende on x^N, v_x^N but not on v_y^N. This solution structure can be used to prove that, if

$$f_0^N(z^N) = \prod_1^N f_0(z_j)$$

with $f_0 \in L^\infty$ with compact support (for simplicity, but weaker conditions are possible) then for all N and j

$$f_{jt}^{N\sharp}(z^j) \leq C_0^{2j} V_T^j,\ t \leq T.$$

So we can use weak L^1 compactness and conclude that there is a subsequence (N_k) of N, such that when $k \to \infty$, then

$$f_{jt}^{N_k\sharp} \text{ converges weakly in } L^1 \text{ to some } f_{jt}, t > 0, j \in N,$$

$$f_j^{N_k\sharp}(z_1, x_1 + t(v_{x_1} - v_{x_2}), v_{x_2}, v_{y_{2_1}}, z_3, \dots, z_j, t)$$

converges weakly in L^1_{loc} to some L^1 function which in fact turns out to be $f_j^\sharp$. This covers the limit of all terms in the BBGKY hierarchy, and we have proved

THEOREM *The solution of the stochastic Liouville equation for the Lebowitz stick model generates solutions to a BBGKY hierarchy, which converge to L^1 solutions of the Boltzmann hierarchy.*

Remark For λ small and for some types of physically more realistic collision models, the analogous result can be proved in a L^1 setting, just requiring finite mass, energy and entropy. In that case there are no explicit solution formulas, and we have to rely entirely on suitable a priori estimates (see [2]).

If $f(z_1, t)$ is a solution to the Boltmann equation for the sticks with initial value f_0, then the products

$$f_{jt}(z^j) = \prod_{k=1}^j f(z_k, t),\ j \in N,$$

satisfy the Boltzmann hierarchy. So to prove that the solutions of the BBGKY hierarchy converge to factorized solutions of the Boltzmann hierarchy, we only have to prove uniqueness for the Boltzmann hierarchy. Our approach is to exploit an equivalence (due to Spohn) between the Boltzmann hierarchy and a statistical Boltzmann equation, and then to prove uniqueness of the latter, relying on good estimates for the deterministic Boltzmann equation. This approach was previously developed in the space homogenous case in a paper by S. Caprino, N. Ianiro and myself [1], and can be used also for the space dependent stick situation, but for technical reasons not yet for the more general one-dimensional gases referred to above.

THEOREM [2] *Suppose* $f_0 \in L^\infty_+(R^3)$ *has compact support and mass one.*

Then the Boltzmann hierarchy for the Lebowitz stick model with initial values $f_j(z^j,0) = \prod_{k=1}^{j} f_0(z_k)$ *has a unique solution in* $L^1 \cap L^\infty$, *and it factorizes also for* $t > 0$. *In particular the equation for* f_1 *in the Boltzmann hierarchy is the deterministic Boltzmann equation for the sticks.*

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