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EVANESCENT WAVES AND DISCONTINUITIES

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An evanescent wave is a wave disturbance which travels along a direction and attenuates along a different direction. Evanescent waves are typically generated when a plane wave obliquely strikes a boundary between two media possessing different material properties. A detailed analysis of the problem of oblique incidence shows that plane waves are completely determined by the knowledge of the so-called reflection and transmission coefficients. On the contrary the properties of the evanescent waves are summarized by the knowledge of two functions, linearly determined by the amplitude of the incident wave, which are to be chosen as the Hilbert transform of each other; in so doing the evanescent wave exhibits the appropriate behavior at infinity. A thorough comparison between discontinuity waves and evanescent waves makes it evident that the usual information about discontinuities does not suffice for calculating evanescent waves. This proves that the oblique incidence problem does not admit a consistent answer within the sole framework of discontinuity waves.

1. Discontinuity wave theory.

Since the pioneering work of Hadamard [1], the theory of discontinuity wave propagation has become more and more popular mainly because of the inherent mathematical rigor. Indeed, as Truesdell pointed out in [2], although the study of wave propagation casts light upon the nature of the material responses, nevertheless such a study is in general performed by confining the analysis to the case of "small" disturbances. Frequently, the term "small" means that the nonlinear effects are neglected and that wave propagation is examined simply by considering harmonic solutions of the resulting linear system. In so doing, we are in fact dealing with an approximate problem, which fact is unsatisfactory both from the physical and the mathematical point of view.

The fundamental idea underlying the discontinuity wave theory is that of adopting a different concept of wave propagation in which the disturbance is limited, rigorously, to a region of no volume at all, namely a surface, but the disturbance itself may be of any amount [2]. The resulting theory is intrinsically exact and mathematically appealing: as a prominent consequence, nonlinearities do not represent a problem at all and are embodied into the theory in a straightforward manner. Apart from rigor, the difference in concept is great. While in the theory of small disturbances the wave surface is assumed in a special form, usually plane, and the disturbances themselves are assumed to depart little from equilibrium, the discontinuities can be represented by a smooth surface of any shape and the unperturbed state of the material can be any we please.

As a significant example, Truesdell [2] mentions that the speed c of propagation of disturbances in gases has the form

$$c^2 = \frac{dp}{d\rho};$$

however, in the cases of small disturbances the unperturbed state is constant and hence the density ρ and the speed c are constant, whereas for discontinuity waves ρ need not be a constant thereby rendering the speed c a function of the state, namely of the density ρ itself.

Finally it is worth mentioning that the derivation of the evolution law for the amplitude of the jump along the bicharacteristics does not represent a problem and can be done once for all under very general assumptions [3].

Today, the theory of discontinuity wave propagation is considered a powerfully tool applicable to a large class of problems belonging to physical and engineering disciplines. In other words, such a theory represents an interdisciplinary and efficient method for dealing with theoretical and applied problems concerning the propagation of disturbances.

However when this theory is applied to the problem of oblique incidence on a boundary some unsuspected drawbacks and pitfalls arise especially in connection with the generation of evanescent waves.

2. Evanescent waves in a word.

An evanescent wave is essentially a wave disturbance which, while traveling along a direction, attenuates along a different direction; an outstanding example of evanescent wave is the celebrated Rayleigh wave—see, e.g., [4, p. 220]. It is a remarkable result that, for hyperbolic system, such two direction cannot be parallel [5]. Note that most of the properties of evanescent waves do not depend on the actual attenuation; accordingly, when the attenuation is not relevant, harmonic exponential waves of this kind are referred to as inhomogeneous waves in the literature—see, e.g., [6] and references therein. So as to stress that attenuation is explicitly taken into account, in accordance with [4, p. 204] we use the term "evanescent wave".

The relevance of evanescent waves to the problem of oblique incidence of waves on an interface between to different media is well known in the literature—see, e.g., [7]. Our purpose is that of employing and re-elaborating this background material so as to solve the oblique incidence problem in a closed, although formal, form under the very general assumption that the governing equations constitute a linear hyperbolic system of differential equations. As a conclusion we are able to prove that the oblique incidence problem does not admit any solution within the sole framework of discontinuity wave theory.

3. Plane traveling waves.

Consider a physical system whose behavior is described by the following N linear hyperbolic differential equations

$$\frac{\partial U}{\partial t} + A^x \frac{\partial U}{\partial x} + A^y \frac{\partial U}{\partial y} + A^z \frac{\partial U}{\partial z} = 0,$$

where A^x, A^y, and A^z are constant $N \times N$ matrices. A plane wave, traveling along the direction $\mathbf{n}$ ($\mathbf{n} \cdot \mathbf{n} = 1$) at speed c, is a solution of the form

$$U = U(\varphi),$$

where the phase φ is a function of time t and space coordinates $\mathbf{x}$ of the form

$$\varphi(\mathbf{x}, t) = t - \mathbf{n} \cdot \mathbf{x}/c.$$

Split the solution $U(\varphi)$ as the product of the amplitude $\mathcal{U}(\varphi)$ and the polarization vector Π, namely

$$U(\varphi) = \mathcal{U}(\varphi)\Pi;$$

evidently such a decomposition is not unique. Then, on substituting the ansatz $U = U(\varphi)$ into the linear system considered and on letting $\mathbf{n} = (\mu\ \sigma\ \nu)$, we conclude that the propagation speeds are determined by the algebraic equation

$$\det |A_n - cI| = 0,$$

where I stands for the N dimensional identity matrix and $A_n = \mu A^x + \sigma A^y + \nu A^z$. Meanwhile, the polarization vector is a solution to the algebraic system

$$(A_n - cI)\Pi = 0.$$

Note that, in general, $c = c(\mathbf{n})$ and $\Pi = \Pi(\mathbf{n})$.

It is worth noticing that the previous conditions formally coincide with those relevant to the propagation of discontinuity waves thereby confirming the well-known result that plane discontinuities (or acceleration waves), plane shocks, and plane traveling waves possess the same wave fronts. In this sense, studying the behavior of plane waves is tantamount to studying the behavior of discontinuity waves.

4. The oblique incidence problem.

Suppose that the plane $y = 0$ is the boundary between two different media: the first one occupies the lower half space $y \leq 0$ and its behavior is described by the previous linear system, whereas the second medium fills the upper half space $y \geq 0$ and is governed by the linear system

$$\frac{\partial \tilde{U}}{\partial t} + \tilde{A}^x \frac{\partial \tilde{U}}{\partial x} + \tilde{A}^y \frac{\partial \tilde{U}}{\partial y} + \tilde{A}^z \frac{\partial \tilde{U}}{\partial z} = 0,$$

where $\tilde{A}^x$, $\tilde{A}^y$, and $\tilde{A}^z$ are constant $N \times N$ matrices. Suppose also that a plane wave, traveling into the lower medium along the direction $\mathbf{n}_\mathrm{I}$ at speed $c(\mathbf{n}_\mathrm{I})$, impinges on the boundary $y = 0$ thereby generating reflected and transmitted waves.

As is well known [8,9], the problem of oblique incidence can fully be analyzed by confining ourselves on the so-called plane of incidence, namely the plane spanned by the normal to the interface and the propagation direction of the incidence wave. Accordingly the situation may be summarized as follows.

$$\mathbf{n}_\mathrm{I} = (\mu_\mathrm{I}\ \sigma_\mathrm{I}\ 0),$$

$$\mu_\mathrm{I}, \sigma_\mathrm{I} > 0, \quad \mu_\mathrm{I}^2 + \sigma_\mathrm{I}^2 = 1.$$

Incident Wave:

$$U^\mathrm{I} = \mathcal{U}^\mathrm{I}\left(t - \frac{\mu_\mathrm{I} x + \sigma_\mathrm{I} y}{c(\mu_\mathrm{I}, \sigma_\mathrm{I})}\right)\Pi(\mu_\mathrm{I}, \sigma_\mathrm{I}).$$

Reflected Waves ($\mathrm{R} = 1, \ldots, p$):

$$U^\mathrm{R} = \mathcal{U}^\mathrm{R}\left(t - \frac{\mu_\mathrm{R} x + \sigma_\mathrm{R} y}{c(\mu_\mathrm{R}, \sigma_\mathrm{R})}\right)\Pi(\mu_\mathrm{R}, \sigma_\mathrm{R}).$$

Transmitted Waves ($\mathrm{T} = 1, \ldots, q$):

$$\tilde{U}^\mathrm{T} = \tilde{\mathcal{U}}^\mathrm{T}\left(t - \frac{\mu_\mathrm{T} x + \sigma_\mathrm{T} y}{\tilde{c}(\mu_\mathrm{T}, \sigma_\mathrm{T})}\right)\tilde{\Pi}(\mu_\mathrm{T}, \sigma_\mathrm{T}).$$

As is well known [9], determining the reflection-transmission pattern consists in solving Snell's law and in evaluating the amplitudes of all the emergent waves.

In details, Snell's law relies on a geometric analysis of the problem and establishes that

$$\frac{\mu_\mathrm{I}}{c(\mu_\mathrm{I}, \sigma_\mathrm{I})} = \frac{\mu_\mathrm{R}}{c(\mu_\mathrm{R}, \sigma_\mathrm{R})} = \frac{\mu_\mathrm{T}}{\tilde{c}(\mu_\mathrm{T}, \sigma_\mathrm{T})}.$$

Since the quantity σ is subjected to the only constraint $\mu^2 + \sigma^2 = 1$, the reflection-transmission pattern is not unique. To restore uniqueness we must impose the requirement of causality [8]; however the presence of evanescent waves makes this point a bit subtle [10].

On the other hand, the amplitude evaluation is dynamical in nature and requires the knowledge of interfacial conditions. On assuming that no fields are localized on the interface, we are allowed to stipulate that the fields on both sides are connected by the Rankine-Hugoniot conditions [11], which establish that, on the boundary $y = 0$,

$$A^y U^{\mathrm{I}} + \sum_{\mathrm{R}=1}^{p} A^y U^{\mathrm{R}} = \sum_{\mathrm{T}=1}^{q} \tilde{A}^y \tilde{U}^{\mathrm{T}},$$

or, explicitly,

$$\sum_{\mathrm{R}=1}^{p} A^y \Pi(\mu_{\mathrm{R}}, \sigma_{\mathrm{R}}) \mathcal{U}^{\mathrm{R}}(\varphi_0) - \sum_{\mathrm{T}=1}^{q} \tilde{A}^y \tilde{\Pi}(\mu_{\mathrm{T}}, \sigma_{\mathrm{T}}) \tilde{\mathcal{U}}^{\mathrm{T}}(\varphi_0) + A^y \Pi(\mu_{\mathrm{I}}, \sigma_{\mathrm{I}}) \mathcal{U}^{\mathrm{I}}(\varphi_0) = 0;$$

note that, owing to Snell's law, the argument of all amplitudes coincide with the quantity $\varphi_0 = t - x\mu_{\mathrm{I}}/c(\mu_{\mathrm{I}}, \sigma_{\mathrm{I}})$.

5. Real waves and evanescent waves.

Suppose now that Snell's law gives rise only to real values of μ subject to the condition $\mu \leq 1$; then we say that all the emergent waves are real. On assuming that the propagation speeds are positive, we must choose

$$\sigma_{\mathrm{R}} = -\sqrt{1 - \mu_{\mathrm{R}}^2} < 0, \qquad \sigma_{\mathrm{T}} = \sqrt{1 - \mu_{\mathrm{T}}^2} > 0.$$

Moreover, we must suppose that the algebraic system for determining the amplitudes of the emergent waves admits a unique solution for the $p + q$ quantities $\mathcal{U}^{\mathrm{R}}$ and $\tilde{\mathcal{U}}^{\mathrm{T}}$. Note, however, that this is not the case for grazing incidence in linear elastic crystals [12].

On the other hand, it often happen that Snell's law gives rise to values of μ which can be either greater than 1 or even complex quantities [13]. In both cases $\sigma = \sqrt{1 - \mu^2}$ turns out to be a complex quantity and the corresponding wave does not travel at a characteristic speed [5]. The wave associated with these values is called an "evanescent wave".

For the sake of generality, suppose that the quantity μ takes a complex value. As an immediate consequence of the previous analysis the polarization vector, the Rankine-Hugoniot conditions, and hence the wave amplitude involve complex quantities thereby rendering their meaning not immediate. The main idea is that of exploiting such formal complex solutions as a convenient tool for arriving at the explicit (real) form of the evanescent solutions we are looking for. The appearance of complex quantities can fruitfully be exploited.

To fix notation denote the real and imaginary part of a complex quantity ψ in accordance with the formula

$$\psi = \psi_1 + i\psi_2, \qquad i = \sqrt{-1}.$$

As a consequence of Snell's law, even when μ and $c(\mu, \sigma)$ are complex quantities it turns out that

$$\frac{\mu}{c} = \frac{\mu_1}{c_1} = \frac{\mu_2}{c_2} \in \mathrm{I\!R};$$

accordingly, the phase φ formally becomes

$$\varphi = t - \frac{\mu_1}{c_1} x - \alpha y,$$

where

$$\alpha = \frac{\sigma}{c} = \frac{c_1\sigma_1 + c_2\sigma_2}{c_1^2 + c_2^2} + i\,\frac{c_1\sigma_2 - c_2\sigma_1}{c_1^2 + c_2^2}.$$

Equivalently

$$\varphi = \tau - iY,$$

where

$$\tau = t - \frac{\mu_1}{c_1} x - \alpha_1 y,$$

$$Y = \alpha_2 y.$$

The fact that the phase φ is a complex function is to be interpreted, in term of real quantities, as the suggestion that evanescent-wave solutions to the governing system are in fact functions of the two variables τ and Y. Hence, on assuming that $U = U(\tau, Y)$, substitution into the original system yields

$$A^\tau \frac{\partial U}{\partial \tau} + A^Y \frac{\partial U}{\partial Y} = 0,$$

where

$$A^\tau = I - \frac{\mu_1}{c_1} A^x - \alpha_1 A^y ,$$

$$A^Y = \alpha_2 A^y .$$

So as to find a solution $U(\tau, Y)$ to the previous system, we can profitably use formally the presence of complex quantities in the following way. Consider the complex column vector

$$\widehat{U} = \mathcal{U}(\tau - iY)\Pi(\mu, \sigma),$$

where, in the present case,

$$\Pi = \Pi_1 + i\Pi_2,$$

$$\mathcal{U}(\tau - iY) = \mathcal{U}_1(\tau, Y) + i\mathcal{U}_2(\tau, Y).$$

Explicitly, we can write

$$\widehat{U} = (\mathcal{U}_1 + i\mathcal{U}_2)(\Pi_1 + i\Pi_2) = (\mathcal{U}_1\Pi_1 - \mathcal{U}_2\Pi_2) + i(\mathcal{U}_1\Pi_2 + \mathcal{U}_2\Pi_1).$$

It is a straightforward matter to ascertain that, whatever function $\mathcal{U}$ we choose, the real and imaginary part of $\widehat{U}$ separately satisfy the system

$$A^\tau \frac{\partial U}{\partial \tau} + A^Y \frac{\partial U}{\partial Y} = 0.$$

In view of the linearity of the differential system under consideration, the general form of the solution is obtained as a superposition of the real and imaginary part of the complex quantity $\widehat{U}$.

The last step in solving the problem of the oblique incidence is that of calculating the amplitude of the emergent waves through the use of the Rankine-Hugoniot conditions. Since

$$\widehat{U} = (\mathcal{U}_1\Pi_1 - \mathcal{U}_2\Pi_2) + i(\mathcal{U}_1\Pi_2 + \mathcal{U}_2\Pi_1)$$

the Rankine-Hugoniot conditions split as

$$\sum_{\mathrm{R}=1}^{p} A^y(\mathcal{U}_1^{\mathrm{R}}\Pi_1^{\mathrm{R}} - \mathcal{U}_2^{\mathrm{R}}\Pi_2^{\mathrm{R}}) - \sum_{\mathrm{T}=1}^{q} \tilde{A}^y(\tilde{\mathcal{U}}_1^{\mathrm{T}}\tilde{\Pi}_1^{\mathrm{T}} - \tilde{\mathcal{U}}_2^{\mathrm{T}}\tilde{\Pi}_2^{\mathrm{T}}) + A^y\mathcal{U}^{\mathrm{I}}\Pi^{\mathrm{I}} = 0,$$

$$\sum_{\mathrm{R}=1}^{p} A^y(\mathcal{U}_1^{\mathrm{R}}\Pi_2^{\mathrm{R}} + \mathcal{U}_2^{\mathrm{R}}\Pi_1^{\mathrm{R}}) \sum_{\mathrm{T}=1}^{q} \tilde{A}^y(\tilde{\mathcal{U}}_1^{\mathrm{T}}\tilde{\Pi}_2^{\mathrm{T}} + \tilde{\mathcal{U}}_2^{\mathrm{T}}\tilde{\Pi}_1^{\mathrm{T}}) = 0.$$

As the solution to the system of the Rankine-Hugoniot condition will depend linearly on the amplitude $\mathcal{U}^{\mathrm{I}}$ of the incident wave, we set

$$\mathcal{U}^{\mathrm{R}} = k^{\mathrm{R}}\mathcal{U}^{\mathrm{I}}, \qquad \tilde{\mathcal{U}}^{\mathrm{T}} = k^{\mathrm{T}}\mathcal{U}^{\mathrm{I}},$$

where the constants k^{R} and k^{T} are complex quantities. By substitution, we arrive at the following algebraic system

$$\sum_{\mathrm{R}=1}^{p} A^y(\Pi_1^{\mathrm{R}}k_1^{\mathrm{R}} - \Pi_2^{\mathrm{R}}k_2^{\mathrm{R}}) - \sum_{\mathrm{T}=1}^{q} \tilde{A}^y(\tilde{\Pi}_1^{\mathrm{T}}k_1^{\mathrm{T}} - \tilde{\Pi}_2^{\mathrm{T}}k_2^{\mathrm{T}}) + A^y\Pi^{\mathrm{I}} = 0,$$

$$\sum_{\mathrm{R}=1}^{p} A^y(\Pi_2^{\mathrm{R}}k_1^{\mathrm{R}} + \Pi_1^{\mathrm{R}}k_2^{\mathrm{R}}) - \sum_{\mathrm{T}=1}^{q} \tilde{A}^y(\tilde{\Pi}_2^{\mathrm{T}}k_1^{\mathrm{T}} + \tilde{\Pi}_1^{\mathrm{T}}k_2^{\mathrm{T}}) = 0,$$

which allows us to determine the $2(p+q)$ quantities k_1^{R}, k_2^{R}, k_1^{T}, and k_2^{T}. We explicitly assume that the such a system admits a unique solution.

Look once more at the Rankine-Hugoniot conditions. It is of fundamental importance to realize that, whereas the real part of $\widehat{U}$ must explicitly depend on the amplitude $\mathcal{U}^{\mathrm{I}}$ of the incident wave, the imaginary part of $\widehat{U}$ satisfy a homogeneous Rankine-Hugoniot equation, which loses track of the quantity $\mathcal{U}^{\mathrm{I}}$. Hence the imaginary part of $\widehat{U}$ can be determined by means of an arbitrary function, say $\mathcal{V}^{\mathrm{I}}$. In conclusion, on defining the two functions

$$\mathcal{U} = (k_1 + ik_2)(\mathcal{U}_1^{\mathrm{I}} + i\mathcal{U}_2^{\mathrm{I}}),$$

$$\mathcal{V} = (k_1 + ik_2)(\mathcal{V}_1^{\mathrm{I}} + i\mathcal{V}_2^{\mathrm{I}}),$$

in which the quantities k_1 and k_2 single out the relevant reflected or transmitted wave, the general solution to the oblique-incidence problem can be written as

$$U = (\mathcal{U}_1\Pi_1 - \mathcal{U}_2\Pi_2) + (\mathcal{V}_1\Pi_2 + \mathcal{V}_2\Pi_1),$$

or

$$U = (\mathcal{U}_1 + \mathcal{V}_2)\Pi_1 - (\mathcal{U}_2 - \mathcal{V}_1)\Pi_2.$$

As a result the evanescent-wave solution depends not only on the amplitude $\mathcal{U}^{\mathrm{I}}$ of the incident wave but also on an arbitrary function $\mathcal{V}^{\mathrm{I}}$ which will be chosen by imposing the regularity of the solution itself.

6. Making the solution an evanescent wave.

Observe first that the evanescent-wave solution is the real part of a suitable complex N-dimensional column vector, precisely

$$U = \Re[(\mathcal{U}^{\mathrm{I}} - i\mathcal{V}^{\mathrm{I}})k\Pi],$$

where the values of the complex number k single out the specific emergent wave. Therefore, the regularity of the solution, which must be "evanescent" at "infinity" in the pertinent half space of definition, is accounted for through the regularity of the complex quantity $\mathcal{U}^{\mathrm{I}} - i\mathcal{V}^{\mathrm{I}}$, as function of $\tau - iY$. Fortunately, problems of this kind are already solved in the literature by having recourse to the Hilbert transforms [14]. Specifically, the Hilbert transform $g(x)$ of a function $f(x) \in L^2(-\infty, \infty)$ is defined as

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\chi)}{\chi - x}\, d\chi,$$

P denoting a principal value at $\chi = x$. In [14, p. 128] it is proved the following

THEOREM (Titchmarsh, 1937): *Let $z = x + iy$ and assume that $\Phi(x)$ is a complex function of the real variable x belonging to $L^2(-\infty, \infty)$. Then $\Phi(x)$ is the limit as $z \to x$ of an analytic function $\Phi(z)$, regular for $y > 0$, if and only if $\Phi(x) = f(x) - ig(x)$, where $g(x)$ is the Hilbert transform of $f(x)$.*

As an immediate consequence of this theorem the evanescent wave solutions are regular functions provided that the arbitrary function $\mathcal{V}^{\mathrm{I}}$ is chosen as the Hilbert transform of $\mathcal{U}^{\mathrm{I}}$.

7. A comparison between discontinuities and evanescent waves.

This section is devoted to a detailed comparison between discontinuities (D) and evanescent waves (E) in order to establish the differences and the similarities between these two kinds of waves, both of which are necessary for a consistent solution of the oblique incidence problem.

PHASE:

D: the phase is real;

E: the phase is complex.

PHASE SPEED:

D: the phase speed is a characteristic speed;

E: if the speed c is real, the phase speed is less than or equal to a characteristic speed [5].

WAVE FRONT:

D: only one distinguished plane:

the plane at constant phase $\varphi = \text{const}$;

E: two distinguished planes:

the plane at constant phase $\tau = \text{const}$,

the plane at constant amplitude $Y = \text{const}$.

AMPLITUDE OF THE INCIDENT WAVE:

D: in the case of linear systems, the amplitude is unimportant;

E: the amplitude is of fundamental importance for evaluating the form of the wave.

We are now in a position of drawing the conclusions which follow from this work. Here we have examined a physical problem where evanescent waves are of vital importance, namely the oblique incidence of a plane wave on a boundary. Indeed, we have proved that calculation of the amplitudes of the emergent waves requires the account of possible evanescent waves. Owing to the equivalence between plane waves and discontinuities, the same is true when the incident wave is a discontinuity wave. Unfortunately, in this latter case the discontinuity wave does not provide any information on the amplitude $\mathcal{U}^{\mathrm{I}}$ of the associated plane wave; therefore it is not possible to determine the explicit form of the possible emergent evanescent waves. Accordingly, the result is that such kind of problems cannot be solved within the sole framework of discontinuity waves which, in spite of being a very powerful approach to wave propagation, fail in the problem of the oblique incidence.

8. Further remarks.

It is worth pointing out that the our approach can easily embody the Stoneley problem [8], whose importance has been recognized since a long time—see, e.g., [15] and reference therein. In fact the Stoneley problem can be viewed as a problem of oblique incidence in the special case when the incident wave has a vanishing amplitude. With this interpretation, the relevant solution is assumed to be a function of the form $U = U(\tau, Y)$ where now

$$\tau = t - \lambda x - \alpha_1 y, \qquad Y = \alpha_2 y.$$

Of course, this procedure does not guarantee that the Rankine-Hugoniot conditions—with $\mathcal{U}^{\mathrm{I}} = 0$—are solvable; imposing their solvability results in suitable conditions between the real quantities λ, α_1, and α_2 which are exactly the so-called Stoneley conditions.

Finally we briefly discuss the influence of nonlinearities on the determination of evanescent waves. Indeed, unlike the case of a single discontinuity and the case of normal incidence, nonlinearities strongly affect the problem of oblique incidence. In essence, nonlinearity plays a twofold role. First, calculation of evanescent waves is not so clear and straightforward also because of the lack of a sort of Hilbert transform technique valid for nonlinear problems. Second, genuine nonlinearity, in the Lax sense [16], makes the amplitude of a discontinuity wave blow up in a finite time, called the critical time. Accordingly, when the incident wave is a discontinuity wave, its amplitude, as well as the amplitude of the emergent waves, can blow up in a finite time thereby rendering the problem intrinsically ill posed. It seems that this point has been overlooked in the literature [9,17]. Although of formidable difficulty, we believe that the problem of oblique incidence in presence of nonlinearity deserves further special attention and analysis.

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