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BREAKING SOLITONS. SYSTEMS OF HYDRODYNAMIC TYPE

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1. Introduction.

I. Certain class of integrable $n+1$-dimensional equations was studied by F. Calogero and A. Degasperis in works [1-3] by using the generalized Wronskian relations. Some general Lax type operator equation was proposed by V.E. Zakharov [4] for constructing of $n+1$-dimensional integrable equations. These constructions were discussed also in the monograph by R.K. Dodd, J.C. Ellbeck, J.D. Gibbon and H.C. Morris [5].

The equations which are studied in this paper and in works [6,7] are not integrable for the general initial data, but their N-soliton solutions may be found explicitly and they possess the breaking behaviour. We consider the differential equations, which are equivalent to the following equation in space of linear operators L and A:

$$L_t = P(L) + \sum_{k=1}^{n} R_k(L, L_{y_k}) + [L, A], \tag{1.1}$$

where $P(L)$ and $R_k(L, L_{y_k})$ are certain meromorphic functions of operator L, fucntions $R_k(L, L_{y_k})$ are linear with respect to L_{y_k}. We assume that operators L and A depend on the variable $t, y_1, \dots, y_n$

and $L_{y_k} = \partial L/\partial y_k$. L and A are supposed to be $n \times n$ matrices or 1-dimensional differential operators (in the last case L is self-adjoint operator, A is skew-symmetric operator).

Coefficients of meromorphic functions $P(L)$, $R_k(L, L_{y_k})$ are assumed to depend on invariants of operators L and their derivatives with respect to variables t, $y_1, \dots, y_n$, that is the coefficients do not change after the transformation $L \to QLQ^{-1}$.

LEMMA 1. *In view of equation (1.1) the eigenvalues $f(t, y_1, \dots, y_n)$ of the operator L satisfy the system of equations*

$$f_t = P(f) + \sum_{k=1}^{n} R_k(f, f_{y_k}). \tag{1.2}$$

If coefficients of functions $P(L)$, $R_k(L, L_{y_k})$ are constants, the system (1.2) is splited into noninteracting equations for each eigenvalue $f_j(t, y_1, \dots, y_n)$:

$$f_{jt} = P(f_j) + \sum_{k=1}^{n} R_k(f_j, f_{jy_k}). \tag{1.3}$$

Proof. is done in work [7].

In the case

$$P(L) = 0, \quad R_k(L, L_{y_k}) = \sum_{0 \le i \le m}^{\infty} c_{ik}^{m} L^{m-i} L_{y_k} L^{i} \tag{1.4}$$

the equation (1.3) is the conservation law and we get as a consequence

$$(f_j^p)_t = \sum_{k=1}^{n} \left(\sum_{0 \le i \le m}^{\infty} \frac{p c_{ik}^{m}}{m+p} f_j^{m+p} \right)_{y_k}. \tag{1.5}$$

Hence, assuming that the eigenvalues $f_j(t, y_1, \dots, y_n)$ tend to zero rapidly enough for $|y_k| \to \infty$ and applying the Gauss-Ostrogradskij theorem, we obtain the conserved quantities

$$\begin{aligned} \frac{dJ_p}{dt} &= 0, \quad J_p = \int_{R^n} (f_j)^p dy_1, \dots, dy_n, \\ \frac{dI_p}{dt} &= 0, \quad I_p = \int_{R^n} Tr(L^p) dy_1, \dots, dy_n. \end{aligned} \tag{1.6}$$

These properties distinguish essentially the equation (1.1) from Lax equation $L_t = [L, A]$. Equation (1.1) as $P(L) \neq 0$, $R_k(L, L_{y_k}) = 0$ has the attractors in the phase space, see work [6].

2. System of hydrodynamic type, connected with the Toda lattice.

Let us consider the operator equation

$$L_t = LL_y + L_yL + [L, A], \tag{2.1}$$

where matrices L and A have the form

$$L = \begin{pmatrix} p_1 & a_1 & 0 & & 0 \\ a_1 & p_2 & a_2 & & \\ 0 & a_2 & p_3 & & \\ & & & \ddots & a_{n-1} \\ 0 & & & a_{n-1} & p_n \end{pmatrix},$$
$$A = \begin{pmatrix} 0 & x_1 & & & 0 \\ -x_1 & 0 & x_2 & & \\ & -x_2 & 0 & & \\ & & & \ddots & x_{n-1} \\ 0 & & & -x_{n-1} & 0 \end{pmatrix}. \tag{2.2}$$

The operator equation (2.1), (2.2) after the substitution

$$a_i = \exp(q_{i+1} - q_i)$$

is reduced to the following system of equations

$$\begin{aligned} p_{i_t} &= 2p_ip_{iy} + 4(q_{i+1,y} + \beta)e^{2(q_{i+1}-q_i)} - 4(q_{i-1,y} + \beta)e^{2(q_i-q_{i-1})}, \\ q_{i_t} &= 2p_iq_{iy} + p_{i_y} + 2\sum_{k=1}^{k=i-1} p_{k_y} + 2\beta p_i. \end{aligned} \tag{2.3}$$

where β is an arbitrary function of t, y. For solutions, independent on y, system (2.3) turns into the famous Toda lattice [9-13], system (2.3) as $\beta = 0$ is a system of hydrodynamic type, following terminology of works [8,14].

According to the Lemma 1, the eigenvalues $f_k(t, y)$ of the matrix L (2.2) due to the system (2.3) satisfy the equation

$$f_{k_t} = 2f_kf_{ky}. \tag{2.4}$$

Hence the eigenvalues $f_k(t,y)$ are the Riemman invariants for the system (2.3). Obviously they possess the breaking behaviour.

3. System of hydrodynamic type, connected with the Volterra model.

I. Let us consider the operator equation

$$L_t = LL_yL + [L,A] \tag{3.1}$$

for martices L and A of the following form

$$L = \begin{pmatrix} 0 & \sqrt{a_1} & & & 0 \\ \sqrt{a_1} & 0 & \sqrt{a_2} & & \\ & \sqrt{a_2} & 0 & & \\ & & & \ddots & \\ & & & & \sqrt{a_{n-1}} \\ 0 & & & \sqrt{a_{n-1}} & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & x_1 & & 0 \\ 0 & 0 & 0 & x_2 & \\ -x_1 & 0 & 0 & \ddots & x_{n-2} \\ & -x_2 & \ddots & \ddots & 0 \\ 0 & & -x_{n-2} & 0 & 0 \end{pmatrix}. \tag{3.2}$$

Operator equation (3.1), (3.2) is equivalent to the system of equations

$$\begin{aligned} a_{i_t} = a_i\Bigg(&\beta(a_{i+1} - a_{i-1})+ \\ &+ a_{iy} + a_{i+1,y} + a_{i+1}\sum_{k=1}^{k=i} a_{ky}a_k^{-1} - a_{i-1}\sum_{k=1}^{k=i-2} a_{ky}a_k^{-1}\Bigg), \end{aligned} \tag{3.3}$$

where β is an arbitrary function of t, y. System (3.3) for solutions, independent on y, turns into Volterra model [13], system (3.3) as $\beta = 0$ is system of hydrodynamic type.

Due to the Lemma 1 we get that eigenvalues $f_k(t,y)$ of the matrix L (3.2) in view of system (3.3) satisfy the equation

$$f_{k_t} = f_k^2 f_{ky}. \tag{3.4}$$

Hence they are the Riemann invariants for the system (3.3) and have braking behaviour.

II. System (3.3) after transformation $a_i = \exp u_i$ turns into the system

$$u_{i_t} = \beta(e^{u_{i+1}} - e^{u_{i-1}}) + e^{u_i} u_{i_y} + e^{u_{i+1}} \sum_{k=1}^{k=i+1} u_{ky} + e^{u_{i-1}} \sum_{k=1}^{k=i-2} u_{ky}. \tag{3.5}$$

System (3.5) has the following form

$$u_{i_t} = \sum_{j=1}^{n-1} A^{ij} \frac{\partial H}{\partial u_j}, \quad H = \frac{1}{2} TrL^2 = e^{u_1} + e^{u_2} + \ldots + e^{u_{n-1}}. \tag{3.6}$$

The operators A^{ij} are skew-symmetric and have the form

$$A^{ij} = g^{ij} \frac{\partial}{\partial y} + \sum_{k=1}^{n-1} b_k^{ij} u_{ky} + \beta I^{ij}. \tag{3.7}$$

Here the coefficients g^{ij}, b_k^{ij}, I^{ij} are constants and non-zero only in the following cases

$$\begin{gathered} g^{ii} = g^{i,i+1} = g^{i+1,i} = 1, \quad I^{i,i+1} = -I^{i+1,i} = 1, \\ b_k^{i,i+1} = -b_k^{i+1,i} = 1, \quad \text{as} \quad 1 \le k \le i. \end{gathered} \tag{3.8}$$

Scalar product of two functionals $F(u)$ and $G(u)$

$$\langle F, G \rangle = \int_{-\infty}^{\infty} \frac{\delta F}{\delta u_i} A^{ij} \frac{\delta G}{\delta u_j} dy \tag{3.9}$$

is skew-symmetric. As a consequence of (3.6) we get the conservation law

$$\int_{-\infty}^{\infty} H dy = \frac{1}{2} \int_{-\infty}^{\infty} TrL^2 dy = const. \tag{3.10}$$

The same conservation law (3.10) and representation of the form (3.6), (3.7) do exist also for the system (2.3).

III. Let us suppose that there exists a smooth function $v(t,x,y)$, such that

$$a_j(t,y) = 1 - \varepsilon^2 v(t,x_j,y), \quad x_j = j\varepsilon, \quad \beta = 3\beta_0\varepsilon^{-1}. \tag{3.11}$$

System (3.3) after the substitution of

$$t' = -\varepsilon^2 t, \quad y' = y + 4t, \quad x' = x + 6\beta_0 t$$

and passing to the continuous limit $\varepsilon \to 0$ is transformed into the equation

$$v_t = 4vv_y + 2v_x \int_0^x v_y(t,\xi,y)d\xi - v_{xxy} + \beta_0(6vv_x - v_{xxx}). \tag{3.12}$$

This equation belongs to the class of equations, studied by F. Calogero and A. Degasperis [1-3]. In the following paragraph we study the concrete properties of the equation (3.12).

4. Equation of interaction of Riemann breaking wave with transversal KdV long waves.

I. *Physical sense.* Equation (3.12) for functions $v = v(t,y)$ takes the form of the Riemann breaking wave equation

$$v_t = 4vv_y. \tag{4.1}$$

For the functions $v = v(t,z)$, $z = x + cy$ equation (3.12) turns into KdV equation

$$v_t = (c + \beta_0)(6vv_z - v_{zzz}). \tag{4.2}$$

So equation (3.12) describes interaction between Riemann breaking waves (4.1), running in y-direction and KdV long waves (4.2), travelling in transversal directions.

II. *Hamiltonian structure.* Equation (3.12) after substitution $v = u_x$ takes the form

$$u_{tx} = 4u_xu_{xy} + 2u_yu_{xx} - u_{xxxy}. \tag{4.3}$$

This equation has Hamiltonian form

$$u_t = \partial_x^{-1}\frac{\delta H}{\delta u}, \ H = \iint_{-\infty}^{\infty} \left(\frac{1}{2}u_{xxx} - u_x^2\right) u_y \, dx dy. \tag{4.4}$$

III. *Operator representation.* Equation (4.3) is equivalent to the following operator equation

$$L_t = 2(LL_y + L_y L) + [L, A], \ L = -\partial_x^2 + u_x, \ A = -u_y \partial_x - \partial_x u_y. \tag{4.5}$$

So according to the Lemma 1 the eigenvalues $f_k(t, y)$ of the Schrödinger operator $L = -\partial_x^2 + u_x$ in view of equation (4.3) satisfy the Riemann breaking wave equation

$$f_{k_t} = 4 f_k f_{ky}. \tag{4.6}$$

If one takes the operator A of the form

$$A = -u_y \partial_x - \partial_x u_y - 2F(t, y)\partial_x, \tag{4.7}$$

where $F(t, y)$ is an arbitrary function of t, y, then from equation (4.5) one gets

$$u_{xt} = 4u_x u_{xy} + 2(u_y + F(t, y))u_{xx} - u_{xxxy}. \tag{4.8}$$

Operator equation (4.5) may be written also in the Lax form

$$L_t = [L, A - L\partial_y - \partial_y L].$$

Equations (4.3), (4.8) possess also the operator representation analogous to the zero-curvature representation

$$U_t - V_x + [U, V] = 4\lambda^2 U_y, \tag{4.9}$$

$$U = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ u_x & 0 \end{pmatrix},$$

$$V = 2i\lambda \begin{pmatrix} u_y + F & 0 \\ u_{xy} & -u_y - F \end{pmatrix} + \begin{pmatrix} -u_{xy} & 2(u_y + F) \\ 2u_x(u_y + F) - u_{xxy} & u_{xy} \end{pmatrix}.$$

Here λ is an arbitrary spectral parameter. Equation (4.9) is the compatibility condition for linear system of equations

$$\psi_x = U\psi, \quad \psi_t = 4\lambda^2 \psi_y + V\psi \tag{4.10}$$

and has commutator form

$$[\partial_x - U, \partial_t - 4\lambda^2\partial_y - V] = 0. \tag{4.11}$$

IV. *Evolution of scattering data.* We consider the one-dimensional scattering problem, associated with the one-dimensional Schrödinger operator $L = -\partial_x^2 + u_x$. We suppose that potential $u_x(t,x,y)$ tends to zero as $x \to \pm\infty$. The primitive function $u(t,x,y)$ has the following asymptotics:

$$\begin{aligned} &u(t,x,y) \to g(t,y), \quad \text{as} \quad x \to -\infty, \\ &u(t,x,y) \to h(t,y), \quad \text{as} \quad x \to +\infty. \end{aligned} \tag{4.12}$$

Evolution of the scattering data $a(k,t,y)$, $b(k,t,y)$, $f_k(t,y)$, $b_k(t,y)$ due to equation

$$u_{tx} = 4u_x u_{xy} + 2(u_y + F(t,y))u_{xx} - u_{xxxy} + \gamma(6u_x u_{xx} - u_{xxxx}) \tag{4.13}$$

is described by equations

$$\begin{aligned} &a_t - 4k^2 a_y = 2ik(g_y - h_y)a, \\ &b_t - 4k^2 b_y = (2ik(g_y + h_y + 2F(t,y)) + 8\gamma i k^3)b, \\ &b_{k_t} + 4\lambda_k b_{k_y} = (-2\lambda_k(g_y + h_y + 2F(t,y)) + 8\gamma\lambda_k^3)b_k, \\ &f_{k_t} = 4f_k f_{ky}, \quad f_k = -\lambda_k^2. \end{aligned} \tag{4.14}$$

These equations as $g(t,y) \equiv 0$, $h(t,y) \equiv 0$ coincide with equations obtained by F. Calogero and A. Degasperis [1-3], and in this case are integrable linear equations.

In general case $g(t,y) \neq 0$, $h(t,y) \neq 0$ equations (4.14) are nonlinear and even are not closed, because they include asymptotics $g(t,y)$, $h(t,y)$ of unkown primitive function $u(t,x,y)$.

V. *N-soliton solutions.* One-soliton solution of equation (4.13) was found in [1-3] and has the form

$$u_x = \frac{-2\lambda^2}{\cosh^2(\lambda x - \varphi)}, \tag{4.15}$$

$$\lambda_t + 4\lambda^2\lambda_y = 0, \ \varphi_t + 4\lambda^2\varphi_y = 4\gamma\lambda^3, \ \lambda^2 = -f$$

N-soliton solutions, found in [6], are determined in accordance with Hirota's method by the formulae

$$(4.16)\qquad u(t,x,y) = -2\frac{d}{dx} ln \det A(t,x,y) - 2\sum_{n=1}^{N} \lambda_n(t,y),$$

$$A_{kj}(t,x,y) = \delta_{kj} + \frac{\beta_k(t,y)}{\lambda_k+\lambda_j} e^{-(\lambda_k+\lambda_j)x},$$

$$\beta_n(t,y) = \frac{b_n(t,y)}{ia'(i\lambda_n)},\ a(k,t,y) = \prod_{n=1}^{N} \frac{k-i\lambda_n}{k+i\lambda_n},$$

$$b_{n_t} + 4\lambda_n b_{n_y} = (-2\lambda_k(g_y+h_y) + 8\gamma\lambda_n^3)b_n,\ \lambda_{n_t} + 4\lambda_n^2\lambda_{n_y} = 0,\ -\lambda_n^2 = f_n.$$

These formulae describe breaking N-solition solutions. The breaking of the graph of the function $u(t,x,y)$ takes place simultaneously on all axis x with the breaking of the graph for one of the functions $\lambda_n(t,y)$. A single valued branch of the function $u(t,x,y)$ corresponds to each choice of single-valued branches of the functions $\lambda_n(t,y)$. Derivative $u_x(t,x,y)$ has the form of the N-soliton solution of the KdV equation for each branch.

Solutions (4.16) are localized on the plane x,y if functions $\lambda_n(t,y)$ exponentially tend to zero as $|y| \to \infty$. The function $\lambda(t,y)$ in formula (4.15) may be taken as smooth solution of equation $\lambda_t + 4\lambda^2\lambda_y = 0$ identically equal zero as $|y| > C$; corresponding function $u_x(t,x,y) = 0$ as $|y| > C$. The main difference with localized multisoliton solutions of Davey-Stewartson equation [15-18] consists of phenomenon of breaking for solutions (4.16).

VI. *Shock N-soliton solutions.* Equation (4.16) for each eigenvalue $f_k(t,y)$ is the conservation law

$$(4.17)\qquad f_{k_t} - (2f_k^2)_y = 0.$$

It is possible to consider discontinuous solutions of (4.17) with Rankine-Hugoniot condition

$$(4.18)\qquad S[f_k] = [-2f_k^2],\quad S = -2(f_{k+} - f_{k-}),$$

where S denotes the speed of propagation of line of discontinuity $y = y(t)$, that is $s = dy/dt$. Corresponding N-soliton solution (4.16) has N shock waves, travelling with different speeds.

VII. *Modified 2+1-dimensional equation.* Equation (4.3) after Miura transformation

$$u_x = v^2 + \sigma v_x \tag{4.19}$$

gets the modified form

$$v_t = 4v^2 v_y + 2v_x \int_0^x (v^2)_y(t,\xi,y)d\xi - S v_{xxy}, \quad S = \sigma^2. \tag{4.20}$$

Modified equation (4.20) possesses the operator representation

$$L_t = \alpha(L_y L^2 + L^2 L_y) + [L, A] \tag{4.21}$$

with matrix operator L:

$$L = \begin{pmatrix} ip_1 & 0 \\ 0 & ip_2 \end{pmatrix} \partial_x + v \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.22}$$

where $\alpha = -(p_1 - p_2)^2/2p_1p_2$;

Breaking soliton of equation (4.20) as $S = -1$ has the form

$$v = \frac{\lambda}{\cosh(\lambda x - \varphi)}, \ \lambda_t = \lambda^2 \lambda_y, \ \varphi_t = \lambda^2 \varphi_y. \tag{4.23}$$

Equations for scattering data evolution are found in [6].

VIII. *Countable set of conservation laws.* Equation (4.3) after the Gardner transformation

$$u_x = W + \varepsilon W_x + \varepsilon^2 W^2 \tag{4.24}$$

turns into the equation

$$W_t = 2(W(u_y - \varepsilon W_y))_x + (W^2 - W_{xx})_y. \tag{4.25}$$

Substituting into this equation the formal power series

$$W(t,x,y,\varepsilon) = \sum_{n=0}^{\infty} P_n(u_x)\varepsilon^n = u_x - u_{xx}\varepsilon + (u_{xxx} - u_x^2)\varepsilon^2 \ldots, \tag{4.26}$$

$$2W(u_y - \varepsilon W_y) = \sum_{n=0}^{\infty} Q_n(u_x, u_y)\varepsilon^n, \ W^2 - W_{xx} = \sum_{n=0}^{\infty} R_n(u_x)\varepsilon^n,$$

we get the conservation laws

$$\frac{\partial P_n(u_x)}{\partial t} = \frac{\partial Q_n(u_x, u_y)}{\partial x} + \frac{\partial R_n(u_x)}{\partial y}. \tag{4.27}$$

Here $P_n(u_x)$ are the same differential polynomials of u_x as in the theory of KdV equation. From (4.27) the relation follows

$$\frac{d}{dt}\iint_{-\infty}^{\infty} P_n(u_x)dxdy = -\int_{-\infty}^{\infty} P_{n+2}(u_x)dx\Big|_{y=-\infty}^{y=+\infty}. \tag{4.28}$$

IX. *Connection with the integrable Klein-Gordon equations.*

PROPOSITION. *Any solution of Klein-Gordon equation*

$$\varphi_{xy} = f(\varphi), \tag{4.29}$$

where function $f(\varphi)$ satisfies the linear equation

$$Sf''(\varphi) = f(\varphi), \tag{4.30}$$

defines the solution of the modified equation

$$v_t = 4v^2 v_y + 2v_x\partial_x^{-1}(v^2)_y - Sv_{xxy} \tag{4.31}$$

by the formula

$$v(t,x,y) = \frac{1}{2}\varphi_x(x+c(t),y), \tag{4.32}$$

where $c(t)$ is an arbitrary function.

Obviously there are the following non-equivalent cases of equations (4.29)-(4.30):

$$S = +1 : \varphi_{xy} = e^{\varphi},\ \varphi_{xy} = \sin h\varphi,\ \varphi_{xy} = \cos h\varphi;$$

$$S = -1 : \qquad \varphi_{xy} = \sin\varphi.$$

Exact solutions of Liouville equation $\varphi_{xy} = e^{\varphi}$ lead to exact solutions of equation (4.31) ($S = +1$)

$$v(t,x,y) = \frac{1}{2}\frac{a''(x+c(t))}{a'(x+c(t))} - \frac{a'(x+c(t))}{a(x+c(t)+b(y))}, \tag{4.33}$$

which depend on three arbitrary functions $a(x)$, $b(y)$, $c(t)$.

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