LE MATEMATICHE
Vol. XLVI (1991) - Fasc. I, pp. 55-63

ATTRACTIVITY CONDITIONS FOR A PERTURBED LOTKA VOLTERRA MODEL

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A perturbed non-autonomous Lotka Volterra model with a diffusive term is considered. Conditions assuring non linear stability of a biological critical point are obtained also in the case in which the diffusivity coefficient is time periodic.

1. Introduction.

The analysis of perturbed Lotka - Volterra two dimensional predator-prey equations has been widely developped by many authors (see for istance, [3], [4], [5], [9], [10], [11], [12], [14], [15], [16])

Different types of perturbations terms have been introduced in order to account for (small) variations of the idealized hypotheses of the Lotka - Volterra models as well as to put controls on the growth of both predator and prey. In [10], [11], [12], small autonomous perturbations of very general type, depending on a small parameter, have been considered and their influence on the stability of the positive equilibrium point of the classic Lotka - Volterra model, or on the existence both of periodic solutions of the perturbed system or perturbed critical points. Recently, in [6] time dipendent perturbation terms, depending on the small parameter, have been introduced in the classic model and their influence on the stability of the equilibria of this model has been studied by using the multiple scale method.

In [7], [8], Fergola Rionero Tenneriello, consider the same type of nonautonomous perturbation, but they do not make any assumption on the " smallness " of these pertubations.

Precisely they consider the following system :

$$\begin{cases} \dot{x} = ax - bxy + D(t)(y - x) \\ \dot{y} = -cy + dxy + D(t)(x - y) \end{cases} \tag{1}$$

with a, b, c, d positive constants such that

$$\frac{c}{d} = \frac{a}{b}$$

and D=D(t) regular enough to assure global (in time) existence and uniqueness.

Using the Liapunov Direct Method they obtain conditions assuring exponential stability of

$$x^* = \frac{c}{d}\ ,\ y^* = \frac{a}{b}$$

under the foundamental assumption that:

$$\inf_{t \in R+} D(t) = D_0 > 0 \tag{2}$$

Because in the case $D = 0$, $x^* = y^*$ is stable, this restriction appears too strong. In order to overcome this restriction, in this comunication we reconsider the problem and obtain conditions assuring non linear stability under conditions on the function

$$\int_0^t D(\tau) d\tau \tag{3}$$

2. Main theorem.

Setting

$$\begin{cases} x = x^* + u \\ y = y^* + v \end{cases} \tag{4}$$

the problem is reduced to study the stability of $u = v = 0$ for the following system :

$$\begin{cases} \dot{u} = -D(t)u + m(t)v - buv \\ \dot{v} = n(t)u - D(t)v + duv \end{cases} \tag{5}$$

with

$$\begin{cases} m(t) = D(t) - a \\ n(t) = D(t) + c \end{cases} \tag{6}$$

Therefore, introducing the Liapunov function

$$\begin{cases} V(t,\lambda) = u^2 + \lambda^2 v^2 \\ \lambda = \text{ positive constant} \end{cases} \tag{7}$$

it follows:

$$\dot{V} \leq -F(\lambda,t)V + \alpha(\lambda)V^{\frac{3}{2}} \tag{8}$$

with

$$\begin{cases} F = 2D - |g| \\ g = \dfrac{m}{\lambda} + n\lambda = \dfrac{D-a}{\lambda} + (D+c)\lambda \\ \alpha(\lambda) = \dfrac{2}{3\sqrt{3}}(d + \dfrac{b}{\lambda}) \end{cases} \tag{9}$$

Setting

$$\begin{cases} H(\lambda,t) = V \ \mathbf{exp}\left[\int_0^t F(\lambda,\tau)d\tau\right] \\ G(\lambda,t) = \int_0^t \mathbf{exp}\left[-\dfrac{1}{2}\int_0^\tau F(\lambda,\xi)d\xi\right]d\tau \end{cases} \tag{10}$$

then it follows:

$$\dot{H} \leq \alpha(\lambda)H^{\frac{3}{2}}\mathbf{exp}\left[-\frac{1}{2}\int_0^t F(\lambda,\tau)d\tau\right] \tag{11}$$

THEOREM I. *If*

$$\exists\lambda \in R^+ : \lim_{t\to\infty}\int_0^t F(\lambda,\tau)d\tau = \infty \tag{12}$$

then

$$H_0^{\frac{1}{2}} < \frac{2}{\alpha(\lambda)G(\infty)} \tag{13}$$

assures the nonlinear attractivity of (x^*, y^*) *according to*

$$V^{\frac{1}{2}} \leq \frac{exp\left[-\frac{1}{2}\int_0^t F(\lambda,\tau)d\tau\right]}{H_0^{-\frac{1}{2}} - \frac{\alpha(\lambda)}{2}G(\infty)} \tag{14}$$

Proof. Integrating (11) in $[0, t]$ we obtain

$$H^{\frac{1}{2}}\left[H_0^{-\frac{1}{2}} - \frac{\alpha(\lambda)}{2}G(t)\right] \leq 1$$

then, by hypotheses (12) and (13), it follows :

$$V^{\frac{1}{2}} \leq \frac{\exp\left[-\frac{1}{2}\int_o^t F(\lambda,\tau)d\tau\right]}{H_0^{-\frac{1}{2}} - \frac{\alpha(\lambda)}{2}G(\infty)} \tag{14}$$

Remark 1. Because

$$\exp\left[-\frac{1}{2}\int_0^t F(\lambda,\tau)d\tau\right] = \exp\left[-\frac{1}{2}\int_0^{\bar{t}} F(\lambda,\tau)d\tau\right]\exp\left[-\frac{1}{2}\int_{\bar{t}}^t F(\lambda,\tau)d\tau\right]$$

with $t > \bar{t}$, one immediately obtains that , concerning the asympotic stability , the behaviour of $D(t)$ in $[0, \bar{t}]$ is unimportant with respect to the

$$\lim_{t\to\infty}\int_0^t F(\lambda,\tau)d\tau = \infty$$

3. Case D=D(t) periodic function.

Let

$$D : R^+ \to R$$

be a periodic function of time with period T.

Remark 2.

$$\lim_{t\to\infty}\int_0^t F(\lambda,\tau)d\tau=\infty \Longleftrightarrow \int_o^T F(\lambda,\tau)d\tau>0 \tag{15}$$

In fact D(t) is a periodic function, hence F is a periodic function with period T, too.

This implies that :

$$\int_0^t F(\lambda,\tau)d\tau \le (n+1)\int_0^T F(\lambda,\tau)d\tau+\int_{I_2} F(\lambda,\tau)d\tau \tag{16}$$

with:

$$\begin{cases} n\in N : nT\le t<(n+1)T \\ I_1=\{t\in[0,T]:F(\lambda,t)\le 0\} \\ I_2=\{t\in[0,T]:F(\lambda,t)>0\}\end{cases} \tag{17}$$

And hence:

$$\int_0^T F(\lambda,\tau)d\tau \ge \frac{1}{n+1}\left(\int_0^t F(\lambda,\tau)d\tau-\int_{I_2} F(\lambda,\tau)d\tau\right)$$

so that (15) immediately follows.

Remark 3. If $D: R^+\to R$ is a periodic function of time with period T we have :

$$\int_0^T F(\lambda,\tau)d\tau>0 \Rightarrow \int_0^T D(\tau)d\tau>0 \tag{18}$$

Proof. Because of

$$\int_0^T [2D(\tau)-|g|]d\tau=\int_0^T F(\lambda,\tau)d\tau>0$$

then it follows

$$2\int_0^T D(\tau)d\tau>\int_0^T |g|d\tau$$

THEOREM II. *If $D=D(t)$ is a periodic function of time with period T we have :*

$$\int_0^T F(\lambda,\tau)d\tau>0\Rightarrow$$

$$\frac{a - c\lambda^2}{(1+\lambda)^2} < \frac{1}{T}\int_0^T D(\tau)d\tau < \frac{a - c\lambda^2}{(1-\lambda)^2} \tag{19}$$

Proof. Because of

$$|g(\lambda, t)| \geq \pm g(\lambda, t)$$

we have

$$\int_0^T F(\lambda, \tau)d\tau > 0 \Rightarrow 2\int_0^T D(\tau)d\tau \geq \pm \int_0^T g(\lambda, \tau)d\tau$$

but

$$g(\lambda, t) = \frac{D - a}{\lambda} + (D + c)\lambda$$

from which

$$\pm \int_0^T g(\lambda, \tau)d\tau < 2\int_0^T D(\tau)d\tau$$

if and only if

$$\pm\frac{1}{\lambda}\int_0^T \Big[(D - a) + (D + c)\lambda^2\Big]d\tau < 2\int_0^T D(\tau)d\tau$$

i.e. for $\lambda > 0$

$$\pm \int_0^T [(D - a) + (D + c)\lambda^2]d\tau < 2\lambda \int_0^T D(\tau)d\tau$$

hence

$$\frac{a - c\lambda^2}{(1+\lambda)^2} < \frac{1}{T}\int_0^T D(\tau)d\tau < \frac{a - c\lambda^2}{(1-\lambda)^2} \tag{20}$$

Easily follows that if $a \geq c$, (20) does not give restriction to the mean value $\overline{D}$ of D :

$$\overline{D} = \frac{1}{T}\int_0^T D(\tau)d\tau \tag{21}$$

In the case $a < c$, one has :

THEOREM III. *If $D : R^+ \to R$ is a periodic function of time with period T and*

$$a < c$$

then

$$\left\{\begin{array}{c}\int_0^T F(\lambda,\tau)d\tau > 0 \\ \lambda \in R^+\end{array}\right. \Longrightarrow \frac{1}{T}\int_0^T D(\tau)d\tau \leq \frac{ac}{c-a} \tag{22}$$

Proof. If $a < c$, the function $\eta_1(\lambda)$

$$\eta_1(\lambda) = \frac{a - c\lambda^2}{(1-\lambda)^2} \tag{23}$$

is increasing from zero to $\frac{a}{c}$ and decreasing from $\frac{a}{c}$ to $\sqrt{\frac{a}{c}}$ with

$$\left\{\begin{array}{r}\eta_1(\overline{\lambda}) = \dfrac{ac}{c-a} \\ \overline{\lambda} = \dfrac{a}{c}\end{array}\right. \tag{24}$$

PROPOSITION I. *If $D : R^+ \to R$ is a periodic function of time with period T , then :*

$$\int_0^T |D-a|d\tau \int_0^T |D+c|d\tau < \left[\int_0^T D(\tau)d\tau\right]^2 \tag{25}$$

implies:

$$\exists\lambda \in R^+ : \int_0^T F(\lambda,\tau)d\tau > 0 \tag{26}$$

Proof. Because of

$$|g| \leq \frac{|D-a|}{\lambda} + |D+c|\lambda$$

one obtains :

$$\frac{1}{\lambda}\int_0^T |D-a|d\tau + \lambda\int_0^T |D+c|d\tau < 2\int_0^T D(\tau)d\tau \Longrightarrow$$
$$\Longrightarrow \int_0^T |g|d\tau < 2\int_0^T D(\tau)d\tau$$

i.e

$$\lambda^2\int_0^T |D+c|d\tau - 2\lambda\int_0^T D(\tau)d\tau + \int_0^T |D-a|d\tau < 0$$

which is satisfied for

$$\left[\int_0^T D(\tau)d\tau\right]^2 > \int_0^T |D+c|d\tau \int_0^T |D-a|d\tau$$

with $\lambda \in (\lambda_1, \lambda_2)$

REFERENCES

[1] Beretta E. , Fergola P., Tenneriello C. *Ultimate boundedness for nonautonomous diffusive Lotka Volterra patches.* Math. Biosci **92** (1988), 29-53.

[2] Beretta E., Fergola P., Tenneriello C., *Partial ultimate boundness for completely nonautonomous diffusive Lotka - Volterra patches.* R Tappl (ed.) Cybernetics and Systems. Kluwer Academic Publishers (1988), 399-406.

[3] Comins H.N., Blatt D.W.E., *Prey predator models in spatially heterogeneous environments.* J. Theor. Biol (1974), 75-83.

[4] Dutt R., Glosh P.K., *Nonlinear correction to Lotka Volterra oscillations in a prey predator system.* Math. Biosci **27** (1975) 9-16.

[5] Dutt R., Glosh P.K., Karmaker B.B., *Application of perturbation theory to the nonlinear Volterra Gause Witt model for prey predator interaction.* Bull. Math. Biol **37** (1975), 139-146.

[6] Fergola P., Nayfeh A.H., *Application of the method of multiple scales to perturbed Lotka Volterra models.* Procedings of Vth International Conference " Waves and Stability in continuous media " S. Giambo', Editel Cosenza (1987).

[7] Fergola P. , Rionero S., Tenneriello C., *Asymptotic stability of a perturbed Lotka Volterra system.* Procedings of the Vth International Conference " Waves and Stability in continuous media " Series and Advances in Mathematics for Applied Sciences **4** (1990).

[8] Fergola P. , Rionero S., Tenneriello C., *On the stability of Lotka Volterra perturbed equations . Atti Acc. Peloritana Pericolanti Cl. Sci. Fis. Natur.* **48** (1991), 235-256.

[9] Freedman H.I.,*Deterministic mathematical models in population ecology.* Marcel Dekker , Inc. New York (1980).

[10] Freedman H.I., Waltman P., *Perturbation of two dimensional predator prey equations.* Siam J. Appl. Math. **28** (1975), 1-10

[11] Freedman H.I., Waltman P., *Periodic solutions of perturbed Lotka Volterra systems.* International Conference on Differential Equations, H.A. Antosiewicz Academic Press New York (1975), 312-316.

[12] Freedman H.I., Waltman P., *Perturbation of two dimensional predator*

prey equations with an unperturbed critical point. **29** Siam J. Appl. Math (1975), 719-733.

[13] Hahn W., *Stability of Motion*. Springer Verlag Berlin 1967

[14] Levin S.A., *Dispersion and population interactions.* Amer. Nat **108** (1974), 207-228.

[15] Samuelson P.A., *Generalized predator prey oscillations in ecological and economic equilibrium* . Proc. Nat. Acad. Sei. USA **68** (1971), 980-983.

[16] Walter C., *The global asymptotic stability of prey predator systems with second order dissipation.* Bull. Math. Biol **36** (1974), 215-217.

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