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GEOMETRICAL CAUCHY PROBLEM FOR H-J EQUATIONS AND WAVE PROPAGATION

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In the paper [1], a geometrical framework for the Cauchy problem for the Hamilton-Jacobi H-J equation was developed. Briefly, we recall the main result of that work. Let M be a n-dimensional manifold, for example the space-time, and let A be a $(n-1)$-dim manifold. Let $S(q^i, a^A)$, $(q^i, a^A) \in M \times A$, be a *complete solution* of the $H-J$ equation $H\left(q^i, \frac{\partial s}{\partial q_i}(q^k)\right) = 0$; if the Cauchy data $\sigma(\chi^A) = s(\tilde{q}^i(\chi^A))$, where $\chi^A \mapsto \tilde{q}^i(\chi^A)$ is an embedding of the initial surface Σ of co-dimension one into M, are not characteristic, then the function

$$\mathcal{S}(q^i, a^A, \chi^B) = S(q^i, a^A) - S(\tilde{q}^i(\chi^B), a^A) + \sigma(\chi^B) \tag{1}$$

is a global generating function, or Morse family, which generates a Lagrangian submanifold Λ of the symplectic contangent bundle (T^*M, ω_M),

$$\begin{aligned}\Lambda = \Big\{ (q,p) \in T^*M :\ & p_i = \frac{\partial \mathcal{S}}{\partial q^i}(q^i, a^A, \chi^B), 0 = \frac{\partial \mathcal{S}}{\partial a^A}(q^i, a^A, \chi^B) \\ & 0 = \frac{\partial \mathcal{S}}{\partial \chi^B}(q^i, a^A, \chi^B), \text{ for some } (a^A, \chi^B) \in A \times \Sigma \Big\}\end{aligned} \tag{2}$$

The manifold Λ is *solution* of the afore-mentioned geometrical

Cauchy problem.

A Lagrangian submanifold Λ of T^*M is characterized by:

i) $\dim \Lambda = \dim M$,

ii) the canonical 2-form $\omega_M = dp_i \wedge dq^i$ restricted to Λ is vanishing.

We just remember that such a solution Λ is, roughly speaking, a graph in the space of states and momenta of a possibly multivalued function of the state variables, and that it is classical solution whenever it is single-valued.

Furthermore, in [1], I proposed a way of constructing single-valued candidate "solutions" to the H-J equation by means of the same Morse family $\mathcal{S}$ which generates the geometrical solution. This was done by an *inf-sup procedure* over the sets of parameters defining the family,

$$s(q^i) = \inf_{\chi \in \Sigma} \sup_{a \in A} \mathcal{S}(q^i, a^A, \chi^B). \tag{3}$$

This seems to be a natural generalization of the classical procedure of *stationarizing* – see (2) – over the same parameters a^A and χ^B (*Huygens principle*) in order to eliminate them, a procedure which leads locally to classical solutions, but fails around singularities of the solution.

In the case of the eikonal equation of geometric optics (with constant index of refraction), this was shown to lead to the known physical solution, which is indeed the unique *viscosity* solution of the boundary value problem. The theory of viscosity solutions was introduced by Crandall and P.L. Lions [2] and Crandall, Evans, and Lions [3]. This is an analytical theory dealing with boundary value problems on open subsets of $\mathbb{R}^n$, and solutions are continuous functions of the state variables satisfying the fully nonlinear partial differential equations in a suitable weak sense. The most remarkable features of this theory are existence, uniqueness and stability results under very general assumptions on the Hamiltonian function and the initial data.

In a further paper [4], I show that the inf-sup procedure on the Morse family leads to the unique viscosity solution for a more general class of problems, namely for convex Hamiltonian functions depending only on momenta, $H = H(p)$. This is achieved by showing that the function we construct coincides with a representation formula for the solution first derived by Hopf [5], and that was shown to provide the unique viscosity solution by P. L. Lions [6] and Bardi and Evans [7].

Even though our assumptions on the Hamiltonian are rather restrictive (substantially, in both approaches we are concerning with "Liouville-integrable" systems), we think our result gives some hope to link the geometrical and the viscosity theory of H-J equations in more general situations.

Lagrangian manifolds and their generating functions were first introduced by Maslov [8] and Hörmander [9] in connection with the construction of asymptotic (for some vanishing parameter ε) solutions of linear partial differential equations. In that way, the two authors developed the theory of the Canonical Operator and the theory of Fourier Integral Operators.

The aim of the paper [10] is to apply the results in [1] to the concrete construction of asymptotic solutions of Cauchy problems for the Schrödinger equation, when a complete solution of the related H-J equation is known. The solutions so constructed, are as global as the complete solution is.

We shall utilize symplectic techniques for the asymptotic integral solutions of the Schrödinger equation by following the line of thought of Duistermaat [11], and in particular, without to enter and to use the full machinery of Fourier Integral Operators.

Let us consider the problem of finding asymptotic solutions (of accuracy one) of the Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t}(t,x) = -\frac{\varepsilon^2}{2}\Delta\psi(t,x) + V(x)\psi(t,x), \tag{4}$$

with the following rapidly oscillatory Cauchy data

$$\psi(0,x) = \psi_0(x)\exp\left\{\frac{i}{\varepsilon}\sigma(x)\right\}. \tag{5}$$

The idea of obtaining oscillatory asymptotic solutions of (4), up to suitable $O(\varepsilon^m)$, consists in trying to solve (4) by means of integrals of the form ($U \subseteq \mathbb{R}^k$):

$$I(t,x;\varepsilon) = \int_{u \in U} b(t,x,u,\varepsilon) \exp\left\{\frac{i}{\varepsilon}\Phi(t,x,u)\right\} du, \tag{6}$$

for some amplitude b and real-valued phase function Φ. Such integrals can be regarded as a continuous superpositions of oscillatory functions.

According to Duistermaat's idea about the globalization of the *method of stationary phase* (in a weak sense), by recalling the Maslov-Hörmander theorem on the parametrization of the Lagrangian manifolds, and by some asymptotic computations, we obtain the following fact:

the geometrical objects globally characterizing the phase Φ of the above oscillatory integrals (6) are the Lagrangian submanifolds Λ of $T^*\mathbb{R}^{n+1}$, $(t,x) \in \mathbb{R}^{n+1}$, belonging to the surface $H^{-1}(0) \subset T^*\mathbb{R}^{n+1}$, $(x^0 = t)$, where the Hamiltonian function is

$$H(x,p) = p_0 + \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(x). \tag{7}$$

In other words, we are looking for a Lagrangian submanifold solving the geometrical Cauchy problem for the $H-J$ equation $H=0$, where the *initial data* are related to the initial phase $\sigma(x)$ in (5). Now, *suppose* that the above $H-J$ equation admits a *global complete solution* $S(t,x,a)$. We have seen that a global Morse family generating the Lagrangian submanifold solving the above Cauchy problem is given by

$$\mathcal{S}(t,x,a,\chi) = S(t,x,a) - S(0,\chi,a) + \sigma(\chi), \tag{8}$$

where here the auxiliary parameters u in (6) are a and χ. It follows that the oscillatory asymptotic solution will be of the form

$$\begin{aligned} \psi(t,x;\varepsilon) = &\int_{\chi\in\Sigma}\int_{a\in A} b(x,a,\chi,\varepsilon) \\ &\times \exp\left\{\frac{i}{\varepsilon}[S(t,x,a) - S(0,\chi,a) + \sigma(\chi)]\right\} d\chi da, \end{aligned} \tag{9}$$

and b will be choosen in order to $\psi(t,x;\varepsilon)$ satisfies the initial data $\psi_0(x)$ with the same accuracy (there is a suitable transport equation).

The example of a particle in a weak ($\lambda \approx 0$) force field is worked out in [10]; the connection with the known exact solution of the limit case ($\lambda = 0$), as it is presented e.g. in Vladimirov [12], formula 14.1, is pointed out.

An other concrete application (in preparation) of the above ideas concerns the study of asymptotic solutions for the system of the *Linear Elasticity;* a different approach to approximate solutions to linear elasticity as continuous superpositions of oscillatory functions can be found in the paper [13] of M.I. Taylor.

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