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RESULTS AND APPLICATIONS IN THERMOELASTICITY OF MATERIALS WITH VOIDS

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1. Introduction.

We consider the linear theory of a thermoelastic porous solid in which the skeletal or matrix is a thermoelastic material and the interstices are void of material. We assume that the initial body is free from stresses. The concept of a distributed body asserts that the mass density at time t has the decomposition $\gamma\nu$, where γ is the density of the matrix material and $\nu(0 < \nu \leq 1)$ is the volume fraction field (cf. [1,2]).

In the first part, in order to derive some applications of the reciprocity theorem, we recall some results established by same authors in [3]. Then we obtain integral representations of the solution and prove that the solving of the boundary-initial value problem can be reduced to the solving of an associated uncoupled problem and to an integral equation for the volume fraction field.

2. Preliminary.

Throughout this paper we shall employ a rectangular Cartesian system $Ox_i (i = 1, 2, 3)$. Let B be a regular region of space occupied by a thermoelastic material with voids, whose boundary is ∂B. Moreover B is the interior of $\bar{B}$, n_i are the components of the unit outward normal to ∂B.

The basic equations of the theory consists [4] of *the equations of motion*

$$t_{ji,j} + f_i = \rho \ddot{u}_i, \quad h_{i,i} + g + l = \rho\kappa\ddot{\varphi}, \tag{2.1}$$

the energy equation

$$T_0 \dot{\eta} = q_{i,i} + S, \tag{2.2}$$

the constitutive equations

$$\begin{aligned} t_{ij} &= C_{ijrs} e_{rs} + B_{ij}\varphi + D_{ijk}\varphi_{,k} - \beta_{ij}\vartheta, \\ h_i &= A_{ij}\varphi_{,j} + D_{rsi} e_{rs} + d_i \varphi - a_i \vartheta, \\ g &= -B_{ij} e_{ij} - \xi\varphi - d_i \varphi_{,i} + m\vartheta, \\ \eta &= \beta_{ij} e_{ij} + a\vartheta + m\varphi + a_i \varphi_{,i}, \\ q_i &= K_{ij}\vartheta_{,j}, \end{aligned} \tag{2.3}$$

the geometrical equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.4}$$

where the significance of the simbols is obvious.

In the above relations we have used the following notations: u-displacement, t-stress, f-body force per unit volume, ρ-density in the reference configuration, κ-equilibrated inertia, h-equilibrated stress, l-extrinsic equilibrated body force per unit volume, g-intrinsic equilibrated body force, η-entropy per unit volume, $T_0 (> 0)$ constant absolute temperature in the reference state, q-heat flux, S-extrinsic heat supply per unit volume, **A**, **B**, **C**, **D**, **K**, β, ξ, a_i, a, d_i and m-characteristic coefficients of the material.

We introduce the notations

$$\vartheta = T - T_0, \; \varphi = \nu - \nu_0,$$

where T is the absolute temperature and ν_0 is the constant volume distributions function for the reference configuration.

We assume that:

i) constitutive coefficients **A**, **B**, **C**, **D**, **K**, β, ξ, a_i, a, d_i, m, density ρ and equilibrated inertia κ are continuous on $\bar{\mathcal{B}}$;

ii) the coefficients **A**, **B**, **C**, **D**, **K** and β have the following symmetries

$$C_{ijrs} = C_{rsij} = C_{jirs}, \quad A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \ .$$

$$D_{ijk} = D_{jik}, \quad \beta_{ij} = \beta_{ji}, \quad K_{ij} = K_{ji};$$

iii) **f**, l and S are continuous on $\bar{\mathcal{B}} \times [0, \infty]$.

It is known that (see for example Carlson [5])

$$q_i \vartheta_{,i} \geq 0. \tag{2.5}$$

We consider the following boundary conditions

$$\begin{gathered} u_i = \hat{u}_i \text{ on } \bar{S}_1 \times I, \quad t_{ji} n_j = \hat{t}_i \text{ on } S_2 \times I, \\ \varphi = \hat{\varphi} \text{ on } \bar{S}_3 \times I, \quad h_i n_i = \hat{h} \text{ on } S_4 \times I, \\ \vartheta = \hat{\vartheta} \text{ on } \bar{S}_5 \times I, \quad q_i n_i = \hat{q} \text{ on } S_6 \times I. \end{gathered} \tag{2.6}$$

Here S_i $(i = 1, ..., 6)$ denote subsets of $\partial\mathcal{B}$ such that $\bar{S}_1 \cup S_2 = \bar{S}_3 \cup S_4 = \bar{S}_5 \cup S_6 = \partial\mathcal{B}$, $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = \emptyset$, $I = [0, \infty)$ and $\hat{u}_i$, $\hat{t}_i$, $\hat{\varphi}$, $\hat{h}$, $\hat{\vartheta}$, $\hat{q}$ are prescribed.

We assume that

(α) a_i, b_i, c, d and η_0 are continuous on $\bar{\mathcal{B}}$;

(β) $\hat{u}$, $\hat{\varphi}$ and $\hat{\vartheta}$ are continuous on $\bar{S}_1 \times I$, $\bar{S}_3 \times I$ and $\bar{S}_5 \times I$, respectively;

(γ) $\hat{t}_i$, $\hat{h}$ and $\hat{q}$ are continuous in time and piecewise regular on $S_2 \times I$, $S_4 \times I$ and $S_6 \times I$, respectively.

The components of the surface traction $\mathbf{t}$, equilibrated surface traction h, and the heat flux q at regular points of $\partial\mathcal{B} \times [0, \infty]$ are define by

$$t_i = t_{ji} n_j, \quad h = h_i n_i, \quad q = q_i n_i.$$

To the system of field equations we add the initial conditions

(2.7) $$u_i(x,0) = a_i(x), \quad \dot{u}_i(x,0) = b_i(x),$$

$$\varphi(x,0) = c(x), \quad \dot{\varphi}(x,0) = d(x), \quad \eta(x,0) = \eta_0(x), x \in \bar{\mathcal{B}}.$$

3. Main results.

THEOREM 3.1. *Assume that*

(i) ρ *and* κ *are strictly positive;*

(ii) K_{ij} *is positive semidefinite;*

(iii) a *is strictly positive (or negative).*

Then the boundary - initial value problem of thermoelastodynamics has at most one solution.

We denote by $\omega * v$ the convolution of ω and v

$$[\omega * v]_{(x,t)} = \int_0^t \omega(x, t-\tau) v(x,\tau) d\tau,$$

where ω and v are scalar fields on $\mathcal{B} \times I$ continuous in time.

We introduce the functions

(3.1) $$j(t) = 1, \quad p(t) = (j * j)_{(t)} = t, \quad t \in I.$$

In what follows we write $\bar{h}$ for $j * h$, i.e.

(3.2) $$\bar{h}(x,t) = \int_0^t h(x,s) ds.$$

Using (3.2) and the condition $\eta(x,0) = \eta_0(x)$, $x \in \mathcal{B}$, the relation (2.2) may by written as

(3.3) $$T_0 \eta = \bar{q}_{i,i} + W,$$

where $W = \bar{S} + T_0 \eta_0$.

Now we consider two external systems

$$N^{(\alpha)} = (\mathbf{f}^{(\alpha)}, l^{(\alpha)}, S^{(\alpha)}, \hat{\mathbf{u}}^{(\alpha)}, \hat{\mathbf{t}}^{(\alpha)}, \hat{\varphi}^{(\alpha)}, \hat{\mathbf{h}}^{(\alpha)}, \hat{\vartheta}^{(\alpha)},$$

$$\hat{q}^{(\alpha)}, \mathbf{a}^{(\alpha)}, \mathbf{b}^{(\alpha)}, c^{(\alpha)}, d^{(\alpha)}, \eta_0^{(\alpha)}) \quad (\alpha = 1, 2)$$

and we denote by

$$Q^{(\alpha)} = \{\mathbf{u}^{(\alpha)}, \varphi^{(\alpha)}, \vartheta^{(\alpha)}, \mathbf{t}^{(\alpha)}, \mathbf{h}^{(\alpha)}, \eta^{(\alpha)}, \mathbf{e}^{(\alpha)}, g^{(\alpha)}, \mathbf{q}^{(\alpha)}\}$$

a solution corresponding to $N^{(\alpha)}$. We define

(3.4)
$$t_i^{(\alpha)} = t_{ij}^{(\alpha)} n_j, \quad h^{(\alpha)} = h_i^{(\alpha)} n_i, \quad q^{(\alpha)} = q_i^{(\alpha)} n_i,$$

$$W^{(\alpha)} = \bar{S}^{(\alpha)} + T_0 \eta_0^{(\alpha)} \quad (\alpha = 1, 2).$$

LEMMA 4.1. *Let*

(3.5)
$$E_{\alpha\beta}(r,s) = \int_{\partial\mathcal{B}} \Big[t_i^{(\alpha)}(r) u_i^{(\beta)}(s) + h^{(\alpha)}(r)\varphi^{(\beta)}(s) -$$
$$- \frac{1}{T_0}\bar{q}^{(\alpha)}(r)\vartheta^{(\beta)}(s) \Big] da - \int_{\mathcal{B}} [\rho \ddot{u}_i^{(\alpha)}(r) u_i^{(\beta)}(s) +$$
$$+ \rho\kappa\ddot{\varphi}^{(\alpha)}(r)\varphi^{(\beta)}(s)] dV + \frac{1}{T_0}\int_{\mathcal{B}} [\bar{q}_i^{(\alpha)}(r)\vartheta_{,i}^{(\beta)}(s) dV +$$
$$+ \int_{\mathcal{B}} \Big[f_i^{(\alpha)}(r) u_i^{(\beta)}(s) + l^{(\alpha)}(r)\varphi^{(\beta)}(s) -$$
$$- \frac{1}{T_0} W^{(\alpha)}(r)\vartheta^{(\beta)}(s) \Big] dV,$$

for all r, $s \in I$, $(\alpha, \beta = 1, 2)$. *Then*

(3.6)
$$E_{\alpha\beta}(r,s) = E_{\beta\alpha}(s,r).$$

COROLLARY 4.1. *Assume that the assumption (ii) from Section 2 holds. Then*

(3.7)
$$\int_{\partial\mathcal{B}} \Big[t_i^{(1)} * u_i^{(2)} + h^{(1)} * \varphi^{(2)} - \frac{1}{T_0} j * q^{(1)} * \vartheta^{(2)} \Big] da -$$

$$-\int_{\mathcal{B}}[\rho\ddot{u}_i^{(1)} * u_i^{(2)} + \rho\kappa\ddot{\varphi}^{(1)} * \varphi^{(2)}]dV + \frac{1}{T_0}\int_{\mathcal{B}} j * q_i^{(1)} * \vartheta_{,i}^{(2)} dV +$$
$$+\int_{\mathcal{B}}\left[f_i^{(1)} * u_i^{(2)} + l^{(1)} * \varphi^{(2)} - \frac{1}{T_0}W^{(1)} * \vartheta^{(2)}\right] dV =$$
$$= \int_{\partial\mathcal{B}}\left[t_i^{(2)} * u_i^{(1)} + h^{(2)} * \varphi^{(1)} - \frac{1}{T_0} j * q^{(2)} * \vartheta^{(1)}\right] da -$$
$$-\int_{\mathcal{B}}[\rho\ddot{u}_i^{(2)} * u_i^{(1)} + \rho\kappa\ddot{\varphi}^{(2)} * \varphi^{(1)}]dV + \frac{1}{T_0}\int_{\mathcal{B}} j * q_i^{(2)} * \vartheta_{,i}^{(1)} dV +$$
$$+\int_{\mathcal{B}}\left[f_i^{(2)} * u_i^{(1)} + l^{(2)} * \varphi^{(1)} - \frac{1}{T_0}W^{(2)} * \vartheta^{(1)}\right] dV.$$

THEOREM 3.2. *Let $Q^{(\alpha)}$ be a solution corresponding to external data system $N^{(\alpha)}$, $(\alpha = 1,2)$. Then*

$$\int_{\partial\mathcal{B}} p * \left[t_i^{(1)} * u_i^{(2)} + h^{(1)} * \varphi^{(2)} - \frac{1}{T_0} j * q^{(1)} * \vartheta^{(2)}\right] da +$$
$$+\int_{\mathcal{B}}\left[G_i^{(1)} * u_i^{(2)} + L^{(1)} * \varphi^{(2)} - \frac{1}{T_0} p * W^{(1)} * \vartheta^{(2)}\right] dV =$$
$$= \int_{\partial\mathcal{B}} p * \left[t_i^{(2)} * u_i^{(1)} + h^{(2)} * \varphi^{(1)} - \frac{1}{T_0} j * q^{(2)} * \vartheta^{(1)}\right] da +$$
$$+\int_{\mathcal{B}}\left[G_i^{(2)} * u_i^{(1)} + L^{(2)} * \varphi^{(1)} - \frac{1}{T_0} p * W^{(2)} * \vartheta^{(1)}\right] dV, \tag{3.8}$$

where

$$G_i^{(\alpha)} = p * f_i^{(\alpha)} + \rho[t b_i^{(\alpha)} + a_i^{(\alpha)}],$$
$$L^{(\alpha)} = p * l^{(\alpha)} + \rho\kappa[t d^{(\alpha)} + c^{(\alpha)}]\ (\alpha = 1,2). \tag{3.9}$$

4. Applications.

In this section we present some applications of the reciprocity theorem. We restrict our attention to bodies with a centre of symmetry, so that $D_{rsi} = d_i = a_i = 0$. Moreover, we assume that the initial data are zero.

By (3.3) and (3.9) we obtain

$$W^{(\alpha)} = j * S^{(\alpha)}, G_i^{(\alpha)} = p * f_i^{(\alpha)}, L^{(\alpha)} = p * l^{(\alpha)} \tag{4.1}$$

and the reciprocity relation (3.8) becomes

$$\begin{aligned} &\int_{\partial\mathcal{B}} \left[t_i^{(1)} * u_i^{(2)} + h^{(1)} * \varphi^{(2)} - \frac{1}{T_0} j * q^{(1)} * \vartheta^{(2)}\right] da + \\ &+\int_{\mathcal{B}} \left[f_i^{(1)} * u_i^{(2)} + l^{(1)} * \varphi^{(2)} - \frac{1}{T_0} j * S^{(1)} * \vartheta^{(2)}\right] dV = \\ &= \int_{\partial\mathcal{B}} \left[t_i^{(2)} * u_i^{(1)} + h^{(2)} * \varphi^{(1)} - \frac{1}{T_0} j * q^{(2)} * \vartheta^{(1)}\right] da + \\ &+\int_{\mathcal{B}} \left[f_i^{(2)} * u_i^{(1)} + l^{(2)} * \varphi^{(1)} - \frac{1}{T_0} j * S^{(2)} * \vartheta^{(1)}\right] dV. \end{aligned} \tag{4.2}$$

In what follows we restrict our attention to the "traction problem", so that $S_1 = S_3 = S_5 = \phi$.

Let us assume that the external system $N^{(1)}$ in which

$$\begin{aligned} &f_i^{(1)} = \delta(x-y)\delta(t)\delta_{ij}, l^{(1)} = 0, S^{(1)} = 0, \\ &a_i^{(1)} = b_i^{(1)} = c^{(1)} = d^{(1)} = 0, \end{aligned} \tag{4.3}$$

where δ is the Dirac measure, generate the displacement $U_i^{(j)}$, the volume fraction $\Phi^{(j)}$ and the temperature $T^{(j)}$. Let $P_i^{(j)}$, $H^{(j)}$ and $Q^{(j)}$ be, respectively, the traction, the equilibrated surface traction and the heat flux generated by $U_i^{(j)}$, $\Phi^{(j)}$ and $T^{(j)}$. Let u_i, φ and ϑ be the displacement, volume fraction and temperature corresponding to the external system $N^{(2)} = \{f_i, l, S, \hat{t}_i, \hat{h}, \hat{q}, a_i = b_i = c = d = 0\}$. It follows from (4.2) and (4.3) that

$$\begin{aligned} u_k(y,t) = &\int_{\mathcal{B}} \left[f_i * U_i^{(k)} + l * \Phi^{(k)} - \frac{1}{T_0} j * S * T^{(k)}\right] dv + \\ &+ 1\int_{\partial\mathcal{B}} \left[\hat{t}_i * U_i^{(k)} - P_i^{(k)} * u_i + \hat{h} * \Phi^{(k)} - H^{(k)} * \varphi + \right. \\ &\left. + \frac{1}{T_0} j * (\vartheta * Q^{(k)} - T^{(k)} * \hat{q})\right] dv. \end{aligned} \tag{4.4}$$

We now assume that the external system $N^{(1)}$, in which

$$f_i^{(1)} = 0, l^{(1)} = \delta(x-y)\delta(t), S^{(1)} = 0, a_i^{(1)} = b_i^{(1)} = c_i^{(1)} = d_i^{(1)} = 0,$$

generate the displacement $U_i^{(4)}$, the volume fraction $\Phi^{(4)}$ and the temperature $T^{(4)}$. Let $P_i^{(4)}$, $H^{(4)}$ and $Q^{(4)}$ be, respectively, the traction, the equilibrated surface traction and the heat flux generated by $U_i^{(4)}$, $\Phi^{(4)}$ and $T^{(4)}$. Let $(u_i, \varphi, \vartheta)$ the solution corresponding to the external system $N^{(2)} = \{f_i, l, S, \hat{t}_i, \hat{h}, \hat{q}, a_i = b_i = c = d = 0\}$. It follows from (4.2) and (4.3) that

$$\begin{aligned} \varphi(y,t) = & \int_{\mathcal{B}} \left[f_i * U_i^{(4)} + l * \Phi^{(4)} - \frac{1}{T_0} j * S * T^{(4)} \right] dv + \\ (4.5) \qquad & + \int_{\partial\mathcal{B}} \Big[\hat{t}_i * U_i^{(4)} - P_i^{(4)} * u_i + \hat{h} * \Phi^{(4)} - H^{(4)} * \varphi + \\ & + \frac{1}{T_0} j * (\vartheta * Q^{(4)} - T^{(4)} * \hat{q}) \Big] dv. \end{aligned}$$

Similarly we can obtain a formula for the temperature field.

The integral representations (4.4) and (4.5) are relations of Somigliana type.

We now denote by (P_1) the boundary-initial-value problem characterized by the field equations (2.1)-(2.4) and the conditions (2.6),(2.7). If we replace the costitutive equation for the intrinsic equilibrated body force (2.3) by the following equation

$$g = -\xi\varphi, \tag{4.6}$$

then we obtain a boundary-initial-value problem characterized by the equations (2.1), (2.2), $(2.3)_{1,2}$, (4.6), $(2.3)_{4,5}$ and the conditions (2.6), (2.7). We denote this problem by (P_2). It is clear that (P_2) is an uncoupled problem, in the sense that the function φ is independent of u_i and ϑ. Let $C^{(1)}$ and $C^{(2)}$ be the solutions of the problem (P_1) and (P_2), respectively. We assume that $N^{(\alpha)}$ $(\alpha = 1,2)$ are the external data system corresponding to the problem (P_α) $(\alpha = 1,2)$. By a proof analogous to that of Theorem 3.2 in view of (3.7) we obtain the following result.

If a body with a centre of symmetry is subjected to two external systems $N^{(\alpha)}$ $(\alpha = 1,2)$, then between the solutions $C^{(\alpha)}$ corresponding to the problems (P_α) $(\alpha = 1,2)$ there is the following reciprocity relation

$$\begin{aligned}
(4.7)\qquad & \int_B \left[f_i^{(1)} * u_i^{(2)} + l^{(1)} * \varphi^{(2)} - \frac{1}{T_0} j * S^{(1)} * \vartheta^{(2)}\right] dv + \\
& + \int_{\partial B} \left[t_i^{(1)} * u_i^{(2)} + h^{(1)} * \varphi^{(2)} - \frac{1}{T_0} j * q^{(1)} * \vartheta^{(2)}\right] da = \\
& = \int_B \left[f_i^{(2)} * u_i^{(1)} + l^{(2)} * \varphi^{(1)} - \frac{1}{T_0} j * S^{(2)} * \vartheta^{(1)}\right] dv + \\
& = \int_{\partial B} \left[t_i^{(2)} * u_i^{(1)} + h^{(2)} * \varphi^{(1)} - \frac{1}{T_0} j * q^{(2)} * \vartheta^{(1)}\right] da + \\
& + \int_B (B_{ij} e_{ij}^{(2)} - m\vartheta^{(2)}) * \varphi^{(1)} dv.
\end{aligned}$$

We assume that the uncoupled problem $P^{(2)}$ corresponds to external system

$$\begin{aligned}
(4.8)\qquad & f_i^{(2)} = 0, l^{(2)} = \delta(x-y)\delta(t), S^{(2)} = 0, \hat{t}^{(2)} = 0, \hat{h}^{(2)} = 0, \\
& \hat{q}^{(2)} = 0, a_i^{(2)} = b_i^{(2)} = c^{(2)} = d^{(2)} = \eta_0^{(2)} = 0.
\end{aligned}$$

Let U, Φ and T be the displacement, volume fraction and temperature field generated by this system. If we use reciprocity relation (4.7) we obtain

$$\begin{aligned}
(4.9)\qquad & \int_B \left[f_i^{(1)} * U_i + l^{(1)} * \Phi - \frac{1}{T_0} j * S^{(1)} * T\right] dv + \\
& + \int_{\partial B} \left[\hat{t}_i^{(1)} * U_i + \hat{h}^{(1)} * \Phi - \frac{1}{T_0} \hat{q}^{(1)} * T\right] da = \\
& = \varphi^{(1)}(y,t) + \int_B (B_{ij} U_{i,j} - mT) * \varphi^{(1)} dv.
\end{aligned}$$

Thus, we conclude that the solving of the coupled problem (P_1) corresponding to the external system $N^{(1)}$ and to the homogeneous initial conditions is reduced to the solving of the associated uncoupled problem (P_2) corresponding to the external system (4.8) and to the solving of the integral equation (4.9) for the volume fraction field.

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