

**A NEW APPROACH TO THE THEORY  
OF HEAT CONDUCTION  
WITH FINITE WAVE SPEEDS**

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Relations between the physical models describing the heat conduction in solids and a phenomenological model leading to quasi-linear hyperbolic equations and systems of conservation laws are presented. A new semi-empirical temperature scale is introduced in terms of which a modified Fourier law is formulated. The hyperbolicity of the heat conduction equation is discussed together with some wave propagation problems.

**1. Introduction.**

It was Maxwell [1] who was the first to modify Fourier's law. After him there were Cattaneo [2,3] and Vernotte [4]. Hence we will refer to

$$(1.1) \quad T\dot{\mathbf{q}} + \mathbf{q} = -k\nabla\vartheta$$

as to M.C.V. (Maxwell-Cattaneo-Vernotte) equation, where  $T > 0$  is a suitable relaxation time,  $\vartheta$  denotes the absolute temperature,  $k$  is the conductivity and  $\mathbf{q}$  the heat flux vector.

A number of experimental results confirms the validity of (1.1),

[5,6]. However, a proper thermodynamic setup for (1.1) is difficult to form, because of the second law of thermodynamics, [7-9].

The heat conduction phenomenon with finite thermal wave speeds can be analyzed in the context of the theory of materials with memory [10-12], in the framework of the theory of the materials with internal state variables, [13-15], as well as in the scheme of theory of relativity and of the so-called extended thermodynamics, [16,17].

In [18,19] we have introduced a new temperature "scale", related to the absolute one by a suitable initial value problem. In terms of this scale the classical Fourier law has been formulated. Here this model will be applied to wave propagation problems in a rigid conductor and thermo-elastic body.

## 2. A new temperature scale.

We make here some physical considerations, starting from the results proved by Cattaneo in [2].

Consider a gas in macroscopic mechanical equilibrium and let  $Q$  be a given physical quantity related to a molecule of the gas. We will denote by  $G$  the average of  $Q$  at a given point of the gas. For the sake of simplicity we will suppose  $G$  be constant on each plane belonging to a family of parallel planes, orthogonal to a given direction  $r$ . However, the value of  $G$  can change from one to another plane so that, denoting by  $x$  a point of  $r$  and supposing  $G$  be time-dependent, we get

$$G = G(t, x).$$

When  $Q$  is regarded as the average of the kinetic energy of the molecules due to translational and rotational as well as to internal degrees of freedom, the function  $G$  differs from the absolute temperature of the gas for a factor  $\alpha$ , depending on the gas. Moreover, if  $q$  means the component of the heat flux vector in the  $x$  direction, it is easily proved that:

$$(2.1) \quad q = -k\partial\vartheta/\partial x + \sigma\partial^2\vartheta/\partial x\partial t,$$

where  $\sigma = Tk$ . We define

$$(2.2) \quad \beta = \vartheta - T\partial\vartheta/\partial t.$$

Owing to (2.1) we get

$$(2.3) \quad q = -k\partial\beta/\partial x.$$

Moreover, from the relationship

$$(2.4) \quad \alpha\beta = G - T\partial G/\partial t$$

follows that  $\beta$  is related to the average of the kinetic energy as well as to its time derivative. The second term in the right hand side of (2.4), containing the mean acceleration of the molecules, can be regarded as a contribution of internal forces among the molecules, which usually are neglected in the statistical definition of temperature. By (2.2) it follows

$$(2.5) \quad \partial\beta/\partial t = \partial\vartheta/\partial t - T\partial^2\vartheta/\partial t^2.$$

To eliminate  $\partial\vartheta/\partial t$  in (2.5) we can use (2.2) so that

$$(2.6) \quad \partial(\alpha\beta)/\partial t = (\alpha/T)(\vartheta - \beta) - T\partial^2(\alpha\vartheta)/\partial t^2.$$

The last term in (2.6) can be neglected because  $T$  is very small compared with  $1/T$  (the magnitude of  $T$  is of the order  $10^{-9} \div 10^{-13}$  second., [5]). Finally we obtain

$$(2.7) \quad \partial(\alpha\beta)/\partial t = (\alpha/T)(\vartheta - \beta).$$

We call (2.7) *canonical kinetic equation* and the function  $\beta$  defined by (2.7) *semi-empirical temperature*.

From the physical point of view  $\beta$  is the absolute temperature minus a *frictional* term due to the internal Van der Waals type forces. Since the velocities of the molecules are reduced by such forces, the final result is a drop in the temperature. The most important effect which is obtained by assuming  $\beta$  as a new temperature scale is the transformation in the type of the heat equation from parabolic to hyperbolic. In (2.7)  $\partial\beta/\partial t$  is equal to a linear function of  $\vartheta$  and  $\beta$ . It

is worth noticing that we can change the scale of  $\beta$  without affecting its physical meaning. In order to take into account this possibility we generalize (2.7) as follows

$$(2.8) \quad \overline{(\alpha\beta)} = f_{(\alpha,k,T)}(\vartheta, \beta)$$

where  $f_{(\alpha,k,T)}$  is a suitable functional of the material functions (constants)  $\alpha$ ,  $k$  and  $T$ , depending on the semi-empirical temperature as well as on the absolute one.

The downscript  $(\alpha, k, T)$  means that constructing  $f$  we need to take into account those functions; the last equation will be called a *normed kinetic equation*.

Let us notice that performing the spatial differentiation of (2.7) we get the so-called prolonged kinetic equation

$$(2.9) \quad \partial^2(\alpha\beta)/\partial x \partial t = \partial(\alpha/T)(\vartheta - \beta)/\partial x.$$

Using (2.3) to express the gradient of  $\beta$  in terms of  $q$ , we get

$$(2.10) \quad \partial(\alpha k^{-1}q)/\partial t = -(\alpha/T)(\partial\vartheta/\partial x + k^{-1}q)$$

if  $\alpha$  and  $T$  are constant. The derived equation is very similar to the 1D counterpart of the M.C.V. equation (1.1).

Let us notice that the semi-empirical temperature is uniquely defined by (2.8) if and only if a suitable initial condition

$$(2.11) \quad \beta(\mathbf{x}, t_0) = \beta_0$$

is given and  $f$  is Lipschitz continuous. In the case of generalized kinetic equation (2.8) we make some supplementary assumptions on  $f$ :

$$\text{a) } (\partial f/\partial B) \leq 0; \quad \text{b) } (\partial f/\partial \vartheta) > 0.$$

The requirement a) assures the stability of the solution of (2.8), while b) is needed in order  $\beta$  be an increasing function of  $\vartheta$  and the natural order relation  $\leq$  between different temperatures be conserved, when the relaxation time is zero.

### 3. Rigid conductor.

Let us consider a particular rigid heat conductor with the constant relaxation time  $T = \tau_0 \geq 0$ , related to a reference temperature  $\vartheta_0$ ; the factor  $\alpha$  in (2.2) is assumed constant also. Physical properties of the conductor are characterized by the following constitutive relations:

$$(3.1) \quad \psi = \hat{\psi}(\vartheta, \nabla\beta) = \hat{\psi}^1(\vartheta) + (\tau_0 k / 2\rho\vartheta) \nabla\beta^2,$$

$$\eta = \hat{\eta}(\vartheta, \nabla\beta), \quad \mathbf{q} = \hat{\mathbf{q}}(\vartheta, \nabla\beta)$$

and the kinetic equation (2.7). Here  $\nabla\beta$  denotes the 3D gradient of  $\beta$ ,  $\psi$  is the free energy and  $\eta$  the specific entropy; the conductivity is assumed temperature-dependent and of the particular form

$$(3.2) \quad k(\vartheta) = k_0(\vartheta/\vartheta_0)^2$$

where  $k_0$  is the heat conduction coefficient corresponding to the reference temperature  $\vartheta_0$ .

The first and second law in such a case read

$$(3.3) \quad \rho\dot{\varepsilon} = -\text{Div} \mathbf{q} + \rho r,$$

$$\rho\dot{\eta} \geq -\text{Div}(\mathbf{q}/\vartheta) + \rho(r/\vartheta).$$

In (3.3)  $\rho$  is the mass density,  $\varepsilon$  is the specific internal energy, related to the free energy  $\psi$  by the (Legendre's) transformation  $\varepsilon = \psi + \eta\vartheta$ , and  $r$  is the body heat supply.

By substituting (3.1) into the inequality (3.3), we get, by the standard arguments, the following potential relations

$$(3.4) \quad \eta = -\partial\psi/\partial\vartheta, \quad \mathbf{q} = -k(\vartheta)\nabla\beta,$$

together with the residual (heat conduction) inequality

$$(3.5) \quad -\mathbf{q} \cdot \nabla\beta \leq 0.$$

Due to (3.4), the only consequence of the latter is the inequality

$$(3.6) \quad k(\vartheta) \geq 0.$$

Let us notice that as a consequence of the constitutive assumption (3.1) and the relation (3.4), we obtain

$$(3.7) \quad \varepsilon \equiv \hat{\varepsilon}(\vartheta) = \psi^1(\vartheta) - \vartheta d\psi^1(\vartheta)/d\vartheta,$$

i.e. the internal energy is independent of the nonequilibrium variable  $\nabla\beta$ . Moreover, the specific entropy function is composed of an equilibrium term

$$(3.8)_1 \quad \eta_1(\vartheta) = -d\psi^1(\vartheta)/d\vartheta$$

plus the nonequilibrium one

$$(3.8)_2 \quad \eta_2(\vartheta, \nabla\beta) = -\frac{\tau_0 k_0}{2\rho\vartheta_0^2} \nabla\beta \cdot \nabla\beta.$$

From this form it is obvious that the principle of maximum entropy at equilibrium is fulfilled.

By (3.3) together with (3.2), (3.4) and (3.7) we finally obtain the following equation, governing heat propagation in the rigid conductors:

$$(3.9) \quad \begin{aligned} \rho\tau_0 c_v \vartheta_0^2 \ddot{\beta} + \rho c_v \vartheta_0^2 \dot{\beta} - 2k_0 \tau_0 \vartheta \nabla\beta \cdot \nabla\dot{\beta} + \\ - 2k_0 \tau_0 \vartheta \nabla\beta \cdot \nabla\beta - k_0 \vartheta^2 \Delta\beta - \rho r \vartheta_0^2 = 0, \end{aligned}$$

where the heat capacity  $c_v$  given by  $d\hat{\varepsilon}/d\vartheta$ .

It is a quasi-linear hyperbolic second order equation. Well posedness of a Cauchy problem for another version of (3.9) was proved in [20]. The hyperbolicity condition

$$(3.10) \quad c_v k_0 \tau_0 > 0$$

is satisfied if  $k_0$ ,  $\tau_0$ , and  $c_v$  are positive.

Finally we underline some interesting properties of field equations in the case of shock and acceleration waves.

Supposing the body heat supply be continuous across the shock front, the jump conditions read

$$(3.11) \quad \rho V[\varepsilon] = [\mathbf{q}] \cdot \mathbf{N},$$

$$(3.12) \quad V[[\beta]] = 0,$$

where  $V$  and  $N$  are the velocity and the normal versor of the shock wave, respectively, and the symbol  $f \equiv f^+ - f^-$  denotes the jump of  $f$ . The continuity of the new temperature, even in the case of shock wave, is an essential property of the model proposed and it is a consequence of the generalized Rankine-Hugoniot relations.

As far as acceleration waves are concerned, let us quote some of the results proved from [21,22]. It is well known that in the classical model of conductor based on Fourier's law, solutions of heat equation do not blow up in finite time. However, such a behaviour can be found in materials with thermal relaxation. In [21] the 1D case of the rigid conductor described here was considered. The following equation, describing the evolution of the amplitude

$$\alpha(t) \equiv [[\beta_{,tt}]] = \tau_0^{-1} [[\vartheta_{,t}]]$$

of the acceleration waves, was derived

$$(3.13) \quad d\alpha(t)/dt - n\tau_0\alpha^2 + (2\tau_0)^{-1}\alpha = 0$$

where

$$(3.14) \quad n = 2/\beta^+ - (2c_v)^{-1} dc_v/d\vartheta$$

and  $\beta^+$  is the value of  $\beta$  at the shock front. Equation (3.13) is of Bernoulli type and, for  $n > 0$ , its solution  $\alpha(t)$  blows up in a finite time, if the initial condition  $\alpha_0 = \alpha(0)$  satisfies the following inequality

$$(3.15) \quad 2n\tau_0^2\alpha_0 - 1 > 0.$$

The finite blow-up time  $t_1$  can be calculated to be

$$t_1 = 2\tau_0 \log |(2n\alpha_0\tau_0^2)/(2n\alpha_0\tau_0^2 - 1)|.$$

For the special case of the specific heat  $c_v(\vartheta) = c_0(\vartheta/\vartheta_0)^4$  the amplitude decays along the wave according to the law

$$\alpha(t) = \alpha_0 \exp(-t/2\tau_0)$$

and blow-up does not occur.

In such a case the equilibrium part of the free energy function  $\psi$  is

$$(3.16) \quad \psi^1(\vartheta) = e_0 + e_1\vartheta + e_2\vartheta^5,$$

where the  $e_i$ 's are constant.

#### 4. Thermoelastic solids.

We recall the balance laws for linear momentum and energy:

$$(4.1) \quad \rho\dot{\mathbf{v}} = \text{Div } \mathbf{S} + \rho\mathbf{b},$$

$$\rho(\varepsilon + 0.5\dot{\mathbf{v}}\mathbf{v}) - \text{Div}(\mathbf{S} \cdot \mathbf{v}) + \text{Div } \mathbf{q} - \rho\mathbf{b}\mathbf{v} - \rho r = 0,$$

where:  $\rho$  is the reference mass density;  $\mathbf{S}$  is the Piola Kirchhoff stress tensor;  $\mathbf{v}$  and  $\dot{\mathbf{v}}$  denote the particle velocity and acceleration;  $\mathbf{q}$  denotes the heat flux calculated in the reference configuration. Here the kinetic equation in its more general form (2.8) is assumed. Together with (4.1) we face with the unilateral differential constraint

$$(4.2) \quad \rho\dot{\eta} \geq -\text{Div}(\mathbf{q}/\vartheta) + \rho(r/\vartheta),$$

representing the second law of thermodynamics. As usual, we rewrite (4.1) in a more suitable form by using the free energy

$$(4.3) \quad \psi = \varepsilon - \vartheta\eta$$

and assume the following constitutive equations:

$$(4.4) \quad \begin{aligned} \psi &= \psi^*(\vartheta, \mathbf{F}, \nabla\beta), & \eta &= \eta(\vartheta, \mathbf{F}, \nabla\beta) \\ \mathbf{q} &= \mathbf{q}^*(\vartheta, \mathbf{F}, \nabla\beta), & \mathbf{S} &= \mathbf{S}^*(\vartheta, \mathbf{F}, \nabla\beta) \end{aligned}$$

Owing to (4.2), (4.4) we get

$$(4.5) \quad \partial\psi/\partial\vartheta = -\eta, \quad \rho\partial\psi/\partial\mathbf{F} = \mathbf{S}, \quad -\rho\vartheta\partial\psi/\partial\nabla\beta(\partial f/\partial\vartheta) = \mathbf{q}$$

and the reduced inequality

$$(4.6) \quad \mathbf{q}(\partial f/\nabla\beta)(\partial f/\partial\vartheta)^{-1} \cdot \nabla\beta \geq 0.$$



Let us make the additional constitutive assumptions

$$(4.7) \quad \psi = a(\vartheta)(\text{tr} \mathbf{F} \mathbf{F}^T - 3) + b(\vartheta) + 0.5(k_0 \tau_0 / \rho \vartheta_0) \nabla \beta^2,$$

$$\mathbf{q} = -k_0 \nabla \beta,$$

where  $a(\vartheta)$  and  $b(\vartheta)$  are suitable functions of the absolute temperature  $\vartheta$ . If, as in the case discussed in [23], we assume

$$(4.8) \quad T = \tau_0, \quad f(\vartheta, \beta) = \vartheta^0 \tau_0^{-1} \log(\vartheta / \vartheta_0) - (\beta - \beta_0) / \tau_0,$$

then we can rewrite the balance of linear momentum (when the body forces vanish and  $\rho$  is constant) as follows

$$(4.9) \quad \ddot{\mathbf{u}} + A \Delta \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{u} + \mathbf{B} = \mathbf{0},$$

where:

$$A = -2a, \quad \mathbf{B} = -2a' \nabla \vartheta,$$

$\mathbf{u}$  is the displacement vector; the symbol ' means the derivative with respect to  $\vartheta$ .

On the other hand, owing to (4.1)<sub>2</sub> and (4.8) we get

$$(4.10) \quad \tau_0 D \ddot{\beta} + D \dot{\beta} + k_0 \tau_0 \nabla \beta \cdot \nabla \dot{\beta} - k_0 \vartheta_0 \Delta \beta - \rho r \vartheta_0 - \vartheta_0 A \nabla \mathbf{u} \cdot \nabla \dot{\mathbf{u}} = 0,$$

$$(4.11) \quad D = -\rho \vartheta^2 [a''(\text{tr} \mathbf{F} \mathbf{F}^T - 3) + b''].$$

The coupled system (4.9)-(4.10) can be written as a first order system if we put

$$\mathbf{H} = \nabla \mathbf{u}, \quad \mathbf{z} = \nabla \beta, \quad \gamma = \dot{\beta}.$$

and a material system of coordinates  $X_j$  is assumed. We get so

$$(4.12) \quad \dot{v}_i + A \partial H_{ij} / \partial X_j + B_j H_{ij} + C = 0,$$

$$\tau_0 D \dot{\gamma} + D \gamma + k_0 \tau_0 z_i \dot{z}_i - k_0 \vartheta_0 \partial z_i / \partial X_i - \rho r \vartheta_0 - \vartheta_0 A H_{ij} \dot{H}_{ij} = 0,$$

$$\dot{z}_i - \partial \gamma / \partial X_i = 0, \quad \dot{H}_{ij} - \partial v_i / \partial X_j = 0.$$

Some straightforward calculations show that (4.12) is a quasi-linear first order symmetric system provided  $A$  is positive. This proves the well posedness of the local Cauchy problem for (4.12).

As far as acceleration waves are concerned, in thermoelastic materials blow-up could occur even if  $\tau_0$  is equal to zero, [24]. In [22] an evolution equation for the amplitude was derived for thermoelastic solids with thermal relaxation.

This is again of Bernoulli type and similar to the equation obtained in [24] when  $\tau_0$  is equal to zero.

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### REFERENCES

- [1] Maxwell J.C., *On the dynamical theory of gases*, Phil. Trans. Roy. Soc. Lon., 1867, **157**, 49-101.
- [2] Cattaneo C., *Sulla conduzione del calore*, Atti Sem. Mat. Fis. Univ. Modena, **3**, (1948), 83-101.
- [3] Cattaneo C., *Sur une forme d'equation de la chaleur, eliminant le paradoxe d'une propagation instantanéé*, Comp. Rend. Sci., **247**, (1958), 431-433.
- [4] Vernotte M.P., *Les paradoxes de la theorie continue de l'equation de la chaleur*, Comp. Rend. Sc., **246**, (1958), 3154-3155.
- [5] Chester M., *Second sound in solids*, Phys. Rev., **131**, (1963), 2013-2015.
- [6] Ackerman C.C., Guyer R.A., *Temperature pulses in dielectric solids*, Ann. Phys., **50**, (1968), 128-185.
- [7] Morro A., Ruggeri T., *Propagazione del calore ed equazioni costitutive*, Quaderno CNR, Editrice Pitagora, Bologna, 1980.
- [8] Ruggeri T., *Struttura dei sistemi alle derivate parziali compatibili con un principio di entropia e termodinamica estesa*, Suppl.B.U.M.I. (Fis. Matem.) **4** (1985), 261-279.
- [9] Ruggeri T., *Thermodynamics and Symmetric Hyperbolic Systems*, Rend Sem. Mat. Univ. Pol. Torino, Fascicolo Speciale, (1989), 1167-183.
- [10] Gurtin M.E, Pipkin A.C., *A general theory of heat conduction with finite wave speeds*, Arch. Rat. Mech. Anal., **31**, (1968), 113-126.
- [11] Meixner J., *On the Linear Theory of Heat Conduction*, Arch. Rat. Mech. Anal., **39**, (1970), 108-130.

- [12] Morro A. *Temperature waves in rigid materials with memory*, Meccanica, 1977, **12**, 73-77.
- [13] Kosinski W., *Thermal waves in inelastic bodies*, Arch.Mech. (A.M.S.), **237** (1975), 733-748.
- [14] Mihailescu M., Suliciu I., *Finite and symmetric thermomechanical waves in materials with internal state variables*, Int. J. Sol. Struct., **2**, (1976), 559-575.
- [15] Morro A. Wave propagation in thermo-viscous materials with hidden variables, Arch. Mech., (1980), 145-161.
- [16] Müller I., *Toward relativistic thermodynamics*, Arch. Rat. Mech. Anal., **34** (1969), 259, 282.
- [17] Müller I, *Zum Paradoxen der Wärmeleitungstheorie*, Zeit. Physik, **198**, (1967), 329-344.
- [18] Kosinski W., *Elastic waves in the presence of a new temperature scale*, in Proc. IUTAM Symp. on Elastic wave propagation, M.F.McCarthy and M.Hayes (eds.), Galway, Ireland. North Holland 1989.
- [19] Cimmelli V.A., Kosinski W., *Evolution Hyperbolic Equations for Heat Conduction, Proceed. V Polish-Italian Meeting on Thermomechanics and Kinetic Theory*, W.Kosinski, W.Larecki, A. Morro, H. Zorski eds, Warszawa 1991, World Scientific, in press.
- [20] Cimmelli V.A., Kosinski W., *Well-posedness results for a non-linear hyperbolic heat equation*, to appear.
- [21] Kosinski W., Saxton K., *Weak discontinuity waves in materials with semi-empirical temperature scale*, Arch. Mech., **41** (4), (1991), in press.
- [22] Kosinski W., Saxton K., *The effect on finite time breakdown due to modified Fourier laws*, to appear in Quart. Appl. Math.
- [23] Cimmelli V.A., Kosinski W., *Nonequilibrium semi-empirical temperature in materials with thermal relaxation*, to appear, in Arch. Mech.
- [24] Dafermos C.M., Hsiao L., *Development of singularities in solutions of the equations of nonlinear thermoelasticity*, Quart.Appl.Math., **44**, (1986), 463-47.

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