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LINEAR ASYMPTOTIC STABILITY OF GEOPHYSICAL CHANNELED FLOWS IN THE PRESENCE OF ARBITRARY LONGITUDE-SHAPED PERTURBATIONS

FULVIO CRISCIANI (Trieste) - RENZO MOSETTI (Trieste)

Sufficient conditions for the linear asymptotic stability of large scale wind-driven oceanic flows are derived in the presence of arbitrary longidude-shaped perturbations. Criteria work when both bottom dissipation and lateral diffusion of relative vorticity are simultaneously present. The stability is controlled by the maximum of the shear of the basic flow and by the maximum of its meridional derivative and involves the dissipation-diffusion coefficients.

1. Introduction.

The geophysical relevance of channeled flows, i.e. flows ideally confined between two "rigid walls" coinciding with a couple of Earth paralles, comes from the fact that, in a rotating planet, steady zonal flows are able to mantain themselves without any external forcing and moreover they are the simplest flow configurations in the presence of a longitude-independent, (in case unsteady) forcing. The main features of the large-scale dynamics of such flows are based on the hydrostatic equilibrium and the geostrophic balance. With the aid of these observational facts, the classical Navier-Stokes equations referred to a rotating system can be properly scaled to obtain the so called "quasi-geostrophic" vorticity balance of the fluid bulk inferior that, in the absence of density stratification, takes the standard form:

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi + \beta y) = F - r\nabla^2\psi + A\nabla^4\psi \tag{1}$$

where ψ is the streamfunction (x being longitude and y latitude), $\nabla^2\psi$ is the local vorticity that is the vertical component of the curl of the geostrophic current

$$u = -\frac{\partial\psi}{\partial y} \quad v = \frac{\partial\psi}{\partial x},$$

β is the planetary vorticity gradient coming from the latitudinal variation of the Coriolis parameter, F is a forcing term that may be the wind-stress curl hereafter assumed longitude- independent (in case unsteady) $-r\nabla^2\psi$ is the bottom dissipation term coming from the Ekman benthic layer and finally $A\nabla^4\psi$ describes the lateral vorticity diffusion.

For a review of this dynamics see for instance Hendershott, 1987 and Pedlosky, 1979.

In what follows, both $r > 0$ and $A > 0$ will be assumed.

Boundary conditions on ψ come from the physical requirement of no mass flux across the walls, i.e.

$$\left(\frac{\partial\psi}{\partial x}\right)_{y_1} = \left(\frac{\partial\psi}{\partial x}\right)_{y_2} = 0 \tag{2}$$

where y_1 and y_2 are the wall latitudes, while the lateral vorticity diffusion term implies further boundary conditions: here the choice

$$(\nabla^2\psi)_{y_1} = (\nabla^2\psi)_{y_2} = 0 \tag{3}$$

will be made.

An important feature of problem (1), (2) and (3) is that, once that the input $F(y)$ is specified, zonal solutions $\psi_0(y)$ exist. In fact, putting $\nabla^2\psi_0 = q_0$, the problem above is simplified into the following one:

$$F(y) - rq_0 + Aq_{0yy} = 0$$

$$q_0(y_1) = q_0(y_2) = 0$$

which trivially admits a unique, solution, from which $\psi_0(y)$ follows, apart from a vorticity-vanishing term, unrelevant from the stability point of view.

In this context the linear asymptotic stability in the energy norm of the zonal solution, say $\psi_0(y)$ of the so posed problem is investigated from the point of view of the "a priori" properties of the related linear perturbation equation

$$\frac{\partial}{\partial t}\nabla^2\phi - \frac{\partial\psi_0}{\partial y}\frac{\partial\nabla^2\phi}{\partial x} + \left(\frac{\partial^3\psi_0}{\partial y^3} + \beta\right)\frac{\partial\phi}{\partial x} = -r\nabla^2\phi + A\nabla^4\phi \tag{4}$$

where $\phi = \psi - \psi_0$.

We recall that a basic state is asymptotically stable if the kinetic energy of every perturbation superimposed to it satisfies the following relationship:

$$\lim_{t\to\infty} K(t) = 0 \tag{5}$$

Relation (5) is clearly stronger than the one required for the stability in the energy norm, wich only requires

$$\frac{dK}{dt} < 0$$

Asymptotic stability criteria can be obtained by using "a priori" estimates through inequalities which in turn imply relation (5). In this work the following conditions are employed:

$$\frac{dK}{dt} + a^2K < 0$$

and

$$K(t) \leq A^2(t) \text{ where } \lim_{t\to\infty} A^2(t) = 0$$

a^2 and A^2 being definite quantities.

We want to stress that the corresponding non-linear problem cannot be fitted into the Arnold's method. On the other hand, the use the direct method of Lyapunov, apart from some specific problems (Benard convection and Couette flow), have not proved very useful yet (Drazin and Reid, 1981; p. 431). This justifies the investigation of different methods in the hydrodynamical stability theory. In a non-linear context, stability criteria for flows confined in closed basins have been obtained by Crisciani and Mosetti, 1990. For zonal flows criteria for stability in the energy norm independent from the perturbation wave-number are reported in Crisciani and Mosetti, 1991.

2. Wavenumber-dependent stability criteria.

As the basic state ψ_0 does not depend on the latitude x, the perturbation can be written as

$$\phi = B(y,t)e^{ikx}$$

and inside the volume element $(0 \leq x \leq L,\ -D \leq z \leq 0,\ y_1 \leq y \leq y_2)$ the perturbation kinetic energy takes the form

$$K = \frac{1}{2} DL(||B_y||^2 + k^2||B||^2) \tag{6}$$

where:

$$||\cdot||^2 = \int_{y_1}^{y_2} |\cdot|^2 dy$$

Now, if equation (4) is multiplied by B^* (the complex conjugate of B), the complex conjugate of Eq. (4) is multiplied by B, the results are added and then integrated from y_1 to y_2, the following equation holds:

$$\frac{d}{dt}(||b_y||^2 + k^2||B||^2) + 2r(||B_y||^2 + k^2||B||^2) =$$
$$= -2k\int_{y_1}^{y_2} \psi_{0yy} Im(B^* B_y) dy - 2A(||B_{yy}||^2 +$$
$$+ 2k^2||B_y||^2 + k^4||B||^2)$$

Taking into account the inequality

$$\left|\int_{y_1}^{y_2} \psi_{0yy} Im(B_y^* B) dy\right| \leq \alpha^{-1} \max_Y |\psi_{0yy}|\ ||B_y||^2$$

where $\alpha = \frac{\pi}{y_2 - y_1}$ we obtain

$$\left|\frac{1}{DL}\frac{dK}{dt} + r(||B_y||^2 + k^2||B||^2) + A(||B_{yy}||^2 + 2k^2||B_y||^2 + k^4||B||^2)\right| \leq \tag{7}$$
$$\leq \alpha^{-1}|k| \max_Y |\psi_{0yy}|\ ||B_y||^2$$

Due to the definition of the kientic energy of the perturbation, the following equality holds

$$||B_{yy}||^2 + 2k^2||B_y||^2 + k^4||B||^2 = ||B_{yy}||^2 + k^2||B_y||^2 + k^2\frac{2K}{DL}$$

So, by substituting this into inequality (7) we have

$$\frac{1}{DL}\frac{dK}{dt}+r(||B_y||^2+k^2||B||^2)+A(||B_{yy}||^2+k^2||B_y||^2+\frac{2k^2}{DL}K)\leq$$

$$\leq \alpha^{-1}|k|\mu_2||B_y||^2$$

and, from this the following inequality holds:

$$\frac{1}{DL}\left(\frac{dK}{dt}+2Ak^2K\right)\leq(|k|\alpha^{-1}\mu_2-Ak^2-r)||B_y||^2 \tag{8}$$

where: $\mu_2=\max_y|q_0|$.

Now, if

i) $(\alpha^{-1}\mu_2)^2-4Ar<0$

then the r.h.s. of Inequality (8) is always negative and therefore

$$\lim_{t\to\infty}K=0;$$

ii) if $|k|\leq\frac{1}{2A}[\alpha^{-1}\mu_2-\sqrt{((\alpha^{-1}\mu_2)^2-4Ar)}]$
again the r.h.s. of Inequality (8) is negative and

$$\lim_{t\to\infty}K=0.$$

Observe that condition i) already is a criterion independent from the perturbation wavenumber k.

To proceed further, if Equation (4) is multiplied by

$$B^*_{yy}-k^2B^*$$

and the same procedure as above is adopted, putting for shortness

$$M=B_{yy}-k^2B$$

the inequalities

$$\frac{d}{dt}||M||^2+2r||M||^2\leq\left(\frac{\mu_3}{|k|}-Ak^2\right)||M||^2$$

and

$$\frac{d}{dt}||M||^2+2Ak^2||M||^2\leq\left(\frac{\mu_3}{|k|}-r\right)||M||^2$$

follow, where

$$\mu_3 = \max_y |q_{oy}|.$$

Since

$$||M||^2 \geq \frac{2k^2}{DL} K$$

if:

iii) $|k| \geq \left(\frac{\mu_3}{A}\right)^{1/3}$

or if

iv) $|k| \geq \frac{\mu_3}{r}$ then, recalling what has been stated in section 1,

$$\lim_{t \to \infty} K = 0$$

follows.

3. Wavenumber-independent stability criteria.

At this point it is useful to denote the set of real numbers where Inequalities ii), iii) and iv) are separately verified, that is to say:

$$D_a = \left\{ x : |x| \leq \frac{1}{2A} [\alpha^{-1}\mu_2 - \sqrt{((\alpha^{-1}\mu_2)^2 - 4Ar)}] \right\}$$

$$D_b = \left\{ x : |x| \geq \left(\frac{\mu_3}{A}\right)^{1/3} \right\}$$

$$D_c = \left\{ x : |x| \geq \frac{\mu_3}{r} \right\}$$

Now, if $D_a \cup D_b = \mathbb{R}$ or $D_a \cup D_c = \mathbb{R}$, every wavenumber k satisfies at least one of the above inequalities and the asymptotic stability holds not only for plane wave perturbations of the kind:

$$\phi = B(y,t)e^{ikx}$$

where k is an arbitrary wavenumber but, due to the linearity of the perturbation equation, also for perturbations of the kind

$$\phi = \int B(y,t,k)e^{ikx} dk$$

the integral being any Fourier superposition of plane waves in the configuration space. It is easily seen that criteria deduced in this way take respectively the form

$$\text{a)}\quad \frac{1}{2A}\left[\alpha^{-1}\mu_2 - \sqrt{((\alpha^{-1}\mu_2)^2 - 4Ar)}\right] \geq \frac{\mu_3}{r}$$

$$\text{b)}\quad \frac{1}{2A}\left[\alpha^{-1}\mu_2 - \sqrt{((\alpha^{-1}\mu_2)^2 - 4Ar)}\right] \geq \left(\frac{\mu_3}{A}\right)^{1/3}$$

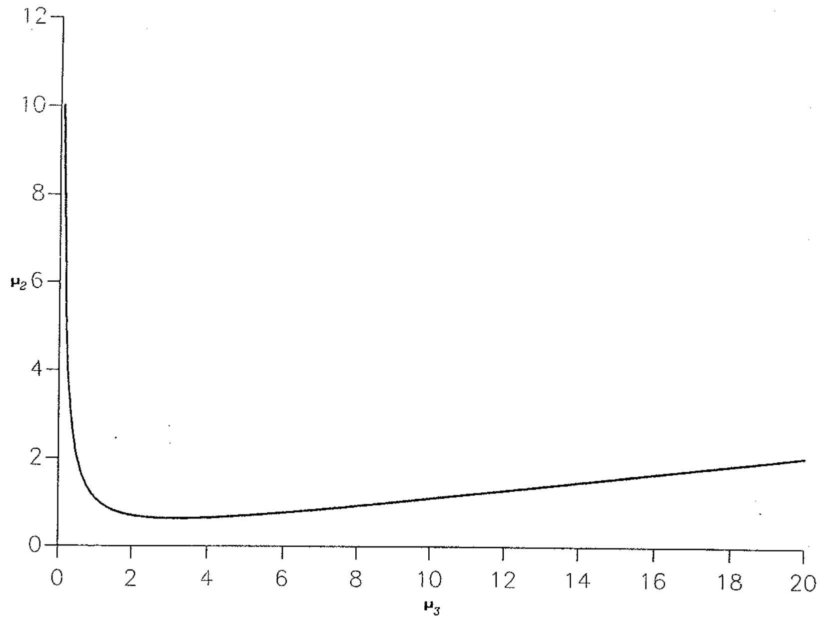

Fig. 1 - Relation between μ_2 and μ_3 obtained by applying the stability criterion a) for Reynolds number = 10 and $r = 1$. The region below the curve is that in which the stability is ensured.

In Figures 1 and 2, the stability regions in the plane μ_2, μ_3 are shown for inequalities a) and b). The results are in nondimensional units obtained by scaling a) and b) after the definition of the Reynolds number.

4. An example.

Let us consider a typical channeled basic flow as in Kuo, 1973

$$u_0 = U_M \cos^2 \frac{2\pi y}{L}.$$

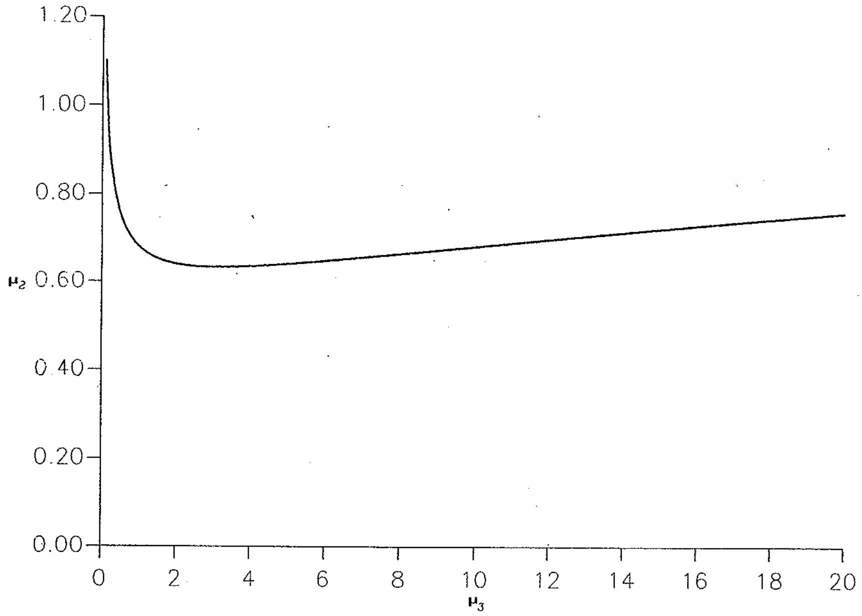

Fig. 2 - Relation between μ_2 and μ_3 obtained by applying the stability criterion b) for Reynolds number $= 10$ and $r = 1$. The region below the curve is that in which the stability is ensured.

In this case

$$\mu_2 = 2\pi \frac{U_M}{L} \qquad \alpha = \frac{\pi}{L}$$

If we apply criterion i) by taking some typical oceanic values for the coefficients of lateral diffusion of vorticity and bottom dissipation (in S.I. units)

$$A \sim 10^3 \quad r \sim 10^{-7}$$

the following inequality guarantees the linear asymptotic stability of the basic flow

$$U_M < 10^{-2}$$

This is the typical order of magnitude for quasi-geostrophic currents and so this kind of flow can reach stability for reasonable values of the diffusion-dissipation parameters.

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Fulvio Crisciani
C.N.R. Istituto Sperimentale Talassografico
Viale R. Gessi 2
34123 Trieste (Italy)

Renzo Mosetti
Osservatorio Geofisico Sperimentale
P.O. Box 2011
34016 Trieste (Italy)