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HYPERBOLIC-PARABOLIC SINGULAR PERTURBATIONS IN GENERAL REGIONS

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We consider a singular perturbations problem for the *inhomogeneous damped wave equation* and the *inhomogeneous heat equation* with moving boundary. We give rigorous and explicit estimates and show the uniform convergence.

1. Introduction.

Hyperbolic-parabolic singular perturbations problems have been considered some years ago by several Authors [8, 9, 5]. They analyzed both the Cauchy problem and the initial-boundary values problem. However, in all these papers the boundary is assumed to be fixed.

More recently, it has been emphasized that hyperbolic heat transfer models could play an important role in technological problems when high energies are involved [4]. Thus, some questions related to hyperbolic free boundary problems have been discussed [6, 7, 1].

In this connection, in some works we studied hyperbolic-parabolic singular perturbations problems related to initial-boundary values

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problems with moving boundary [2, 3]. Following these researches we consider, now, the singular perturbations for the equations

$$(\varepsilon\partial_t^2 + \partial_t)u_\varepsilon = (\alpha\partial_x^2 + c)u_\varepsilon + f,$$

$$\partial_t u = (\alpha\partial_x^2 + c)u + f,$$

in an initial-boundary values problem with moving boundary of equation $x = r(t)$. We deduce a rigorous and explicit estimate for the difference of the solutions and, as consequence, the uniform convergence. More precisely, denoting by $u_\varepsilon(x,t)$ and $u(x,t)$ the two solutions, we obtain the following estimate

$$|u_\varepsilon - u| \leq k\varepsilon^p,$$

where k is a constant independent of x, t, ε and p is a positive rational number.

2. Statement of the problem.

We want to discuss the singular perturbations problem that arises when we study the convergence of the solutions of the inhomogeneous damped wave equation to the solutions of the inhomogeneous heat equation for the boundary values problems defined in (2.1)-(2.6). The hyperbolic problem is the following

$$(\varepsilon\partial_t^2 + \partial_t)u_\varepsilon = (\alpha\partial_x^2 + c)u_\varepsilon + f(x,t), \;\; x > r(t), 0 < t < T, \tag{2.1}$$

$$u_\varepsilon(x,0) = \phi(x), \;\; u_{\varepsilon,t}(x,0) = \psi(x), \;\; x > 0, \tag{2.2}$$

$$u_\varepsilon(r(t),t) = a(t), \;\; 0 < t < T, \tag{2.3}$$

In the parabolic case the same boundary value problem, obviously, requires only one initial condition

$$\partial_t u = (\alpha\partial_x^2 + c)u + f(x,t), \;\; x > r(t), \;\; 0 < t < T, \tag{2.4}$$

$$u(x,0) = \phi(x), \;\; x > 0, \tag{2.5}$$

$$u(r(t),t) = a(t), \; 0 < t < T. \tag{2.6}$$

Here α and ε are assumed to be constant. The small parameter ε is called material relaxation time. Moreover, $x = r(t)$ is the equation of the moving boundary.

We consider the singular perturbations problem on the region

$$\Omega = \{(x_0, t_0) 0 < t_0 < T, \; r(t_0) < x_0 < t_0\sqrt{\alpha/\varepsilon}\}, T > 0, \tag{2.7}$$

since for $x_0 > t_0\sqrt{\varepsilon/\alpha}$ the hyperbolic boundary values problem becames a Cauchy problem and the solution converges to the corresponding solution of the parabolic initial value problem [5].

By using the *fundamental solution* of (2.1)

$$V^c(x_0 - x, t_0 - \tau) = \frac{e^{-\frac{t_0-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}} I_0\left(\sqrt{(1+4c\varepsilon)\left[\frac{(t_0-\tau)^2}{4\varepsilon^2} - \frac{(x_0-x)^2}{4\alpha\varepsilon}\right]}\right), \tag{2.8}$$

where $I_n (n \geq 0)$ is the modified Bessel function of order n the solution u_ε of (2.1)-(2.3) can be written as

$$u_\varepsilon(x_0, t_0) = \frac{e^{-t_0/2\varepsilon}}{2}\varphi\left(x_0 + t_0\sqrt{\frac{\alpha}{\varepsilon}}\right) + \tag{2.9}$$

$$+ \int_0^{x_0 + t_0\sqrt{\frac{\alpha}{\varepsilon}}} [\varepsilon\psi(x) + \varphi(x)(1 + \varepsilon\partial_{t_0})]V^c(x_0 - x, t_0)dx +$$

$$+ \frac{a(t)}{2}e^{-\frac{t_0-t}{2\varepsilon}} - \int_0^t a(\tau)[\dot{r}(\tau) - \varepsilon\dot{r}(\tau)\partial_\tau - \alpha\partial_x]V^c(x_0 - r(\tau), t_0 - \tau)d\tau -$$

$$- \int_0^t V^c(x_0 - r(\tau), t_0 - \tau)[\varepsilon\dot{r}(\tau)\dot{a}(\tau) + \alpha w_\varepsilon(\tau)(1 - \varepsilon\dot{r}^2(\tau)/\alpha)]d\tau +$$

$$+ \int_0^{t_0} d\tau \int_{s(\tau)}^{x_0 + \sqrt{\frac{\alpha}{\varepsilon}}(t_0-\tau)} f(x,\tau)V^c(x_0 - x, t_0 - \tau)dx, (x_0, t_0) \in \Omega.$$

Here t and $s(\tau)$ are respectively defined by

$$t = t_0 - \sqrt{\varepsilon/\alpha}[x_0 - r(t)],$$

$$s(\tau) = r(\tau) \text{ for } 0 < \tau < t, s(\tau) = x_0 - \sqrt{\frac{\alpha}{\varepsilon}}(t_0 - \tau) \text{ for } t < \tau < t_0,$$

and the function

$$w_\varepsilon(t) = u_{\varepsilon,x}(r(t), t) \tag{2.10}$$

satisfies the following Volterra integral equation

$$
\begin{aligned}
(2.11)\quad & w_\varepsilon(t)\left[1-\dot r(t)\sqrt{\frac{\varepsilon}{\alpha}}\right]=e^{-t/2\varepsilon}\varphi'\left(r(t)+t\sqrt{\frac{\alpha}{\varepsilon}}\right)+2\varepsilon\psi(0)V^c(r(t),t)+\\
&+2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}[\varepsilon\psi'(x)+\varphi'(x)(1+\varepsilon\partial_{t_0})]V^c(r(t)-x,t)dx-\\
&-2\int_0^t[\dot a(\tau)(1+\varepsilon\partial_{t_0}+\varepsilon\dot r(\tau)\partial_{x_0})-ca(\tau)]V^c(r(t)-r(\tau),t-\tau)d\tau-\\
&-\sqrt{\frac{\varepsilon}{\alpha}}\dot a(t)+2\int_0^t V_x^c(r(t)-r(\tau),t-\tau)\alpha w_\varepsilon(\tau)[1-\varepsilon\dot r^2(\tau)/\alpha]d\tau+\\
&+2\int_0^t f(r(\tau),\tau)V^c(r(t)-r(\tau),t-\tau)d\tau+\\
&+2\int_0^t d\tau\int_{r(\tau)}^{r(t)+\sqrt{\frac{\alpha}{\varepsilon}}(t-\tau)} f_x(x,\tau)V^c(r(t)-x,t-\tau)dx,\ 0<t<T.
\end{aligned}
$$

Moreover, introducing the *fundamental solution* of (2.4) given by

$$
(2.12)\qquad E^c(x_0-x,t_0-\tau)=\frac{e^{c(t_0-\tau)}e^{-\frac{(x_0-x)^2}{4\alpha(t_0-\tau)}}}{\sqrt{4\pi\alpha(t_0-\tau)}},
$$

the solution u of (2.4)-(2.6) is

$$
\begin{aligned}
(2.13)\qquad & u(x_0,t_0)=\int_0^\infty\varphi(x)E^c(x_0-x,t_0)dx-\\
&-\int_0^{t_0}[a(\tau)(\dot r(\tau)-\alpha\partial_x)E^c(x_0-r(\tau),t_0-\tau)+\alpha w(\tau)E^c(x_0-r(\tau),t_0-\tau)]d\tau+\\
&+\int_0^{t_0}d\tau\int_{r(t)}^\infty f(x,\tau)E^c(x_0-x,t_0-\tau)dx,\ (x_0,t_0)\in\Omega,
\end{aligned}
$$

where the function

$$
(2.14)\qquad w(t)=u_x(r(t),t)
$$

verifies the following integral equation

$$
\begin{aligned}
(2.15)\qquad & w(t)=2\int_0^\infty\varphi'(x)E^c(r(t)-x,t)dx-\\
&-2\int_0^t\{[\dot a(\tau)-ca(\tau)]E^c(r(t)-r(\tau),t-\tau)-\alpha w(\tau)E_x^c(r(t)-r(\tau),t-\tau)\}d\tau+
\end{aligned}
$$

$$+2\int_0^t f(r(\tau),\tau)E^c(r(t)-r(\tau),t-\tau)d\tau+$$
$$+2\int_0^t d\tau\int_{r(\tau)}^{\infty} f_x(x,\tau)E^c(r(t)-x,t-\tau)dx,\ 0<t<T.$$

3. Singular perturbations.

We study the singular perturbations problem with the following hypotheses on the data

$$\varphi\in C^2([0,+\infty[),\ \varphi(0)=a(0), \tag{3.1}$$

$$|\varphi(x)|<M_\varphi,\ |\varphi'(x)|<M'_\varphi,\ |\varphi''(x)|<M''_\varphi, \tag{3.2}$$

$$\psi\in C^1([0,2T\sqrt{\alpha/\varepsilon}]),\ |\psi(x)|<M_\psi,\ |\psi'(x)|<M'_\psi, \tag{3.3}$$

$$a\in C^2([0,T]),\ |a(t)|<M_a,\ |\dot a(t)|<M'_a,\ |\ddot a(t)|<M''_a, \tag{3.4}$$

$$f\in C^1(\{(x,t)\ :\ 0<t<T, r(t)<x<\infty\}), \tag{3.5}$$

$$|f(x,t)|<M_f,\ |f_x(x,t)|<M'_f. \tag{3.6}$$

In (3.1))-(3.6)) $M_\varphi, M'_\varphi, M''_\varphi, M_\psi, M'_\psi, M_a, M'_a, M''_a, M_f, M'_f$ are positive constants. Moreover, on the function $r(t)$ describing the moving boundary, we assume

$$r(t)\in C^2([0,T]),\ r(0)=0,\ |\dot r(t)|\le r_1,\ r_1=\text{constant}<\sqrt{\alpha/\varepsilon}. \tag{3.7}$$

Finally, we recall two inequalities [5, Sec.2] about the fundamental solutions (2.8), (2.12) when $t>0$ and $|x|<t\sqrt{\alpha/\varepsilon}$

$$V^c(x,t)\le C_0E^c(x,t),\ d=\sqrt{1+4c\varepsilon},\ C_0=\sqrt{2\pi(2/d+1/e)}<4, \tag{3.8}$$

$$\frac{de^{-\frac{t}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}}\frac{I_1\left(d\sqrt{\frac{t^2}{4\varepsilon^2}-\frac{x^2}{4\alpha\varepsilon}}\right)}{\sqrt{1-\varepsilon x^2/\alpha t^2}}\le C_1E^c(x,t),\ C_1=d^2\sqrt{2\pi}(2/d+3/e)^{3/2}. \tag{3.9}$$

Now, we consider the two solutions (2.9), (2.13) and get

$$u_\varepsilon(x_0,t_0)-u(x_0,t_0)=m(x_0,t_0,\varepsilon)+m_b(x_0,t_0,\varepsilon)+m_f(x_0,t_0,\varepsilon)+ \\ +\int_0^t \alpha\{w(\tau)-w_\varepsilon(\tau)[1-\varepsilon\dot r^2(\tau)/\alpha]\}V^c(x_0-r(\tau),t_0-\tau)d\tau, \tag{3.10}$$

where

$$m(x_0,t_0,\varepsilon)=-\alpha\int_0^t w(\tau)V^c(x_0-r(\tau),t_0-\tau)d\tau+ \\ +\alpha\int_0^{t_0} w(\tau)E^c(x_0-r(\tau),t_0-\tau)d\tau, \tag{3.11}$$

$$m_b(x_0,t_0,\varepsilon)=\frac{e^{-\frac{t_0}{2\varepsilon}}}{2}\varphi\left(x_0+t_0\sqrt{\frac{\alpha}{\varepsilon}}\right)-\int_0^\infty \varphi(x)E^c(x_0-x,t_0)dx+ \\ +\int_0^{x_0+t_0\sqrt{\frac{\alpha}{\varepsilon}}}[\varepsilon\psi(x)+\varphi(x)(1+\varepsilon\partial_{t_0})]V(x_0-x,t_0)dx, \\ +\frac{a(t)}{2}e^{-\frac{t_0-t}{2\varepsilon}}-\int_0^{t_0} a(\tau)[\alpha\partial_x-\dot r(\tau)]E^c(x_0-r(\tau),t_0-\tau)d\tau+ \\ +\int_0^t[\varepsilon a(\tau)\dot r(\tau)\partial_\tau+a(\tau)\alpha\partial_x-a(\tau)\dot r(\tau)-\varepsilon\dot r(\tau)\dot a(\tau)]V^c(x_0-r(\tau),t_0-\tau)d\tau, \tag{3.12}$$

$$m_f(x_0,t_0,\varepsilon)=\int_0^{t_0}d\tau\int_{s(\tau)}^{x_0+\sqrt{\frac{\alpha}{\varepsilon}}(t_0-\tau)} f(x,\tau)V^c(x_0-x,t_0-\tau)d\tau- \\ -\int_0^{t_0}d\tau\int_{r(\tau)}^\infty f(x,\tau)E^c(x_0-x,t_0-\tau)d\tau. \tag{3.13}$$

First, we discuss $m(x_0,t_0,\varepsilon)$ and show

THEOREM 3.1. *Under the hypotheses* (3.1) – (3.6) *there exists a constant* K_m *independent of* ε, x_0, t_0, *such that*

$$|m(x_0,t_0,\varepsilon)|\le K_m\varepsilon^q, \tag{3.14}$$

where q *is a strictly positive rational number.*

Proof. Setting

$$(3.15)\qquad m_1 = \alpha \int_0^t w(\tau)(E^c - V^c)(x_0 - r(\tau), t_0 - \tau)d\tau,$$

$$(3.16)\qquad m_2 = \alpha \int_t^{t_0} w(\tau)E^c(x_0 - r(\tau), t_0 - \tau)d\tau,$$

from (3.11) we have

$$(3.17)\qquad m(x_0, t_0, \varepsilon) = m_1 + m_2.$$

Considering that $|w(t)| \leq M_w, 0 < t < T$, where M_w is a constant independent of ε (see [2, 3]), we obtain

(3.18)

$$|m_2| \leq \sqrt{\frac{\alpha}{\pi}} M_w e^{cT} \int_t^{t_0} \frac{e^{-\frac{[x_0 - r(\tau)]^2}{4\alpha(t_0-\tau)}}}{\sqrt{4(t_0 - \tau)}} d\tau \leq \sqrt{\frac{\alpha}{\pi}} M_w e^{cT} \int_t^{t_0} \frac{e^{-\frac{t_0-\tau}{4\varepsilon}}}{\sqrt{4(t_0-\tau)}} d\tau,$$

since $\varepsilon[x_0 - r(\tau)]^2 \geq \alpha(t_0 - \tau)^2$ for $t \leq \tau < t_0$. Hence,

$$(3.19)\quad |m_2| \leq \sqrt{\frac{\alpha}{\pi}} M_w \frac{e^{cT}}{2} \left(\frac{\varepsilon}{e}\right)^{1/4} \int_t^{t_0} \frac{d\tau}{(t_0-\tau)^{3/4}} \leq \sqrt{\frac{\alpha}{\pi}} M_w 2e^{cT} \left(\frac{T\varepsilon}{e}\right)^{1/4}.$$

Then, we consider m_1 and suppose $t \leq \varepsilon$; recalling (3.8) we have

$$(3.20)\qquad |m_1| \leq \sqrt{\frac{\alpha}{\pi}} 5M_w e^{cT} \int_0^t \frac{d\tau}{\sqrt{4(t_0-\tau)}} \leq \sqrt{\frac{\alpha}{\pi}} 5M_w e^{cT} \varepsilon^{1/2}.$$

If $t > \varepsilon$, we introduce

$$D_1 = \{0 < \tau < t - \varepsilon \ : \ \frac{\varepsilon}{\alpha}\left[\frac{x_0 - r(\tau)}{t_0 - \tau}\right]^2 \leq 1/4\},$$

$$D_2 = \{0 < \tau < t - \varepsilon \ : \ \frac{\varepsilon}{\alpha}\left[\frac{x_0 - r(\tau)}{t_0 - \tau}\right]^2 > 1/4\},$$

and note that

$$(3.21)\qquad |m_1| \leq m_{11} + m_{12} + m_{13}$$

with

$$m_{11} = \alpha M_w \int_{D_1} |(E^c - V^c)(x_0 - r(\tau), t_0 - \tau)| d\tau,$$
$$m_{12} = \alpha M_w \int_{D_2} |(E^c - V^c)(x_0 - r(\tau), t_0 - \tau)| d\tau,$$
$$m_{13} = \alpha M_w \int_{t-\varepsilon}^{t} |(E^c - V^c)(x_0 - r(\tau), t_0 - \tau)| d\tau.$$

The last integral is easily estimated as

$$m_{13} \leq \sqrt{\frac{\alpha}{\pi}} 5 M_w e^{cT} \int_{t-\varepsilon}^{t} \frac{d\tau}{\sqrt{4(t_0 - \tau)}} \leq \sqrt{\frac{\alpha}{\pi}} 5 M_w e^{cT} \varepsilon^{1/2}. \tag{3.22}$$

Moreover,

$$m_{12} \leq \sqrt{\frac{\alpha}{\pi}} 5 M_w e^{cT} \int_{D_2} \frac{e^{-\frac{t_0-\tau}{16\varepsilon}}}{\sqrt{4(t_0 - \tau)}} d\tau,$$

hence

$$m_{12} \leq \sqrt{\frac{\alpha}{\pi}} \frac{5}{2} M_w e^{cT} \int_0^{t-\varepsilon} \left(\frac{4\varepsilon}{e}\right)^{1/4} \frac{d\tau}{(t_0 - \tau)^{3/4}},$$

and therefore

$$m_{12} \leq \sqrt{\frac{\alpha}{\pi}} 10 M_w e^{cT} \left(\frac{4T\varepsilon}{e}\right)^{1/4}. \tag{3.23}$$

Finally, we consider m_{11} and observe that

$$m_{11} \leq m_{111} + m_{112} + m_{113} + m_{114}, \tag{3.24}$$

where

$$m_{111} = M_w \alpha \int_{D_1} \frac{e^{-\frac{t_0-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}} \left| I_0\left(\frac{d(t_0-\tau)}{2\varepsilon}\sqrt{1 - \frac{\varepsilon}{\alpha}\left[\frac{x_0 - r(\tau)}{t_0 - \tau}\right]^2}\right) - \right.$$
$$\left. - \frac{e^{\frac{d(t_0-\tau)}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{x_0-r(\tau)}{t_0-\tau}\right)^2}}}{\sqrt{\pi d(t_0-\tau)/\varepsilon}\left[1 - \frac{\varepsilon}{\alpha}\left(\frac{x_0-r(\tau)}{t_0-\tau}\right)^2\right]^{1/4}} \right| d\tau,$$

$$m_{112} = M_w \alpha \int_{D_1} \frac{e^{-\frac{t_0-\tau}{2\varepsilon}} e^{\frac{d(t_0-\tau)}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}(\frac{x_0-r(\tau)}{t_0-\tau})^2}}}{\sqrt{4d\pi\alpha(t_0-\tau)}} \left| \frac{1}{[1 - \frac{\varepsilon}{\alpha}(\frac{x_0-r(\tau)}{t_0-\tau})^2]^{1/4}} - 1 \right| d\tau,$$

$$m_{113} = M_w \frac{\alpha}{\sqrt{d}} \int_{D_1} \left| \frac{e^{-\frac{t_0-\tau}{2\varepsilon}} e^{\frac{t_0-\tau}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{x_0-r(\tau)}{t_0-\tau}\right)^2}}}{\sqrt{4\pi\alpha(t_0-\tau)}} - E^c(x_0-r(\tau),t_0-\tau) \right| d\tau,$$

$$m_{114} = M_w \alpha \int_{D_1} E^c(x_0-r(\tau),t_0-\tau)|1/\sqrt{d}-1|d\tau.$$

Afterwards, m_{111} can be evaluated by applying the following property of the modified Bessel functions [10]

$$|I_n(\xi) - e^{\xi}/\sqrt{2\pi\xi}| \leq K/\xi, \;\; K = \text{constant}, \xi > 0.$$

Indeed, recalling also that $1 - [\varepsilon(x_0 - r(\tau))^2/\alpha(t_0-\tau)^2] \geq 3/4$ on D_1 , we have

$$\text{(3.25)} \qquad m_{111} \leq M_w \frac{2K}{d}\sqrt{\frac{\alpha\varepsilon}{3}} \int_0^{t-\varepsilon} \frac{e^{-\frac{t_0-\tau}{2\varepsilon}}}{t_0-\tau} d\tau \leq 4KM_w\sqrt{\frac{2\alpha\varepsilon}{3}},$$

Now, we estimate m_{112} obtaining

$$m_{112} \leq \frac{M_w\alpha}{\sqrt{d}} \left(\frac{4}{3}\right)^{1/4} \int_0^{t-\varepsilon} E^c(x_0-r(\tau),t_0-\tau) \frac{\varepsilon[x_0-r(\tau)]^2}{\alpha(t_0-\tau)^2} d\tau,$$

and, therefore, using (3.8),

$$\text{(3.26)} \qquad m_{112} \leq 4e^{cT} M_w \left(\frac{4}{3}\right)^{1/4} \left(\frac{\alpha\varepsilon}{\pi}\right)^{1/2}.$$

Similarly,

$$\text{(3.27)} \qquad m_{113} \leq \frac{e^{cT}M_w}{e^2}\sqrt{\frac{\alpha\varepsilon}{\pi}}.$$

Lastly,

$$\text{(3.28)} \quad m_{114} \leq M_w\alpha \int_{D_1} E^c(x_0-r(\tau),t_0-\tau)(\sqrt{d}-1)d\tau \leq M_w\sqrt{\frac{T\alpha}{\pi}} e^{cT} 4c\varepsilon.$$

Recalling (3.15)-(3.28) we see that the theorem is proved.

Analogous results can be shown for each of the integrals at right-hand side of (3.10) by using the methods introduced in [2, 3] and rearranged to this situation following Th.3.1. Thus, we achieve

THEOREM 3.2. *If the hypotheses* (3.1) – (3.6) *are satisfied there exists a constant k independent of* ε, x_0, t_0, *such that*

$$|u_\varepsilon(x_0, t_0) - u(x_0, t_0)| \leq k\varepsilon^p,$$

where p is a strictly positive rational number. Therefore, u_ε converges uniformly to u.

Finally, we remark that Th.3.2 gives also a rigorous and explicit estimate of the difference of the solutions.

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