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TRANSPORT PHENOMENA IN A PLASMA. QUASILINEAR THEORY

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Making use of a recently developed quasi-linear formulation of 1D Vlasov equation, we derive the balance relations for the space-averaged distribution function and spectral power density. The validity-range in the short-time behavior as well as in the time asymptotic limit is discussed. The formalism is perturbative but non-markovian in character, as it formally generalizes, and in the appropriate limit reproduces, Pocobelli's kinetic theory.

Si presentano le equazioni di bilancio per la funzione di distribuzione media e la densità di potenza spettrale, sulla base di una recente formulazione quasi-lineare per l'equazione 1D di Vlasov. Si discutono le condizioni di validità sia nel breve termine che nel limite asintotico. Il formalismo è di tipo perturbativo ma non-markoviano, in quanto formalmente generalizza, e nel limite appropriato si identifica con la teoria cinetica di Pocobelli.

The application of the quasi-linear (QL) approximation to transport processes in weakly-turbulent plasmas, as well as the fundations of the theory, have been investigated since the early sixties [1]. In order to extend the investigations to more general situations, involving, e. g,. trapping-particle dynamics and radiation scattering phenomena, a detailed knowledge of the orbit perturbation during a sufficiently long time of the particle motion is in order. In the Dupree-Weinstock theory of strong turbulence [2, 3], the role of the time-propagator $U(t,0)$ on the transition between Eulerian (x,v)-variables and Lagrangian $(x(t),v(t))$-Vlasov-orbits, in an arbitrary function $G(x,v)$, is basic to understand the description of turbulent transport within the framework of Vlasov-Poisson equations description [4, 5]. The same is true when using the Fokker-Planck equation analysis. [6] Explicit "orbit function" calculations may be found, e. g, in Ref. [7] and [8]. A common situation in the formalism we are reffering to, is the truncation of the Vlasov-cumulant hierarchy and, more or less questionable cut-off techniques in treating phase-mixing terms. As a result one is led to consider with novel interest the possibilities provided by the QL approach, in order to avoid the ensemble averaging technique when dealing with problems which are rather coherent in character. Actually, the question of how exact the QL theory is, has been investigated. [1, 9, 10] Still, we feel that the discussion is, in some aspects, not conclusive. On the other hand, we appreciate the advantage, in simplicity and time saving, by avoiding -possibly- all the details of the perturbed particle trajectory. Looking for a time-evolution problem involving the one-particle, space- averaged distribution function $Sf(x,v,t) = F(v,t)$, in which $f(x,v,t) : R \otimes R \otimes R^+ \to R^+$ and $S = (1/\mathrm{L})\int_0^{\mathrm{L}} dx$, Vlasov equation is replaced by the set

$$(\partial_t + SL)F = -S(L'f') \tag{1}$$

$$(\partial_t + L')f' = -a'\partial_v F, \tag{2}$$

where quantity a denotes the electrostatic field-acceleration of electrons, $SL = Sa\partial_v$ and $L' = v\partial_x + a'\partial_v$. Solving eq. (2) for the fluctuating perturbation $f'(x,v,t)$, it is our aim to derive an explicit expression of the correlation function on the RHS in (1). Using standard notations [3], the formal solution of (2) should be expressed in terms of the operator $U_S(t',t) = \exp\left[-\int_{t'}^{t} L'(t'')dt''\right]$, but, according to the spirit of this work, we will solve (2) within the framework of the QL approximation. According to, the solution reads

$$f' = U(to,t)f'(to) - \int_{t_o}^{t} U(t',t)(a'\partial_v F)(t')dt'$$

(3) $$U(t',t'') = \exp[(t'-t'')v\partial_x].$$

Of course, informations on the following questions should possibly be extracted: i) to what extent, if any, wave-particle interactions, can be described by solution (3) and, ii) what kind of transport equations will result from the formalism.

Question i) implies that, w. r. to the conventional QL treatment, our findings should be sufficiently non-markovian in character. We know [2] how this point is associated with a correct treatment of the operator $U(t',t'')$ in eq. (3). In more recent jears, non-markovian calculations in kinetic theory of a $1D$ plasma model have been performed [11, 12], following two different approaches of perturbative type. The main aim of this lecture is to show that the two routes are fully equivalent, in the sense we will discuss herewith.

We start by following first the approach presented in paper 11. In virtue of the theorem of conservation of Vlasov solution along the characteristics

(4) $$f(x,v,t) = f(x_o,v_o,t_o),$$

where x_o and v_o (both functions of the arguments x,v,t_o,t) solve the characteristic equations $v = dx/dt$ and $a = dv/dt$. Defining the Eulerian velocity- and space- perturbations to the free-flight $v' = v_o - v$, $x' = x_o - x$, and introducing a Taylor expansion in the Fourier-series of the initial (t_o) perturbation, one obtains from eq. (4) the following expression for the Vlasov solution, correct to first order in the velocity perturbation v'

(5) $$f(t) = F(t_o) + v'\partial_v F(t_o) + \sum_{k'} f_k(t_o)\exp[ik(x - v(t-t_o))].$$

In this paper k' means $k \neq 0$. Introducing a similar Fourier-expansion for the stochastic acceleration $a'(x,t)$

$$a' = \sum_{k'} a_k(t)\exp(ikx), \tag{6}$$

the resulting correlation function is

$$S(a'\partial_v f') = \sum_{k'} a_{-k}(t)\partial_v(f_k(to)\exp[-ikv(t-to)] + v_k\partial_v F(to)), \tag{7}$$

in which v_k is the Fourier transform of quantity v'.

As pointed out in [11], perturbation v' plays a crucial role in the theory. At this stage, looking for an explicit expression for v_k, we will proceed by introducing an alternative approach of perturbative type. To this end, we go back to consider (3), noting that, with the same periodic b. c. leading to (7), Fourier-transforming expression (3) gives,

$$f_k(t) = f_k(t_o)\exp[-ikv(t-to)] - \int_{to}^{t} a_k(t')\partial_v F(t')\exp[ikv(t'-t)]dt'. \tag{8}$$

Assuming that envelope amplitudes and space-averaged distribution function are varying on the same time-scale, it is advantageous to introduce a two-time-scale analysis, by eliminating the HF Langmuir oscilations from the evolution problem. To this end we put

$$\begin{gathered} (f_k(t);a_k(t)) = (F_k(t);A_k(t))\exp[i\omega(to-t)] \\ \omega = \omega' + i\gamma \quad (\omega'_k = -\omega'_{-k}, \gamma_k = \gamma_{-k} \text{ real}), \end{gathered} \tag{9}$$

and perform the integration w. r. to time by expanding amplitudes and equilibrium distribution function according to

$$A_k(t')\partial_v F(t') = \sum_{n=o} C_n(t'-t'')^n, \ (t'' \in [to,t]). \tag{10}$$

As a result we get

$$\begin{gathered} \int_{to}^{t} a_k(t')\partial_v F(t')\exp[ikv(t'-t)]dt' = \\ I_k(t_o,t'',t)A_k(t'')\partial_v F(t''), \\ I_k(t_o,t'',t) = \sum_{n=o} I_k^{(n)}\partial_t^n, \ I_k(t_o,t''=t_o=t) = 0. \end{gathered} \tag{11}$$

The expressions of the first two terms are the following

$$I_k^{(o)} = \frac{1 - \exp[i\Delta(to - t)]}{i\Delta}, \quad (\Delta = kv - \omega), \tag{12}$$

and, respectively

$$I_k^{(1)} = \frac{1 - \exp[i\Delta(to - t)}{\Delta^2} + \frac{t - t'' + (t'' - t_o) \exp[i\Delta(to - t)]}{i\Delta}. \tag{13}$$

Collecting the expressions (8) to (11), we arrive at the following expression for the correlation function

$$\begin{aligned} S(a'\partial_v f') = \sum_{k'} \exp[2\gamma(t - to)] A_{-k}(t) \partial_v (F_k(t_o) \exp[i\Delta(to - t)] - \\ - \sum_{n=o} I_k^{(n)} \partial_t^n A_k(t) \partial_v F(t)|_{t=t''}). \end{aligned} \tag{14}$$

Concerning the convergence of the time-series and truncation criteria in expression (14), they will depend on the choice of time t'' (we are free to select for t'' any value between t_o and t), which in turn is dictated by the physical situation at hand, diffusion-($t'' \approx t$) or, evolution-like ($t'' \approx to$) process. We find interesting to examine separately the two cases $t'' = t_o$ and $t'' = t$. For $t'' = t_o$, a direct comparison of two expressions (14) and (7) is possible. It is easily seen that, at the lowest significant order ($n = 0$) in expansion (10), the expression of velocity perturbation is

$$v_k = -I_k^{(o)} A_k(t_o) \exp[i\omega(t_o - t)]. \tag{15}$$

To the same order, the diffusion coefficient D has the form

$$D = \sum_{k'} \exp[2\gamma(t - to)] I_k^{(o)} A_{-k}(t_o) A_k(t_o). \tag{16}$$

Quantities (15)and (16) correspond, in our notation, to formulas (24)and, respectively (30), of paper [11]. After insertion of our correlation function (14) into (1), we see that the resulting evolution equation reduces, to the lowest order and for the same b. c., exactly to the evolution equation as given by formula (39) in the same paper. [11] According to this argument, we will consider expression (14) as a generalized, non-markovian correlation function, useful to properly describing wave-particle interactions such as trapping-particle dynamics, as discussed in Pocobelli's paper 11.

To that extent, we also notice that expression (16) of the diffusion tensor, emerges in studying Landau-ballistic correlations. [13] Next we consider the case $t'' = t$. This case allows for a simple comparison of our findings with conventional continuity equations in transport phenomena. In the long-wave-lenght (radiation-) region of the wave-spectrum [1], we should also take into account in (14) first order ($n = 1$) terms. To see this, let us define the spectral wave-energy density

$$W = \sum_{k'} W_k(t) \exp[2\gamma(t - to)], \qquad (17)$$
$$W_k(t) = A_{-k}(t) A_k(t)/2\Omega^2,$$

Ω the Langmuir frequency of the electrons.

Recalling the energy conservation

$$\partial_t \left(W + \int_R (v^2 F/2) dv \right) = 0, \qquad (18)$$

it is not difficult to chek that the following single-mode evolution holds

$$\Gamma_1 \partial_t W_k + 2\Gamma_2 W_k = \mathrm{Re}\Gamma_o A_{-k} + \Gamma_3 \mathrm{Im} A_{-k} \partial_t A_k, \qquad (19)$$

in which

$$\Gamma_1 = 1 - i\Omega^2 \int_R v \partial_v F(t) \mathrm{Re} \partial_\omega I_k^{(o)} dv, \qquad (20)$$

$$\Gamma_2 = \gamma - \Omega^2 \int_R (\partial_v F(t) \mathrm{Re} I_k^{(o)} + i \partial_v \partial_t F(t) \mathrm{Re} \partial_\omega I_k^{(o)}) v dv), \qquad (21)$$

$$\Gamma_3 = -i\Omega^2 \int_R v \partial_v F(t) \mathrm{Im} \partial_\omega I_k^{(o)} dv \qquad (22)$$

and

$$\Gamma_o = \int_R \exp[i\Delta(to - t)] F_k(to) v dv. \qquad (23)$$

Looking at the time-behavior of the coefficients listed above, calculations show that we are faced by a situation which is typical in QL theory, i. e. the occurrence of resonant secularities and the development in velocity-space of growing oscillations. In this respect, the analytical validity of present analysis is restricted to a sufficiently short time-range.

We have some remarks. Balance eq. (19) has been written in a form which is amenable of applications in studying: 1) the inclusion of mode-coupling terms, as given by the coupled-mode theory of coherent interactions. Eq. (19) then, will contain fourth-(and higher-) order terms in the field amplitudes; 2) effects which are lost in time-asymptotic calculations, and, possibly, new effects. In particular we are referring here to the inclusion of transients associated with the initial data as given by the evolution function (23).

As a conclusion we observe that even in the asymptotic time-limit eq. (19) is consistent with the well-known result concerning the partition of the spectral power-density between slow ($v \ll \omega/k$) and resonant particles. [14]. To see this, it is sufficient to assume that the envelope amplitudes are saturated, in that limit. Using the asymptotic formula

$$\lim_{\mathrm{Re}\Delta(t-t_o)\to\infty} I_k^{(o)} = \pi\delta(\mathrm{Re}\Delta) + P\frac{\gamma}{(\mathrm{Re}\Delta)^2}, \tag{24}$$

δ and P denoting the Dirac-delta function and, respectively the Cauchy-Principal value, one gets, for resonant particles, the balance relation

$$2W = -\int_{\mathrm{Re}\Delta=0} Fv^2 dv/2 \tag{25}$$

expressing the fact that the energy gain of the spectrum is just one half the energy loss of the resonant particles. For energy conservation then, the energy gain of slow particles is given by

$$W = \int_{v\ll\omega/k} Fv^2 dv/2 \tag{26}$$

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