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STABILITY OF NON-PARALLEL FLOW IN A CHANNEL

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This is a review of several generalizations of Hiemenz's classic solution for steady two-dimensional flow of a uniform incompressible viscous fluid near a stagnation point on a bluff body. These generalizations are diverse exact solutions, steady and unsteady, two-and three-dimensional, of the Navier-Stokes equations. The solutions exhibit many types of instability and bifurcation. There are turning points, transcritical bifurcations, pitchfork bifurcations, Hopf bifurcations and Takens-Bogdanov bifurcations. The solutions also take the period-doubling and Ruelle-Takens routes to chaos.

1. Introduction.

In 1911 Hiemenz [14], investigating the flow of a uniform viscous incompressible fluid near a stagnation point on a bluff body, assumed that flow was steady and two-dimensional with Cartesian velocity components of the form

$$u = xF'(y), \qquad v = -F(y), \qquad w = 0, \tag{1.1}$$

for some function F. He substituted this similarity solution into the Navier-Stokes equations, and deduced that

$$\nu F''' + FF'' - F'^2 + \beta = 0, \tag{1.2}$$

where ν is the kinematic viscosity of the fluid and β is some constant related to the pressure gradient. He further assumed that there is a rigid plane wall (representing the surface of the body near the stagnation point at the origin) at $y=0$ and that the flow is bounded far from the body; this gives the boundary conditions that

$$F = F' = 0 \text{ at } y = 0 \qquad (1.3)$$
$$F' \text{ is bounded as } y \to \infty.$$

See, e.g., [22] p. 152 for more details, including computational results, of this classic exact solution of the Navier-Stokes equations.

We shall discuss several generalizations of this similarity solution, especially some recent ones. In this section we shall give the equations of the generalizations. In the next section we shall give various physical problems and their associated boundary conditions, and in the following section we shall review some of the recent literature on the solutions of the various boundary-value problems.

Homann [15] obtained the analogous solutions for axisymmetric steady flow at a stagnation on a flat plate by assuming that

$$u = xF'(z), \qquad v = yF'(z), \qquad w = -2F(z). \qquad (1.4)$$

Howarth [16] went further, obtaining solutions which are not necessarily two-dimensional or axisymmetric, by taking

$$u = xF'(z), \qquad v = yG'(z), \qquad w = -F(z) - G(z) \qquad (1.5)$$

where G may differ from F to give a three-dimensional flow. This leads to a coupled pair of ordinary differential equations for F and G. If G is identically zero then we regain the problem of Hiemenz, and if $G = F$ for all z we regain the problem of Homann.

Danberg and Fansler [10] considered a wall which moves and also extrudes steadily, taking essentially the similarity form,

$$u = xF'(z) + G(y), \qquad v = -F(y). \qquad (1.6)$$

Terrill and Shrestha [30] applied Hiemenz's similarity form (1.1) to steady flows with a transverse magnetic field. Rajagopal *et al.* [20]

have applied the form to steady flows of a class of non-Newtonian fluids.

In 1962 Proudman and Johnson [18] considered *unsteady* two-dimensional flow near a stagnation point, taking

$$u = xf_y(y,t), \qquad v = -f(y,t), \qquad w = 0, \tag{1.7}$$

where f may depend upon time t as will as y, and a subscript denotes partial differentiation. This and the Navier-Stokes equations lead to the equation

$$f_{yt} = \nu f_{yyy} + ff_{yy} - f_y^2 + \beta, \tag{1.8}$$

where β may now be a function of t. We differentiate this equation partially with respect to y and find

$$f_{yyt} = \nu f_{yyyy} + ff_{yyy} - f_y f_{yy}, \tag{1.9}$$

calling this the *Proudman-Johnson equation*. It can be seen that this is like a nonlinear diffusion equation; it is also a fundamental equation analogous to the Kuramoto-Sivashinsky equation. It and its steady form,

$$\nu F^{iv} + FF''' - F'F'' = 0, \tag{1.10}$$

which is the differential of equation (1.2), are the chief subject of this paper. (The Howarth problem also can be generalized for unsteady flows [27].)

We shall here ignore other similarity forms, such as that of Blasius, and their generalizations, interesting though they are.

A few elementary solutions of these equations are well known. It can be seen by inspection that $f = a(t) + b(t)y$ is a solution of the Proudman-Johnson equation for all functions a, b. Also $F = \gamma\nu + Be^{-\gamma y}$ is a solution of the steady equation (1.10) for all real B and γ. We can extend this a little, by taking $f = a(t) + b(t)e^{-\gamma y}$ for arbitrary a provided that

$$\frac{db}{dt} = \gamma(\gamma\nu - a)b.$$

Similarly we may take $f = a(t) + b(t)e^{-\gamma y} + c(t)e^{\gamma y}$. Boulanger *et al.* [2] §6 elaborate some of these solutions. Cariello and Tabor [unpublished] have shown recently that the Proudman-Johnson equation fails the Painlevé test, so we expect the equation to be nonintegrable, not to have soliton solutions etc.; they also found some special analytical solutions of it in similarity form.

2. Physical Problems and Boundary Conditions.

The need to separate U^{235} from U^{238} in the 1940s posed the problem of separating the isotopes in the form of uranium hexafluoride by gaseous diffusion. Berman [1] modelled this by flow in a two-dimensional channel with uniform suction through porous walls at $y = -h, h$. Thus he used Hiemenz's equation (1.2) and the boundary conditions that

$$(2.1) \qquad \begin{aligned} F = V, \quad F' = 0 \quad \text{at} \quad y = -h, \\ F = -V, \quad F' = 0 \quad \text{at} \quad y = h. \end{aligned}$$

In fact Berman confined attention to flows symmetric about the centre plane, $y = 0$, of the channel, i.e. to those solutions with $F = 0$, $F'' = 0$ at $y = 0$. This problem has also been used to model flow around turbine blades with suction through their surfaces. An asymmetric form of the problem with asymmetric solutions was first considered by Shrestha and Terrill [24] in 1968.

Raithby and Knudsen [19], Brady [3], and Durlofsky and Brady [12] examined, by using various numerical and asymptotic methods, the development downstream of steady perturbations of flows of the Hiemenz similarity form (1.1). They did this for symmetric flows with Berman's boundary conditions at the walls. They thereby put the Berman solutions in a more realistic context, because the development downstream gives the flow in a channel of finite length.

Crane [9] used equation (1.2) and the boundary conditions that

$$(2.2) \qquad \begin{aligned} F = 0, \quad F' = E \quad \text{at} \quad y = 0, \\ F' \quad \text{is bounded as} \quad y \to \infty. \end{aligned}$$

This represents a wall accelerating away from the origin in its own plane. The flow inside a long slender drop in an extensional flow [4] has led to use of equation (1.2) and the boundary conditions that

$$F = 0, \quad F' = -E \quad \text{at} \quad y = -h, \qquad F = 0, \quad F' = E \quad \text{at} \quad y = h; \tag{2.3}$$

again, attention was initially confined to symmetric solutions of conditions (2.3) for a channel.

P. Watson *et al.* [32] generalized these sets of boundary conditions, taking

$$f = \mp V_\pm, \quad f_y = E_\pm \quad \text{at} \quad y = \pm h, \tag{2.4}$$

where V_+, V_-, E_+, E_- are general constants. (They also may, indeed, be functions of t.) In particular, if $E_\pm = 0, V_- = 0$ then there is suction at the wall at $y = h$, and the other wall is impermeable and fixed.

Drazin *et al.* [11] took f or $f - y$ to be a periodic function of y for all t, and mentioned the solution of the Proudman-Johnson equation also over the infinite interval $-\infty < y < \infty$.

3. Some Solutions of the Boundary-value Problems.

The problem of Hiemenz has been re-examined recently in the context of the current interest in the possibility of 'blow-up', i.e. of the development of a local singularity of the Navier-Stokes equations in a finite time. For the Hiemenz problem, or rather the Proudman-Johnson problem, it is clear that blow-up may occur for the special case of an inviscid fluid ([5], [26]), but there is not yet conclusive evidence that blow-up may occur for a viscous fluid (see [8], [6]).

The Berman problem (1.2), (2.1) for symmetric steady flows in a channel was the subject of several numerical and asymptotic papers in the 1960s and 1970s, e.g. [28], [29], [19], [21]. It emerged that there is one solution for all values of the Reynolds number R, which we define by $R = V\nu/h$, together with a pair of solutions for $R > R_3$, where $R_3 \approx 12$. At $R = R_3$ there is a turning point, where the pair of solutions coalesce.

For $R < 0$ (i.e. for injection rather than suction at the walls), Skalak and Wang [25] proved the uniqueness of the steady symmetric solution of the Berman problem. Shih [23] proved that this solution exists. Cox [7] proved that there is no asymmetric steady solution for $R < 0$. Of course, we are confident that the Berman solution is not unique for $R > R_3$.

In 1988 Zaturska *et al.* [33] considered the unsteady Berman problem (1.9), (2.1), allowing asymmetric as well as symmetric solutions. They, in particular, considered the steady solutions and their stability, using a variety of asymptotic, numerical and geometric methods to give some conviction to their results, but proving nothing. They showed that the first symmetric steady solution which had been found is unstable for $R > R_1$, where $R_1 \approx 6$, and that each of the pair of symmetric steady solutions which exists for $R > R_3$ is always unstable. There is a pitchfork bifurcation at $R = R_1$, and two asymmetric steady solutions exist for $R > R_1$. This pair of solutions in turn becomes unstable for $R > R_{11}$, where $R_{11} \approx 13$. There is a Hopf bifurcation at $R = R_{11}$, and two time-periodic solutions for $R > R_{11}$. These periodic solutions become unstable quite soon as R increases further, and a complicated sequence of bifurcations occurs, in which there seem to be quasi-periodic as well as periodic solutions, with period doubling and phase locking. This sequence ends at $R = R_h$, where $R_h \approx 20$, and a symmetric pair of homoclinic connections forms in the phase space of the solutions. As R increases above R_h, chaos ensues; it seems to be low-dimensional, and is reminiscent of the chaos of the Lorenz system (with the standard values of the parameters b and σ) at its onset.

Cox [8] showed that if the Berman problem were rendered more than slightly asymmetric by taking unequal suction velocities at the walls of the channel then the chaos would no longer occur. So the discovery of chaos at as low a value of the Reynolds number as 20 may not be as easy to realize in a laboratory experiment as appears on first thoughts.

Hiemenz's equation (1.2) and boundary conditions (2.2) for accelerating walls were solved by Crane [9], who found the exact explicit solution,

$$F(y) = \gamma\nu(1 - e^{-\gamma y}), \tag{3.1}$$

where $\gamma = (E/\nu)^{\frac{1}{2}}$. McLeod and Rajagopal [17] proved that this is the unique steady solution.

Brady and Acrivos [4] treated symmetric steady flow in a channel with symmetrically accelerating walls, i.e. the problem (1.2), (2.3). It is somewhat similar to the Berman problem.

E.B.B. Watson *et al.* [31] considered unsteady flows in a channel with symmetrically accelerating walls, i.e. the problem (1.9), (2.3), allowing asymmetric as well as symmetric solutions. They found transition to chaos not only as $R = E\nu/h$ increases, but also as it decreases below zero. The sequence of bifurcations as R increases is qualitatively similar to that found by Zaturska *et al.* [33] for the suction problem, but the quantitative details are very different. The sequence of bifurcations as R decreases below zero begins similarly, but chaos ensues after a Feigenbaum sequence of period doubling.

P. Watson *et al.* [32] considered the problem with asymmetrically accelerating walls. They found not only the bifurcations mentioned in the previous paragraph, but also some Takens-Bogdanov bifurcations.

4. Conclusions.

These problems and their solutions not only provide an intellectual challenge to applied mathematicians interested in waves and stability in continuous media, but also serve more important purposes.

The boundary-value problems, although they are somewhat idealized, have the engineering applications we have mentioned. Certainly, it is not easy to make laboratory experiments to compare with the solutions. Only one experiment has been recorded [19]. The results seem inconclusive, perhaps because the experimentalists were looking for steady symmetric flows which we now know to be unstable.

The solutions are all 'exact' solutions of the Navier-Stokes equations. Such solutions are significant as prototypes to be used to understand and interpret a variety of related flows which are similar in at least part, as well as additions to the menagerie of exact solutions. The exact solutions of the Navier-Stokes equations have, of course, a long tradition of use in this way.

Lastly, the problems we have mentioned are excellent for teaching the theory of hydrodynamic stability. They are quite simple problems to illustrate the general concepts and methods of linear stability, nonlinear stability and transition to chaos. The technical difficulties of their solution are much less severe than those for the full Navier-Stokes equations. In this sense the Proudman-Johnson equation serves as a model, as well as a special case, of the Navier-Stokes equations: we use synecdoche. The basic flows we have considered vary with the governing dimensionless parameters, e.g. the Reynolds number, in a way much more typical of realistic flows than the state of rest in Rayleigh-Bénard convection, the plane parallel flows of the Orr-Sommerfeld problem, or the Couette flows of the Taylor problem. This is because the basic flows mentioned here are themselves solutions of nonlinear problems, whereas the state of rest, the plane parallel flows and the Couette flows of the classic theory of hydrodynamic stability are themselves solutions of linear problems.

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