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LARGE-TIME BEHAVIOUR OF SOME FULLY DISCRETE KINETIC MODELS IN BOUNDED DOMAINS

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We investigate the large-time behaviour of the fully discretized versions both of the three velocity Broadwell model and of the four velocity model in a strip. We analize the different behaviours on the light of some recent results by M.Slemrod [7] and C.Cercignani [3].

1. Introduction.

One of the most interesting problems of the kinetic theory of rarefied gases is represented by the analysis of the large-time behaviour of the solution to the Boltzmann equation.

The results available for the fully nonlinear spatially dependent Boltzmann equation in a bounded domain have been proposed in the last years by Arkeryd [1], DesVillettes [4] and Cercignani [2]. These results, which have been made possible by the fundamental work of DiPerna-Lions [5], confirm the old claim that the evolution of the gas in a domain bounded by solid walls kept at constant temperature should tend to a Maxwellian distribution.

The situation is quite different in the case of the discrete velocity models, where only partial answers are known.

In a recent paper, M.Slemrod [7] studied the time-asymptotic behaviour of the solution of the reduced Broadwell model in one-dimension of space, in a strip with boundary conditions of specular reflection. He found that the global solution converges in the weak* topology of a suitable Orlicz-Banach space to travelling waves without interactions. His theorem is not enough to prove decay to equilibrium, and the author himself has not clarify wheter this is a consequence of the failure of the analysis or a patology of the model.

Another interesting example has been proposed by Cercignani [3]. He discovered that the solution to the one-dimensional in space four velocity model by Gatignol, namely the plane Broadwell model with velocities turned out of 45 degrees with respect to the walls, manifests a trend towards travelling waves, and gave the explicit form of the solution.

Both the quoted results confirm that there is a considerable difference between the kinetic theory with a continuous velocity variable, and the discrete kinetic models.

To improve the knowledge about the asymptotic behaviour of the reduced Broadwell model, the authors introduced in [6] a fully discretized kinetic model. In the same paper it was shown that this model possesses a solution that approximates the solution to the true Broadwell model, but, in apparent contrast with the Slemrod's result, the solution to this new model manifests a trend towards the constant state (global equilibrium).

The aim of the present paper is the comparison between the Cercignani's result for the four velocity model, and the trend of the solution to the fully discretized version of this model.
In perfect agreement with the behaviour of the discrete model, the asymptotic state of the fully discrete model we introduced, both in the case of specular reflection and diffusion at the boundaries, is a travelling wave.

This fact push us to conjecture that the asymptotic state of the solution to the model studied by Slemrod is the constant one.

In more detail, in the next section we introduce the Broadwell model, its fully discretized version and the main results we obtained in [6]. The third section deals with the four velocity model, its fully discretized version and the trend towards travelling waves.

Finally, we propose in the fourth section some concluding remarks and numerical computations.

2. The Broadwell model.

The Broadwell model is often used, due to its simplicity, in the study of some specific problems related to the evolution of a rarefied gas. The gas particles have the same mass and speed, and can move along three perpendicular coordinate axes. For flows which depend only on the x space variable, and for which only three densities are different at the initial time, one considers the simpler form

$$\begin{aligned} &\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=w^2-uv \\ &\frac{\partial v}{\partial t}-\frac{\partial v}{\partial x}=w^2-uv \\ &\frac{\partial w}{\partial t}=-\frac{1}{2}\left(w^2-uv\right) \end{aligned} \tag{2.1}$$

where $(u=u(x,t), v=v(x,t), w=w(x,t))$, $x\in\mathbf{R}$, $t\in\mathbf{R}^+$.

The discretization we proposed in [6] consists of the lattice model obtained by considering the gas concentrated at the points

$$x_1=\frac{1}{2(2N)}=\frac{1}{2}\delta \quad , \quad x_i=x_{i-1}+\delta(i=2,\dots,2N) .$$

The iterative scheme corresponding to system (2.1) is the following

$$\begin{aligned} &u_i^{n+1}=u_{i-1}^n+\delta\left[(w_i^n)^2-u_{i-1}^n v_{i+1}^n\right] \\ &v_i^{n+1}=v_{i+1}^n+\delta\left[(w_i^n)^2-u_{i-1}^n v_{i+1}^n\right] \\ &w_i^{n+1}=w_i^n-\frac{1}{2}\delta\left[(w_i^n)^2-u_{i-1}^n v_{i+1}^n\right] \end{aligned} \tag{2.2}$$

$n\geq 0$, and $i=1,2,\dots,2N-1$, with the boundary conditions

$$\begin{aligned} &u_1^{n+1}=v_1^n+\delta\left[(w_1^n)^2-v_1^n v_2^n\right] \\ &v_1^{n+1}=v_2^n+\delta\left[(w_1^n)^2-v_1^n v_2^n\right] \\ &w_1^{n+1}=w_1^n-\frac{1}{2}\delta\left[(w_1^n)^2-v_1^n v_2^n\right] \end{aligned} \tag{2.3}$$

for $i = 1$, and

$$
\begin{aligned}
u_{2N}^{n+1} &= u_{2N-1}^{n} + \delta \left[(w_{2N}^{n})^2 - u_{2N-1}^{n} u_{2N}^{n} \right] \\
v_{2N}^{n+1} &= u_{2N}^{n} + \delta \left[(w_{2N}^{n})^2 - u_{2N-1}^{n} u_{2N}^{n} \right] \\
w_{2N}^{n+1} &= w_{2N}^{n} - \frac{1}{2}\delta \left[(w_{2N}^{n})^2 - u_{2N-1}^{n} u_{2N}^{n} \right]
\end{aligned}
\tag{2.4}
$$

for $i = 2N$.

Let $\{u_i^o, v_i^0, w_i^0\}$, $i = 1, \ldots, 2N$ be nonnegative initial data for the evolution system (2.2), with boundary conditions (2.3) and (2.4), and let $\rho^0 = \delta \sum_{i=1}^{2N} (u_i^0 + v_i^0 + 4w_i^0)$ be the initial total mass. In [6] we proved

THEOREM 1. *We shall assume that $\rho^0 \leq 1$. Then, for any spatial partition (any N) and for all time iterations ($n \geq 1$), the system (2.2), with boundary conditions (2.3) and (2.4)*

a) Maps positive data into positive data;

b) preserves the total mass;

c) the sequences $\{u_i^n, v_i^n, w_i^n\}$, $i = 1, \ldots, 2N$ converge to the equilibrium asymptotic state given by $\left\{ \frac{1}{12N}\rho^0, \frac{1}{12N}\rho^0, \frac{1}{12N}\rho^0 \right\}$.

The above theorem assures that the fully discrete model manifests a trend towards the global equilibrium, that in this case is given by the state in which all the components are equal. On the other hand, with the solution we found in theorem 1, we can approach the solution to the discrete Broadwell model.

Given the sequences $\{u_i^n, v_i^n, w_i^n\}$, $i = 1, \ldots, 2N$, let us define, for $\delta = \frac{1}{2N}$

$$u^\delta(x,t) = u_i^n$$

$$v^\delta(x,t) = v_i^n$$

$$w^\delta(x,t) = w_i^n \qquad i = 1, \ldots, 2N$$

if $x \in \left[\delta\left(i - \frac{1}{2}\right), \delta\left(i + \frac{1}{2}\right) \right)$, and $t \in [n\delta, (n+1)\delta)$. Then [6]

THEOREM 2. *Let $T > 0$. Then, for any $\epsilon > 0$ we can choose an $N^*(T, \epsilon)$ such that, by dividing the interval $(0, 1)$ into $2N$ parts, with $N \geq N^*$, we have*

$$\sup_{x \in (0,1)} \{|u^\delta(x,t) - u(x,t)| + |u^\delta(x,t) - u(x,t)| + |u^\delta(x,t) - u(x,t)\} \leq \epsilon$$

for every $t \leq T$.

It is clear that from the previous thorem we can not conclude that the Broadwell model manifests a trend towards the constant state (global equilibrium). Numerical computations [6] on the other hand show, for many classes of initial values, that the time employed by the fully discrete system to approach the constant state by a fixed error, is independent on the number $2N$.

To understand more deeply the relationships between the discrete kinetic models and their fully discrete versions, we shall consider in the next section the plane four velocity model, which has been recently studied in a bounded domain by Cercignani [3].

3. The plane four velocity model.

The plane four velocity model is one of the simplest discrete velocity models. The evolution equations for the densities corresponding to the velocities $v_1 = v_2 = 1$ and $v_3 = v_4 = -1$ read

$$\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial x} = (-1)^{i+1} (f_2 f_4 - f_1 f_3) \qquad i = 1, 2, 3, 4 \tag{3.1}$$

Here we consider the initial-boundary value problem in the strip $(0, 1)$ with specular reflection at the boundaries.

Cercignani [3] proved that the asymptotic state of the solution is given by travelling waves, and found the explicit form of it. His reasoning can be easily adapted to the fully discrete version of the four velocity model, so obtaining the same result on the asymptotic state.

First, let us define the model. We consider, as in the previous section, the gas confined to the strip $0 < x < 1$, concentrated on the points

$$x_1 = \frac{1}{2(2N)} = \frac{1}{2}\delta \quad , \quad x_i = x_{i-1} + \delta (i = 2, \ldots, 2N)$$

The iterative scheme corresponding to system (3.1) is the following

$$
\begin{aligned}
f_i^{n+1} &= f_{i-1}^n + \delta Q_i^n \\
g_i^{n+1} &= g_{i-1}^n - \delta Q_i^n \\
h_i^{n+1} &= h_{i+1}^n + \delta Q_i^n \\
k_i^{n+1} &= k_{i+1}^n - \delta Q_i^n \quad i = 2, 3, \ldots, 2N-1
\end{aligned} \tag{3.2}
$$

where $Q_i^n = g_{i-1}^n k_{i+1}^n - f_{i-1}^n h_{i+1}^n$.
On the boundary, as $i = 1$

$$
\begin{aligned}
f_1^{n+1} &= k_1^n + \delta Q_1^n \\
g_1^{n+1} &= h_1^n - \delta Q_1^n \\
h_1^{n+1} &= h_2^n + \delta Q_1^n \\
k_1^{n+1} &= k_2^n - \delta Q_1^n
\end{aligned} \tag{3.3}
$$

Here, $Q_1^n = h_1^n k_2^n - k_1^n h_2^n$. On the other boundary, namely for $i = 2N$

$$
\begin{aligned}
f_{2N}^{n+1} &= f_{2N-1}^n + \delta Q_{2N}^n \\
g_{2N}^{n+1} &= g_{2N-1}^n - \delta Q_{2N}^n \\
h_{2N}^{n+1} &= g_{2N}^n + \delta Q_{2N}^n \\
k_{2N}^{n+1} &= f_{2N}^n - \delta Q_{2N}^n
\end{aligned} \tag{3.4}
$$

where $Q_{2N}^n = g_{2N-1}^n f_{2N}^n - f_{2N-1}^n g_{2N}^n$.

The symmetries of this model permit to identify straithforwardly the asymptotic state.

Let us introduce the following notation

$$A_i^n = f_i^n + g_i^n \ ; \qquad B_i^n = h_i^n + k_i^n$$

$$F_i^n = f_i^n; \qquad K_i^n = k_i^n$$

Then, system (3.2), with boundary conditions (3.3) and (3.4) is rewritten as follows

$$\begin{aligned}
& A_i^{n+1} = A_{i-1}^n \\
& B_i^{n+1} = B_{i+1}^n \\
(3.5) \qquad & F_i^{n+1} = F_{i-1}^n + \delta\left[A_{i-1}^n K_{i+1}^n - B_{i+1}^n F_{i-1}^n\right] \\
& A_i^{n+1} = A_{i-1}^n - \delta\left[A_{i-1}^n K_{i+1}^n - B_{i+1}^n F_{i-1}^n\right]
\end{aligned}$$

with the following boundary conditions

$$\begin{aligned}
& A_1^{n+1} = B_1^n \ ; \qquad B_1^{n+1} = B_2^n \\
& B_{2N}^{n+1} = A_{2N}^n; \qquad A_{2N}^{n+1} = A_{2N-1}^n \\
& F_1^{n+1} = B_1^n - K_2^n + \delta\left[B_1^n K_2^n - B_2^n K_1^n\right] \\
(3.6) \qquad & K_1^{n+1} = K_2^n - \delta\left[B_1^n K_2^n - B_2^n K_1^n\right] \\
& F_{2N}^{n+1} = F_{2N-1}^n + \delta\left[A_{2N}^n F_{2N-1}^n - A_{2N-1}^n F_{2N}^n\right] \\
& K_{2N}^{n+1} = A_{2N}^n - F_{2N}^n - \delta\left[A_{2N}^n F_{2N-1}^n - A_{2N-1}^n F_{2N}^n\right]
\end{aligned}$$

When the total initial mass is less than one, we can repeat the analysis of [6] for the Broadwell model, concluding with an analogous of theorem 1, parts (a) and (b). The situation is now different for the asymptotic state. It is clear, from the first two relations in (3.5) and (3.6), that A_i^n and B_i^n, $i = 1, \dots, 2N$ and $n \geq 0$ are completely known in term of the initial values, and represent travelling waves. The same conclusion can be drawn for the asymptotic states of F_i^n and K_i^n, due to the linearity of the relations (3.5 c,d) and (3.6 c,d,e,f). This result is in perfect agreement with the analogous one proved by Cercignani.

In addition to the case of specular reflection at the boundaries, with the fully discrete model we can treat the case of pure diffusion, with the same conclusion about the trend towards travelling waves. In the last section we will show numerically that this last case leads to a regularity in the final form of the state of the system.

4. Final remarks.

We studied in this paper the asymptotic states of some fully discrete model in a bounded domain, looking for connections with the asymptotic states of the corresponding discrete velocity models. The behaviour of the fully discrete Broadwell model towards a constant state is presented in figures 1 a,b,c.

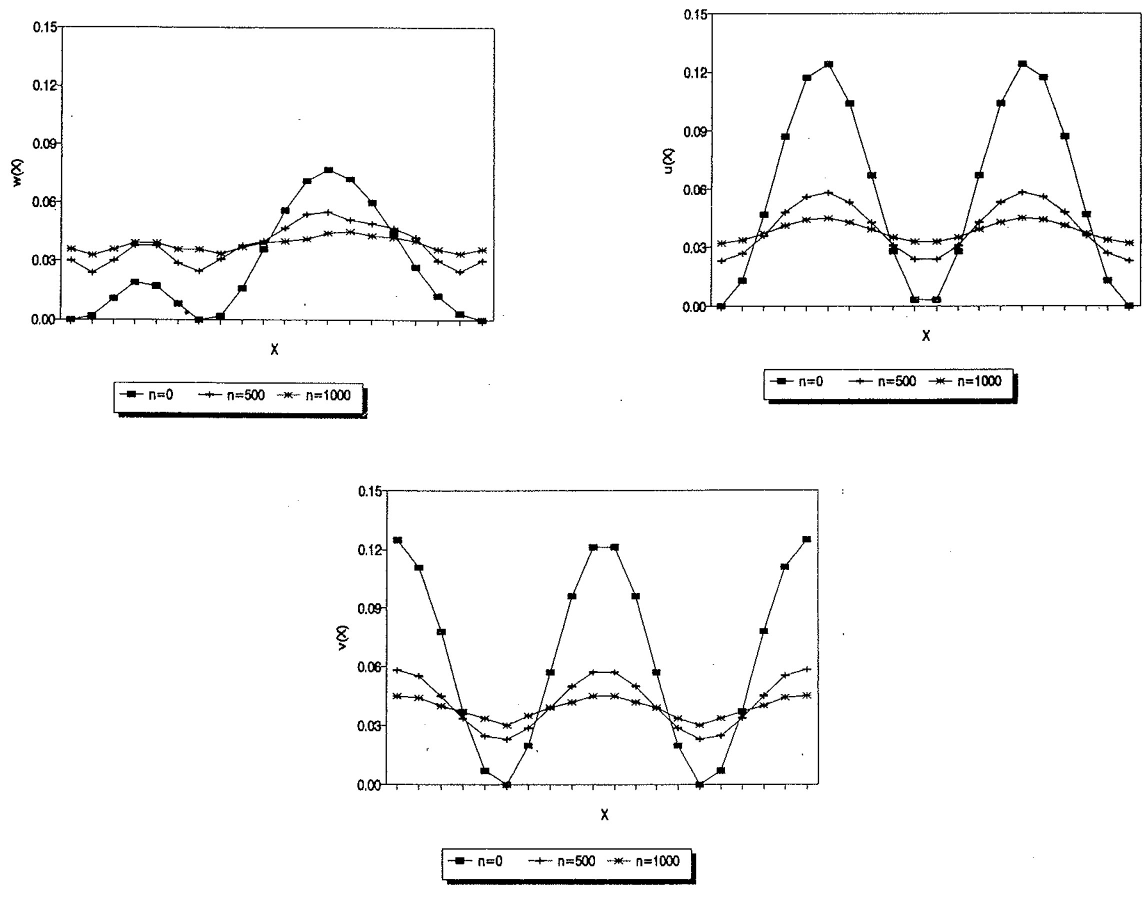

Fig. 1 - a, b, c,

When the four velocity model is considered, the behaviour both of the discrete and of the fully discrete model is identical. With regards to the this last model, we give here some picture obtained by numerical computations that refer both to the case of specular reflection at the boundaries, and to the case of pure diffusion. This second case, which is not so obvious to treat for the discrete model corresponds to system (3.2), where a modification of (3.3) and (3.4) occurs. The boundary conditions at $i = 1$ and $i = 2N$ are substituted by the following

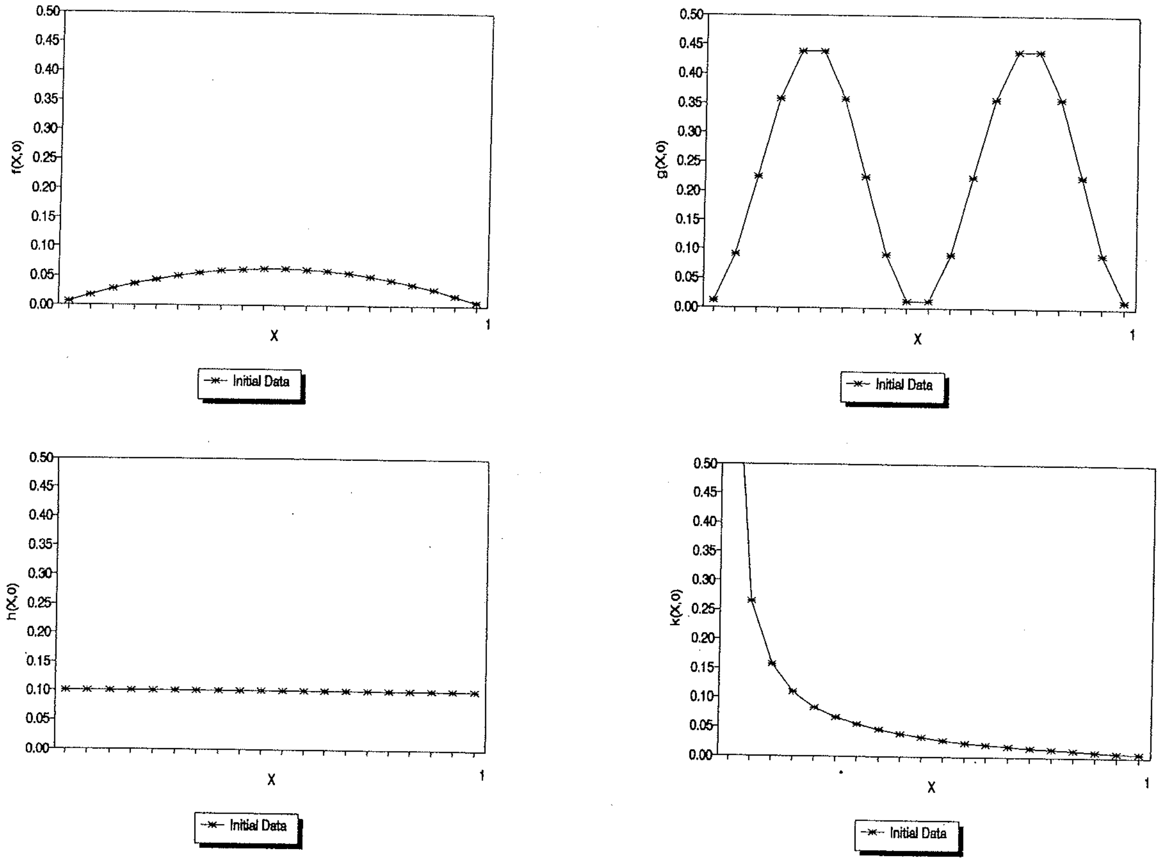

Fig. 2 - a, b, c, d

$$
\begin{aligned}
f_1^{n+1} &= \frac{1}{2}h_1^n + \frac{1}{2}k_1^n + \delta P_1^n \\
g_1^{n+1} &= \frac{1}{2}h_1^n + \frac{1}{2}k_1^n - \delta P_1^n \\
h_1^{n+1} &= h_2^n + \delta P_1^n \\
k_1^{n+1} &= k_2^n - \delta P_1^n
\end{aligned}
\tag{4.1}
$$

where $P_1^n = \left[\frac{1}{2}h_1^n + \frac{1}{2}k_1^n\right][k_2^n - h_2^n]$, and

$$
(4.2)\qquad
\begin{aligned}
f_{2N}^{n+1} &= f_{2N-1}^n + \delta P_{2N}^n \\
g_{2N}^{n+1} &= g_{2N-1}^n - \delta P_{2N}^n \\
h_{2N}^{n+1} &= \frac{1}{2}f_{2N}^n + \frac{1}{2}g2N^n + \delta P_{2N}^n \\
k_{2N}^{n+1} &= \frac{1}{2}f_{2N}^n + \frac{1}{2}g2N^n - \delta P_{2N}^n
\end{aligned}
$$

where $P_{2N}^n = \left[\frac{1}{2}f_{2N}^n + \frac{1}{2}g_{2N}^n\right]\left[f_{2N-1}^n - g_{2N-1}^n\right]$.

Figures 2 a,b,c,d give the initial densities, whenever figures 3 a,b,c,d present the resulting travelling waves.

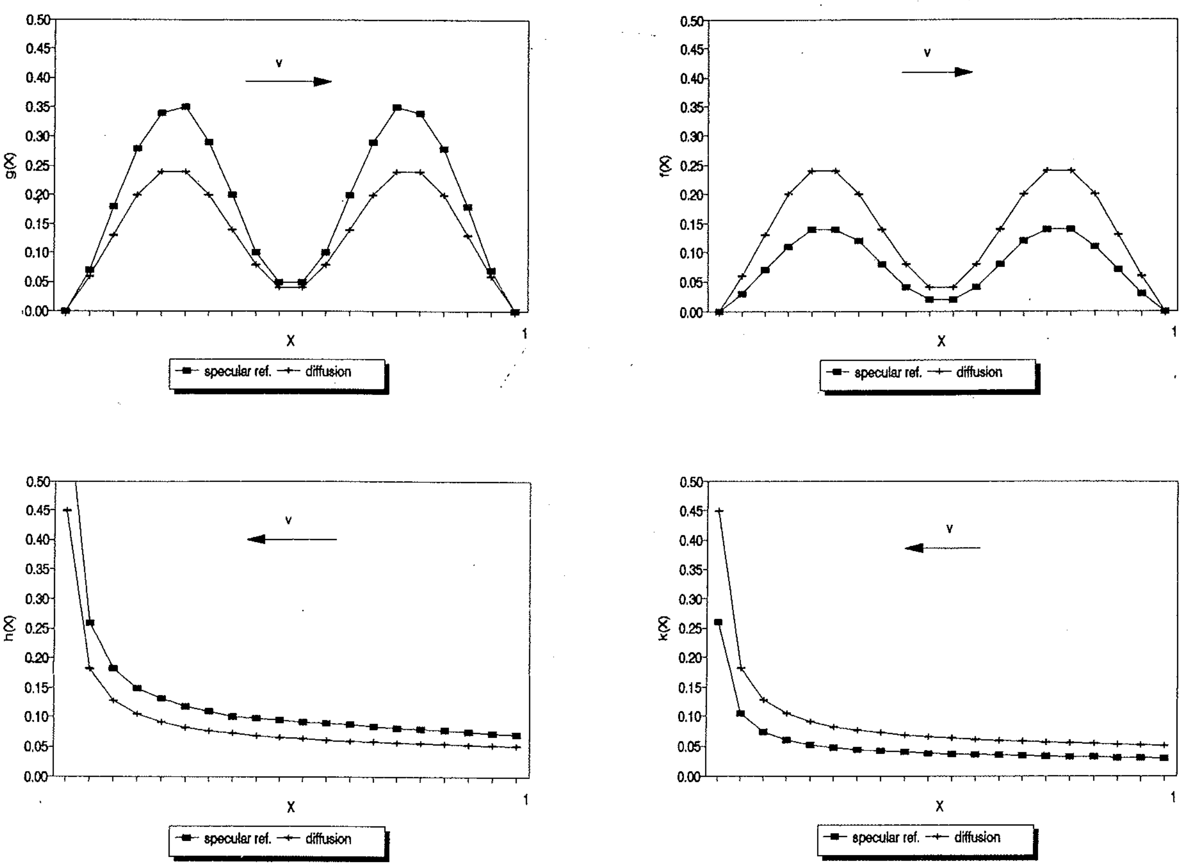

Fig. 3 - a, b, c, d

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