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ONE PARAMETER DUAL LORENTZIAN SPHERICAL MOTIONS AND RULED SURFACES

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In this work, we first introduced one parameter dual Lorentzian spherical motions in three dimensional dual Lorentz space D_1^3 and spacelike and timelike ruled surfaces in three dimensional Lorentz space IR_1^3 corresponding to dual curves on dual Lorentz unit sphere $\tilde{S}_1^2$. After that we have given the relations on the velocities and instantaneous rotation axis for one parameter Lorentzian spherical motions in dual Lorentz space D_1^3, with some examples on these timelike and spacelike ruled surfaces. Finally we have obtained the theorem related to the acceleration, acceleration centres and acceleration axis for these one parameter dual Lorentzian spherical motions.

1. Introduction

Dual numbers were introduced in the 19^{th} century by Clifford [1] and quickly found application in description of movements of rigid bodies in three dimensions [2, 3] and in description of geometrical objects also in three dimensional space [4]. The relevant formalism was developed. It has contemporary application within the curve design methods in computer-aided geometric design and computer modeling of rigid bodies, linkages, robots, mechanism design, modeling human body dynamics etc. [5–8]. For several decades there were attempts to apply dual numbers to rigid body dynamics.

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E. Study devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the vectors of dual unit sphere S^2 and the directed lines of space of lines IR^3 [3]. The existence of the dual numbers has been noticed in some papers concerning supermathematics e.g. [9, 10]. It is worth noting that Gromov, in a series of papers, applied dual numbers in several ways: in contractions and analytical continuations of classical groups [11], then in quantum group formalism [12, 13]. The most interesting use of dual numbers in field theory can be found in a series of papers by Wald et al. [14–16].

Considering one and two parameters spherical motions in Euclidean space, Muller has given the relations for absolute, sliding, relative velocities and pole curves of these motions. If we take Minkowski 3-space IR_1^3 instead of E^3 the E.Study mapping can be stated as follows: The dual timelike and spacelike unit vectors of dual hyperbolic and Lorentzian unit spheres $\tilde{H}_0^2$ and $\tilde{S}_1^2$ at the dual Lorentzian space D_1^3 are in one-to-one correspondence with the directed timelike and spacelike lines of the space of Lorentzian lines IR_1^3, respectively, [17].

This paper is organized as follows: In the first part, basic concepts have been given in dual Lorentzian space D_1^3. In the second part, one parameter dual Lorentzian spherical motions and timelike and spacelike ruled surfaces are defined. In doing so, the orthonormal dual frame of $\left\{\tilde{M};\overrightarrow{\tilde{E}_1},\overrightarrow{\tilde{E}_2},\overrightarrow{\tilde{E}_3}\right\}$ and of $\left\{\tilde{M};\overrightarrow{\tilde{E}'_1},\overrightarrow{\tilde{E}'_2},\overrightarrow{\tilde{E}'_3}\right\}$ are taken representing moving dual Lorentzian sphere $\tilde{S}_1^2$ and fixed dual Lorentzian sphere $\tilde{S}_1^{2'}$, respectively. Furthermore, the relations between absolute, relative and sliding velocities of one parameter dual Lorentzian spherical motions have been obtained. In addition to that, the relations between the velocities and instantaneous rotation axis of one parameter dual Lorentzian spherical motions have been given together with the examples on timelike and spacelike ruled surfaces. Finally, we have obtained the relations for the acceleration, acceleration centres and axis for one parameter dual Lorentzian spherical motions.

2. Basic Concepts

If a and a^* are real numbers and $\mathscr{E}^2 = 0$, the combination $\tilde{a} = a + \mathscr{E}a^*$ is called a dual number, where $\mathscr{E}$ is dual unit. The set of all dual numbers forms a commutative ring over the real number field and denoted by D. Then the set

$$D^3 = \{\tilde{a} = (A_1, A_2, A_3) | A_i \in D, 1 \leq i \leq 3\}$$

is a module over the ring D which is called a D-module or dual space and denoted by D^3. The elements of D^3 are called dual vectors. Thus a dual vector $\overrightarrow{\tilde{a}}$

can be written

$$\overrightarrow{\tilde{a}} = \vec{a} + \mathscr{E}\vec{a}^*$$

where $\vec{a}$ and $\vec{a}^*$ are real vectors in IR^3.

The Lorentzian inner product of dual vectors $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ in D^3 is defined by

$$\left\langle \overrightarrow{\tilde{a}}, \overrightarrow{\tilde{b}} \right\rangle = \left\langle \vec{a}, \vec{b} \right\rangle + \mathscr{E}\left(\left\langle \vec{a}, \vec{b}^* \right\rangle + \left\langle \vec{a}^*, \vec{b} \right\rangle \right)$$

with the Lorentzian inner product $\vec{a}$ and $\vec{b}$

$$\left\langle \vec{a}, \vec{b} \right\rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3$$

where $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$. Therefore, D^3 with the Lorentzian inner product $\left\langle \overrightarrow{\tilde{a}}, \overrightarrow{\tilde{b}} \right\rangle$ is called 3-dimensional dual Lorentzian space and denoted by of D_1^3 [17].

A dual vector $\overrightarrow{\tilde{a}}$ is said to be timelike if $\vec{a}$ is timelike ($\langle \vec{a}, \vec{a} \rangle < 0$), spacelike if $\vec{a}$ is spacelike ($\langle \vec{a}, \vec{a} \rangle > 0$ or $\vec{a} = 0$) and lightlike (or null) if $\vec{a}$ is lightlike ($\langle \vec{a}, \vec{a} \rangle = 0, \vec{a} \neq 0$), where $\langle , \rangle$ is a Lorentzian inner product with signature $(-,+,+)$. The set of all dual vectors such that $\left\langle \overrightarrow{\tilde{a}}, \overrightarrow{\tilde{a}} \right\rangle = 0$ is called the dual lightlike (or null) cone and is denoted by Γ. The norm of a dual vectors $\overrightarrow{\tilde{a}}$ is

$$\left\| \overrightarrow{\tilde{a}} \right\| = \|\vec{a}\| + \mathscr{E}\frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|} = a + \mathscr{E}a^*, \quad \vec{a} \neq 0$$

where $a = \|\vec{a}\|$ and $a^* = \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}$.

Taking $\vec{a} \neq 0$, for the dual vector $\overrightarrow{\tilde{a}} = \vec{a} + \mathscr{E}\vec{a}^*$, $\overrightarrow{\tilde{a}_0} = \frac{\overrightarrow{\tilde{a}}}{\|\overrightarrow{\tilde{a}}\|}$ is a unit dual vector and can be written as follows:

$$\overrightarrow{\tilde{a}_0} = \frac{\vec{a}}{\|\vec{a}\|} + \mathscr{E}\frac{\vec{a}^* - k\vec{a}}{\|\vec{a}\|} = \vec{a}_0 + \mathscr{E}\vec{a}_0^*.$$

Here $\vec{a}_0 = \frac{\vec{a}}{\|\vec{a}\|}$, $\vec{a}_0^* = \frac{\vec{a}^* - k\vec{a}}{\|\vec{a}\|}$ and $k = \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|^2}$. We called unit dual vector $\overrightarrow{\tilde{a}_0}$ and k as the axis of dual vector and pitch, respectively. If $\overrightarrow{\tilde{a}} = \vec{a} + \mathscr{E}\vec{a}^*$ is unit dual vector, then

$$\|\vec{a}\| = 1 \text{ and } \langle \vec{a}, \vec{a}^* \rangle = 0.$$

Let $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ be two future-pointing (resp. past-pointing) unit dual timelike vectors in D_1^3. Then we have [17]

$$\left\langle \overrightarrow{\tilde{a}}, \overrightarrow{\tilde{b}} \right\rangle = -\cosh\tilde{\Theta}.$$

We also consider the time orientation as follows: A dual timelike vector $\overrightarrow{\tilde{a}}$ is futurepointing (resp. past-pointing) if and only if $\vec{a}$ is future-pointing (resp. past-pointing).

The dual hyperbolic and dual Lorentzian unit spheres are

$$\tilde{H}_0^2 = \left\{ \overrightarrow{\tilde{a}} = \vec{a} + \mathscr{E}\vec{a}^* \in D_1^3 \middle| \left\| \overrightarrow{\tilde{a}} \right\| = 1,\ \vec{a}, \vec{a}^* \in IR_1^3 \text{ and } \vec{a} \text{ timelike} \right\}$$

and

$$\tilde{S}_1^2 = \left\{ \overrightarrow{\tilde{a}} = \vec{a} + \mathscr{E}\vec{a}^* \in D_1^3 \middle| \left\| \overrightarrow{\tilde{a}} \right\| = 1,\ \vec{a}, \vec{a}^* \in IR_1^3 \text{ and } \vec{a} \text{ spacelike} \right\}$$

respectively.

The dual Lorentzian cross-product of $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ is defined as

$$\overrightarrow{\tilde{a}} \wedge \overrightarrow{\tilde{b}} = \vec{a} \wedge \vec{b} + \mathscr{E}\left(\vec{a} \wedge \vec{b}^* + \vec{a}^* \wedge \vec{b} \right)$$

with the Lorentzian cross-product $\vec{a}$ and $\vec{b}$

$$\vec{a} \wedge \vec{b} = (a_3b_2 - a_2b_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

where $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, [17]. Let $\overrightarrow{\tilde{u}}, \overrightarrow{\tilde{v}}, \overrightarrow{\tilde{w}}, \overrightarrow{\tilde{t}} \in D_1^3$. Then we have

i) $\left\langle \overrightarrow{\tilde{u}} \wedge \overrightarrow{\tilde{v}}, \overrightarrow{\tilde{w}} \right\rangle = \det\left(\overrightarrow{\tilde{u}}, \overrightarrow{\tilde{v}}, \overrightarrow{\tilde{w}} \right)$

ii) $\left(\overrightarrow{\tilde{u}} \wedge \overrightarrow{\tilde{v}} \right) \wedge \overrightarrow{\tilde{w}} = -\left\langle \overrightarrow{\tilde{u}}, \overrightarrow{\tilde{w}} \right\rangle \overrightarrow{\tilde{v}} + \left\langle \overrightarrow{\tilde{v}}, \overrightarrow{\tilde{w}} \right\rangle \overrightarrow{\tilde{u}}$

iii) $\left(\overrightarrow{\tilde{u}} \wedge \overrightarrow{\tilde{v}} \right) \wedge \overrightarrow{\tilde{v}} = \overrightarrow{\tilde{0}}$

iv)

$$\left\langle \overrightarrow{\tilde{u}} \wedge \overrightarrow{\tilde{v}}, \overrightarrow{\tilde{w}} \wedge \overrightarrow{\tilde{t}} \right\rangle = -\left\langle \overrightarrow{\tilde{u}}, \overrightarrow{\tilde{w}} \right\rangle \left\langle \overrightarrow{\tilde{v}}, \overrightarrow{\tilde{t}} \right\rangle + \left\langle \overrightarrow{\tilde{u}}, \overrightarrow{\tilde{t}} \right\rangle \left\langle \overrightarrow{\tilde{v}}, \overrightarrow{\tilde{w}} \right\rangle \tag{1}$$

v) $\left(\overrightarrow{\tilde{u}} \wedge \overrightarrow{\tilde{v}} \right) \wedge \overrightarrow{\tilde{w}} + \left(\overrightarrow{\tilde{v}} \wedge \overrightarrow{\tilde{w}} \right) \wedge \overrightarrow{\tilde{u}} + \left(\overrightarrow{\tilde{w}} \wedge \overrightarrow{\tilde{u}} \right) \wedge \overrightarrow{\tilde{v}} = 0.$

Let f be a differentiable dual function. Thus, Taylor expansion of dual function f is

$$f(x + \mathscr{E}x^*) = f(x) + \mathscr{E}x^* f'(x)$$

where $f'(x)$ is the derivation of f, [18, 19].

3. Spherical Motions in Dual Lorentz Space D_1^3, Timelike and Spacelike Ruled Surfaces

The two coordinate systems $\{O;\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ and $\{O';\vec{e}'_1,\vec{e}'_2,\vec{e}'_3\}$ are right-handed orthonormal coordinate systems which represent the moving space L and the fixed space L' in IR_1^3, respectively. According to the E. Study theorem, the points of unit dual Lorentzian sphere $\tilde{K}$ and $\tilde{K}'$ with the same centres $\tilde{M}$ correspond one to one to $\vec{e}_i$ and $\vec{e}'_i$ $(1 \le i \le 3)$ axis in dual Lorentzian space D_1^3, respectively. Therefore, L/L' Lorentzian motions can be considered as dual Lorentzian spherical motions $\tilde{K}/\tilde{K}'$.

Let $\tilde{K}$ and $\tilde{K}'$ be the common centre $\tilde{M}$ of unit Lorentzian dual spheres and $\left\{\tilde{M};\overrightarrow{\tilde{E}_1},\overrightarrow{\tilde{E}_2},\overrightarrow{\tilde{E}_3}\right\}$, $\left\{\tilde{M};\overrightarrow{\tilde{E}'_1},\overrightarrow{\tilde{E}'_2},\overrightarrow{\tilde{E}'_3}\right\}$ be the orthonormal coordinate systems, respectively, which are rigidly linked to these spheres. Here

$$\overrightarrow{\tilde{E}_i} = \vec{e}_i + \mathscr{E}\vec{e}_i^{\,*}, \ \ \overrightarrow{\tilde{E}'_i} = \vec{e}'_i + \mathscr{E}\overrightarrow{e_i'^{*}}, \ \ 1 \le i \le 3$$

and

$$\vec{e}_i^{\,*} = \overrightarrow{\tilde{M}O} \wedge \vec{e}_i, \ \ \overrightarrow{e_i'^{*}} = \overrightarrow{\tilde{M}O'} \wedge \vec{e}'_i, \ \ 1 \le i \le 3$$

since each of these systems are oriented to the same direction (see Figure 1).

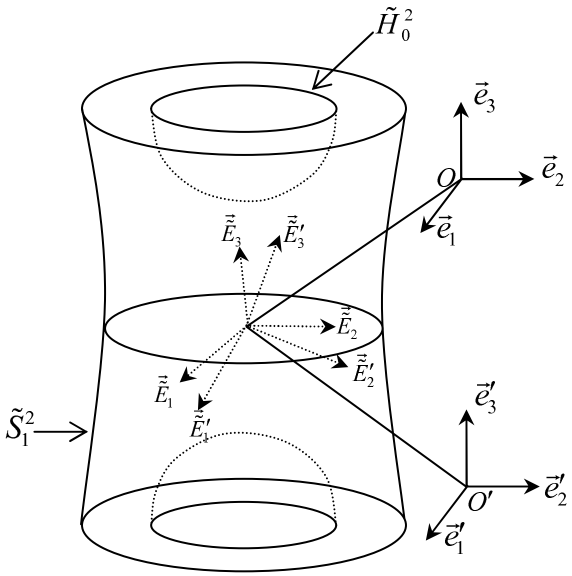

Figure 1: Dual Lorentzian orthonormal Systems

Therefore, one-parameter motion for a rigid body in three dimensional dual

Lorentzian space D_1^3 is given by the following transformation

$$\begin{bmatrix} \tilde{X}' \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{U}' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ 1 \end{bmatrix} \tag{2}$$

where $\tilde{A}$ is an orthonormal matrix in the sense of dual Lorentzian in type 3x3, and $\tilde{X}$, $\tilde{X}'$, $\tilde{U}'$ are dual matrices in type 3x1. Furthermore, the dual matrices $\tilde{A}$ and $\tilde{U}'$ are the functions which are differentiable enough with respect to a dual parameter $\tilde{t} = t + \mathscr{E}t^*$. If otherwise mentioned we take $t^* = 0$. Therefore, we can give one-parameter dual Lorentzian spherical motions in dual Lorentzian space D_1^3. The vectors $\overrightarrow{\tilde{X}}$ and $\overrightarrow{\tilde{X}'}$ represent the base vectors of the same dual point $\tilde{X}$ with respect to dual orthonormal coordinate system in the moving $\tilde{K}$ and dual Lorentzian sphere $\tilde{K}'$. At the initial time $t = t_0$ we suppose that the coordinate systems at $\tilde{K}$ and $\tilde{K}'$ are coincident.

Now, we consider the interconnections for these two coordinate systems. From equation (2) we write

$$\tilde{X}' = \tilde{A}\tilde{X} + \tilde{U}' \tag{3}$$

using $\tilde{U}' = -\tilde{A}\tilde{U}$ and $\tilde{U} = -\tilde{A}^{-1}\tilde{U}'$ equations we reach

$$\tilde{X} = \tilde{A}^{-1}\tilde{X}' + \tilde{U}. \tag{4}$$

Thus equations (3) and (4) express the coordinate transformations between the two coordinate systems.

A dual point $\tilde{X}$ in a (moving) dual Lorentzian sphere $\tilde{K}$ under one-parameter dual Lorentzian spherical motion draws dual timelike curve $\left(\tilde{X}\right)$ with spacelike generator dependent on the parameter $t \in IR$ or dual spacelike curve with spacelike generator on the (fixed) Lorentzian sphere $\tilde{K}'$. Timelike or spacelike ruled surface correspond respectively to these timelike or spacelike spherical curves which are differentiable with respect to parameter t. Here dual timelike or spacelike curves $\left(\tilde{X}\right)$ are called dual spherical picture of ruled surface (see figure II).

From the E. Study theory, at every time t spacelike unit dual vector $\overrightarrow{\tilde{X}} = \vec{x} + \mathscr{E}\vec{x}^*$ produces a directional line for which the main lines are spacelike unit vector $\vec{x} = \vec{x}(t)$ and crossing the point $p(t)$. Here $p(t)$ is a point satisfying the equation

$$\vec{x}^*(t) = \vec{p}(t) \wedge \vec{x}(t) \tag{5}$$

in IR_1^3. Thus the equation for the timelike or spacelike ruled surface (see figure III) is

$$\vec{y}(t,u) = p(t) + u\vec{x}(t). \tag{6}$$

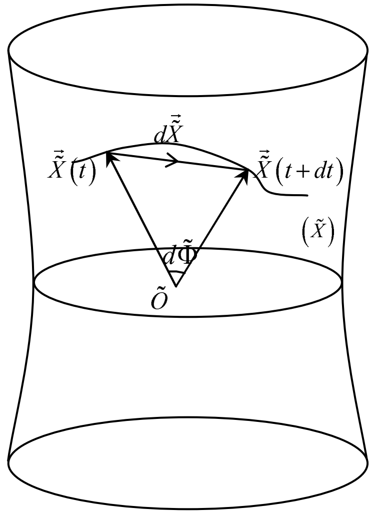

Figure 2: Dual Timelike or Spacelike Curve

Considering equations (1) and (5) we see that

$$\vec{x}^* \wedge \vec{x} = \vec{p}(t) - \langle \vec{x}, \vec{p} \rangle \vec{x}.$$

If we choose $v = u + \langle \vec{x}, \vec{p} \rangle$, then the equation for timelike or spacelike ruled surface given by equation (6) becomes

$$\vec{y}(t,v) = \vec{x}^*(t) \wedge \vec{x}(t) + v\vec{x}(t).$$

The dual arc element

$$d\tilde{\Phi} = d\varphi + \mathscr{E} d\varphi^*$$

for dual timelike or spacelike curves $(\tilde{X})$ we write

$$d\tilde{\Phi}^2 = \left\langle d\overrightarrow{\tilde{X}}, d\overrightarrow{\tilde{X}} \right\rangle$$

or

$$d\varphi^2 + 2\mathscr{E} d\varphi d\varphi^* = \langle d\vec{x}, d\vec{x} \rangle + 2\mathscr{E} \langle d\vec{x}, d\vec{x}^* \rangle.$$

The magnitude $d\tilde{\Phi}$ is the angle between neighboring spacelike unit dual vectors $\overrightarrow{\tilde{X}}(t)$ and $\overrightarrow{\tilde{X}}(t+dt)$. Real part $d\varphi$ and dual part $d\varphi^*$ of $d\tilde{\Phi}$ correspond to an angle between neighboring two main lines and the shortest distance between these lines, respectively. Therefore, since the dual term

$$\left\langle d\overrightarrow{\tilde{X}}, d\overrightarrow{\tilde{X}} \right\rangle = \langle d\vec{x}, d\vec{x} \rangle + 2\mathscr{E} \langle d\vec{x}, d\vec{x}^* \rangle$$

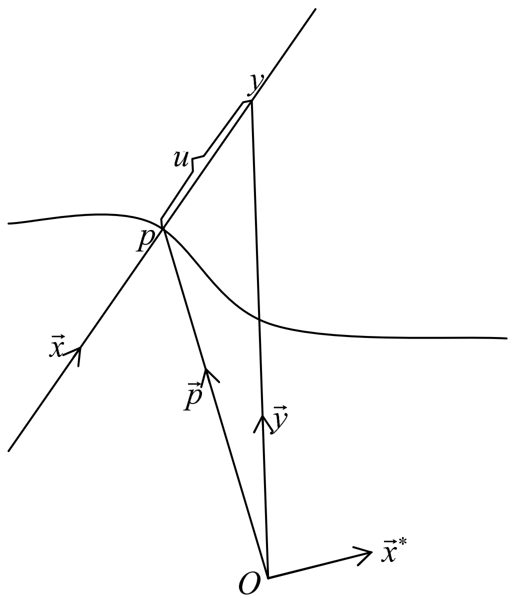

Figure 3: Ruled surfaces

is invariant to the coordinate changes, the ratio

$$\frac{1}{d} = \frac{\langle d\vec{x}, d\vec{x}^* \rangle}{\langle d\vec{x}, d\vec{x} \rangle} = \frac{d\varphi d\varphi^*}{d\varphi d\varphi} = \frac{d\varphi^*}{d\varphi}$$

stays invariant under coordinate changes. This ratio is called distribution parameter of timelike or spacelike ruled surfaces. Here the sign $\langle \rangle$ denotes, inner product in the sense of Lorentz.

4. Velocities and Instantaneous Rotation Axis

Let us now evaluate the velocity of dual point $\tilde{X}$ with respect to the fixed dual Lorentz sphere $\tilde{K}'$. Derivating equation (3) with respect to real parameter t yields

$$\overset{\bullet}{\tilde{X}}{}' = \overset{\bullet}{\tilde{A}}\tilde{X} + \tilde{A}\overset{\bullet}{\tilde{X}} + \overset{\bullet}{\tilde{U}}{}' . \tag{7}$$

Therefore, the absolute velocity of $\tilde{X}$ becomes

$$\tilde{V}_a' = \overset{\bullet}{\tilde{X}}{}' .$$

Assuming that the point $\tilde{X}$ is fixed with respect to moving orthogonal system, and differentiation in equation (3) we reach the sliding velocity of point $\tilde{X}$ as

$$\tilde{V}_f' = \overset{\bullet}{\tilde{A}}\tilde{X} + \overset{\bullet}{\tilde{U}}{}' . \tag{8}$$

The relative velocity of the point $\tilde{X}$ is given if the point $\tilde{X}$ is fixed with respect to the fixed orthonormal system, i.e. sliding velocity of this point is equal to zero. Thus the relative velocity is obtained to be

$$\tilde{V}'_r = \tilde{A}\overset{\bullet}{\tilde{X}}. \tag{9}$$

From these equations we see that

$$\tilde{V}_a = \tilde{V}_f + \tilde{V}_r.$$

Therefore, we give the following theorem.

Theorem 4.1. *The absolute velocity vector of the dual point $\tilde{X}$ in dual Lorentz space D_1^3 is the sum of the relative velocity vector and sliding velocity vector.*

Now, we search the fixed points (dual pole points) which are staying fixed in both of the systems, i.e. fixed and moving systems. These points are characterized by that the sliding velocity is equal to zero. That is,

$$\overset{\bullet}{\tilde{X}'} = \overset{\bullet}{\tilde{A}}\tilde{X} + \overset{\bullet}{\tilde{U}'} = 0.$$

Since $\tilde{U}' = -\tilde{A}\tilde{U}$, considering that $\overset{\bullet}{\tilde{U}'} = -\overset{\bullet}{\tilde{A}}\tilde{U} - \tilde{A}\overset{\bullet}{\tilde{U}}$ we reach

$$\tilde{A}^{-1}\overset{\bullet}{\tilde{A}}\tilde{X} = \tilde{A}^{-1}\overset{\bullet}{\tilde{A}}\tilde{U} + \overset{\bullet}{\tilde{U}}.$$

Therefore, the following two equations are satisfied for the dual pole points

$$\overset{\bullet}{\tilde{X}'} = \overset{\bullet}{\tilde{A}}\tilde{X} + \overset{\bullet}{\tilde{U}'} = 0 \quad or \quad \tilde{A}^{-1}\overset{\bullet}{\tilde{A}}\tilde{X} = \tilde{A}^{-1}\overset{\bullet}{\tilde{A}}\tilde{U} + \overset{\bullet}{\tilde{U}}. \tag{10}$$

The solution for this linear equation is not unique. Here, since $\tilde{A}^{-1}\tilde{A} = \tilde{A}\tilde{A}^{-1} = I_3$ we find

$$\left(\overset{\bullet}{\tilde{A}^{-1}}\right)\tilde{A} + \tilde{A}^{-1}\overset{\bullet}{\tilde{A}} = 0 \ , \ \overset{\bullet}{\tilde{A}}\tilde{A}^{-1} + \tilde{A}\left(\overset{\bullet}{\tilde{A}^{-1}}\right) = 0.$$

If we assume that $\tilde{S} = \left(\overset{\bullet}{\tilde{A}^{-1}}\right)\tilde{A}$ and $\tilde{S}' = \tilde{A}\left(\overset{\bullet}{\tilde{A}^{-1}}\right)$, then the matrices $\tilde{S}$ and $\tilde{S}'$ are anti-symmetric matrices in the sense of dual Lorentz.

Let assume that $\tilde{\Omega}_{ij}$ $(1 \leq i,j \leq 3)$ are the elements of $\tilde{S}$. Let us denote the permutations of the indices $i,j,k = 1,2,3;2,3,1;3,1,2$ by

$$\tilde{\Omega}_{ij} = \tilde{\Omega}_k$$

then we can easily get that

$$\tilde{S} = \begin{bmatrix} 0 & \tilde{\Omega}_3 & -\tilde{\Omega}_2 \\ \tilde{\Omega}_3 & 0 & -\tilde{\Omega}_1 \\ -\tilde{\Omega}_2 & \tilde{\Omega}_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 + \mathscr{E}\omega_3^* & -\omega_2 - \mathscr{E}\omega_2^* \\ \omega_3 + \mathscr{E}\omega_3^* & 0 & -\omega_1 - \mathscr{E}\omega_1^* \\ -\omega_2 - \mathscr{E}\omega_2^* & \omega_1 + \mathscr{E}\omega_1^* & 0 \end{bmatrix}.$$

Similarly

$$\tilde{S}' = \begin{bmatrix} 0 & \widetilde{\bar{\Omega}'_3} & -\widetilde{\bar{\Omega}'_2} \\ \widetilde{\bar{\Omega}'_3} & 0 & -\widetilde{\bar{\Omega}'_1} \\ -\widetilde{\bar{\Omega}'_2} & \widetilde{\bar{\Omega}'_1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega}'_3 + \mathscr{E}\bar{\omega}'^*_3 & -\bar{\omega}'_2 - \mathscr{E}\bar{\omega}'^*_2 \\ \bar{\omega}'_3 + \mathscr{E}\bar{\omega}'^*_3 & 0 & -\bar{\omega}'_1 - \mathscr{E}\bar{\omega}'^*_1 \\ -\bar{\omega}'_2 - \mathscr{E}\bar{\omega}'^*_2 & \bar{\omega}'_1 + \mathscr{E}\bar{\omega}'^*_1 & 0 \end{bmatrix},$$

where

$$\tilde{\Omega}_i = \omega_i + \mathscr{E}\omega_i^* \ \ and \ \ \widetilde{\bar{\Omega}'_i} = \bar{\omega}'_i + \mathscr{E}\bar{\omega}'^*_i \ \ , \ \ 1 \le i \le 3$$

are dual Pfaff forms (1-form).

For any $\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_1 \\ \tilde{\Omega}_2 \\ \tilde{\Omega}_3 \end{bmatrix}$ and $\widetilde{\bar{\Omega}'} = \begin{bmatrix} \widetilde{\bar{\Omega}'_1} \\ \widetilde{\bar{\Omega}'_2} \\ \widetilde{\bar{\Omega}'_3} \end{bmatrix}$ we can easily see that

$$\tilde{S}\tilde{\Omega} = 0 \ \ , \ \ \tilde{S}'\widetilde{\bar{\Omega}'} = 0 \tag{11}$$

where $\det\tilde{S} = 0$ and $\det\tilde{S}' = 0$. Furthermore, from equations $\tilde{A}^{-1} = \varepsilon\tilde{A}^T$ and $\tilde{S} = \left(\overset{\bullet}{\tilde{A}^{-1}}\right)\tilde{A}$, we write

$$\det\tilde{A} \neq 0 \ \ and \ \ \det\overset{\bullet}{\tilde{A}} = 0.$$

Thus $\overset{\bullet}{\tilde{A}}$ is singular and $rank\left(\tilde{S}\right) = rank\left(\overset{\bullet}{\tilde{A}}\right)$. Now, let us find $rank\overset{\bullet}{\tilde{A}}$ for the case $n = 3$ and $\tilde{A} \in SO_1(3)$. Differentiating equations $\tilde{A}^T\varepsilon\tilde{A} = \varepsilon$ with respect to parameter t we reach

$$\left(\overset{\bullet}{\tilde{A}^T}\right)\varepsilon\tilde{A} + \tilde{A}^T\varepsilon\overset{\bullet}{\tilde{A}} = 0.$$

If we choose

$$\widetilde{h} = \left(\overset{\bullet}{\tilde{A}^T}\right)\varepsilon\tilde{A}$$

we get

$$\widetilde{h} + \widetilde{h}^T = 0.$$

This means that $\widetilde{h}$ is anti-symmetric matrix. Therefore, since $\tilde{A} \in SO_1(3)$,

$$\det\widetilde{h} = (-1)^3\det\widetilde{h} = 0.$$

In this situation $rank\tilde{h} \le 2$. If $rank\tilde{h} = r$ then r must be even as the determinant of the all first order quadratic submatrices of $\tilde{h}$ is equal to zero.

Now we take the determinants of the both sides of the equation $\tilde{h} = \left(\overset{\bullet}{\tilde{A}^T}\right)\varepsilon\tilde{A}$ and find

$$\det\overset{\bullet}{\tilde{A}} = 0.$$

From the last result we get that

$$rank\tilde{h} = rank\overset{\bullet}{\tilde{A}} = r.$$

Since $n = 3$ it becomes

$$rank\overset{\bullet}{\tilde{A}} = 2$$

and from the equation (10) we obtain

$$-\tilde{S}\tilde{X} = -\tilde{S}\tilde{U} + \overset{\bullet}{\tilde{U}} \quad or \quad \varepsilon\tilde{S}^T\varepsilon\tilde{X} = \varepsilon\tilde{S}^T\varepsilon\tilde{U} + \overset{\bullet}{\tilde{U}}. \tag{12}$$

Since

$$rank\left(-\tilde{S}\right) = rank\tilde{S} = rank\overset{\bullet}{\tilde{A}} = 2$$

we reach that $rank\left(-\tilde{S}, \overset{\bullet}{\tilde{U}}\right) = rank\left(\tilde{A}^{-1}\overset{\bullet}{\tilde{A}}, \overset{\bullet}{\tilde{U}}\right) = rank\overset{\bullet}{\tilde{A}} = 2$. Under this condition (i.e. $rank\overset{\bullet}{\tilde{A}} = 2$), considering $\tilde{S}\tilde{\Omega} = 0$ and $\tilde{\Omega}^T\tilde{S}^T = 0$ and multiplying the equation(12) from the left-hand side by $\tilde{\Omega}^T\varepsilon$ we get

$$\tilde{\Omega}^T\varepsilon\overset{\bullet}{\tilde{U}} = 0 \tag{13}$$

where $\overrightarrow{\tilde{\Omega}} \neq 0$.

Under these conditions (i.e. $\tilde{\Omega}^T\varepsilon\overset{\bullet}{\tilde{U}} = 0$) we can solve the equation(10) and we reach a line equation. Thus line is an instantaneous rotation axis a in both systems at the t. Assuming that $\tilde{Y}$ is a variable point on this axis, we can write from the system (10) that

$$-\tilde{S}\tilde{Y} = -\tilde{S}\tilde{U} + \overset{\bullet}{\tilde{U}}.$$

From the solution of the system it can be easily seen that the direction of the instantaneous rotation axis is parallel to this dual vector $\overrightarrow{\tilde{\Omega}}$. This vector $\overrightarrow{\tilde{\Omega}}$ is called dual angular velocity vector of the movement $\tilde{K}/\tilde{K}'$. Therefore, the unit dual vector becomes

$$\overrightarrow{\tilde{\Omega}_0} = \frac{\overrightarrow{\tilde{\Omega}}}{\left\|\overrightarrow{\tilde{\Omega}}\right\|} = \frac{\vec{\omega}}{\omega} + \mathscr{E}\left(\frac{\vec{\omega}^*}{\omega} - \frac{\omega^*\vec{\omega}}{\omega^2}\right) \tag{14}$$

where $\left\|\overrightarrow{\tilde{\Omega}}\right\| = \omega + \mathscr{E}\omega^*$, $\|\vec{\omega}\| = \omega$ and $\omega^* = \frac{\langle\vec{\omega},\vec{\omega}^*\rangle}{\omega}$.

The dual number $\left\|\overrightarrow{\tilde{\Omega}}\right\| = \omega + \mathscr{E}\omega^*$ is called dual angular velocity of the movement $\tilde{K}/\tilde{K}'$. In addition to that the pitch of the movement $\tilde{K}/\tilde{K}'$ is

$$k = \frac{\omega^*}{\omega} = \frac{\langle \vec{\omega}, \vec{\omega}^* \rangle}{\|\vec{\omega}\|^2}.$$

In the time t the point on $\tilde{K}$ which coincides with $\overrightarrow{\tilde{\Omega}_0}$ has dual velocity of zero. If we denote this point with $\tilde{P}$ then this point is named a pole point of movement $\tilde{K}/\tilde{K}'$.

Thus, we can imagine that the movement H/H' is an helical motion about an axis. More generally, if we divide the motion into small parts each displacement occurs, related to this axis. The placement and the direction of the instantaneous rotation axis is continuously changes during the motion. Hence one can draw two timelike ruled surface or two spacelike ruled surface, one is on the fixed Lorentz sphere H' and other is on the moving Lorentz sphere H. The first timelike or spacelike ruled surface is called fixed timelike or spacelike axoid whereas the second one is named with moving timelike or spacelike axoid. Fixed or moving timelike or spacelike axoids have, all the times, a common tangent along the instantaneous rotation axis and slide on each other.

To evaluation the components of the vector $\overrightarrow{\tilde{\Omega}}$ easily we express the orthogonal matrix $\tilde{A}$ in the sense of dual Lorentz in terms of the Euler angles with the consecutive non-sliding three rotational motion moving system transforms to the fixed system. If one makes

i) a rotation of dual angle $\tilde{\Phi}$ about the axis $\overrightarrow{\tilde{E}_3}$

ii) a rotation of dual angle $\tilde{\Theta}$ about the axis $\overrightarrow{\tilde{E}_1'}$

iii) a rotation of dual angle $\tilde{\Psi}$ about the axis $\overrightarrow{\tilde{E}_3'}$

then these rotations correspond to the following matrices

$$\tilde{A}_1 = \begin{bmatrix} \cosh\tilde{\Phi} & \sinh\tilde{\Phi} & 0 \\ \sinh\tilde{\Phi} & \cosh\tilde{\Phi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \tilde{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\tilde{\Theta} & -\sin\tilde{\Theta} \\ 0 & \sin\tilde{\Theta} & \cos\tilde{\Theta} \end{bmatrix}$$

$$\tilde{A}_3 = \begin{bmatrix} \cosh\tilde{\Psi} & \sinh\tilde{\Psi} & 0 \\ \sinh\tilde{\Psi} & \cosh\tilde{\Psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we obtain $\tilde{A} = \tilde{A}_3\tilde{A}_2\tilde{A}_1 =$

$$\begin{bmatrix} \cosh\tilde{\Psi}\cosh\tilde{\Phi}+\sinh\tilde{\Psi}\sinh\tilde{\Phi}\cos\tilde{\Theta} & \cosh\tilde{\Psi}\sinh\tilde{\Phi}+\sinh\tilde{\Psi}\cosh\tilde{\Phi}\cos\tilde{\Theta} & -\sinh\tilde{\Psi}\sin\tilde{\Theta} \\ \sinh\tilde{\Psi}\cosh\tilde{\Phi}+\cosh\tilde{\Psi}\sinh\tilde{\Phi}\cos\tilde{\Theta} & \sinh\tilde{\Psi}\sinh\tilde{\Phi}+\cosh\tilde{\Psi}\cosh\tilde{\Phi}\cos\tilde{\Theta} & -\cosh\tilde{\Psi}\sin\tilde{\Theta} \\ \sin\tilde{\Theta}\sinh\tilde{\Phi} & \sin\tilde{\Theta}\cosh\tilde{\Phi} & \cos\tilde{\Theta} \end{bmatrix} \tag{15}$$

Considering the last equation and evaluating the matrix $\tilde{S} = \left(\dot{\tilde{A}^{-1}}\right)\tilde{A}$ we find

$$\tilde{S} = \begin{bmatrix} 0 & -\dot{\tilde{\Phi}} - \dot{\tilde{\Psi}}\cos\tilde{\Theta} & \dot{\tilde{\Psi}}\cosh\tilde{\Phi}\sin\tilde{\Theta} - \dot{\tilde{\Theta}}\sinh\tilde{\Phi} \\ -\dot{\tilde{\Phi}} - \dot{\tilde{\Psi}}\cos\tilde{\Theta} & 0 & -\dot{\tilde{\Psi}}\sinh\tilde{\Phi}\sin\tilde{\Theta} + \dot{\tilde{\Theta}}\cosh\tilde{\Phi} \\ \dot{\tilde{\Psi}}\cosh\tilde{\Phi}\sin\tilde{\Theta} - \dot{\tilde{\Theta}}\sinh\tilde{\Phi} & \dot{\tilde{\Psi}}\sinh\tilde{\Phi}\sin\tilde{\Theta} - \dot{\tilde{\Theta}}\cosh\tilde{\Phi} & 0 \end{bmatrix}$$

If we consider the equations given by equation (11) then we get

$$\begin{aligned} \tilde{\Omega}_1 &= \dot{\tilde{\Psi}}\sinh\tilde{\Phi}\sin\tilde{\Theta} - \dot{\tilde{\Theta}}\cosh\tilde{\Phi} \\ \tilde{\Omega}_2 &= -\dot{\tilde{\Psi}}\cosh\tilde{\Phi}\sin\tilde{\Theta} + \dot{\tilde{\Theta}}\sinh\tilde{\Phi} \\ \tilde{\Omega}_3 &= -\dot{\tilde{\Phi}} - \dot{\tilde{\Psi}}\cos\tilde{\Theta} \end{aligned} \tag{16}$$

and

$$\begin{aligned} \widetilde{\bar{\Omega}}'_1 &= \dot{\tilde{\Phi}}\sinh\tilde{\Psi}\sin\tilde{\Theta} - \dot{\tilde{\Theta}}\cosh\tilde{\Psi} \\ \widetilde{\bar{\Omega}}'_2 &= \dot{\tilde{\Phi}}\cosh\tilde{\Psi}\sin\tilde{\Theta} - \dot{\tilde{\Theta}}\sinh\tilde{\Psi} \cdot \\ \widetilde{\bar{\Omega}}'_3 &= -\dot{\tilde{\Psi}} - \dot{\tilde{\Phi}}\cos\tilde{\Theta}. \end{aligned}$$

Thus, from the relation $\tilde{\Omega}' = \tilde{A}\tilde{\Omega}$ we reach

$$\begin{aligned} \tilde{\Omega}'_1 &= \dot{\tilde{\Phi}}\sinh\tilde{\Psi}\sin\tilde{\Theta} - \dot{\tilde{\Theta}}\cosh\tilde{\Psi} \\ \tilde{\Omega}'_2 &= \dot{\tilde{\Phi}}\cosh\tilde{\Psi}\sin\tilde{\Theta} - \dot{\tilde{\Theta}}\sinh\tilde{\Psi} \\ \tilde{\Omega}'_3 &= -\dot{\tilde{\Psi}} - \dot{\tilde{\Phi}}\cos\tilde{\Theta}. \end{aligned}$$

Therefore, we see that $\tilde{\Omega}'$ and $\widetilde{\bar{\Omega}}'$ are equal to each other.

From the equation (8) we obtain

$$\tilde{V}_f = \tilde{A}^{-1}\tilde{V}'_f = \varepsilon\tilde{S}^T\varepsilon\tilde{X} - \varepsilon\tilde{S}^T\varepsilon\tilde{U} - \dot{\tilde{U}}\,.$$

Multiplying the last equations by $\tilde{\Omega}^T\varepsilon$ from the both sides we get

$$\tilde{\Omega}^T\varepsilon\tilde{V}_f = \tilde{\Omega}^T\tilde{S}^T\varepsilon\tilde{X} - \tilde{\Omega}^T\tilde{S}^T\varepsilon\tilde{U} - \tilde{\Omega}^T\varepsilon\dot{\tilde{U}}\,.$$

From the last equation we see that the relation

$$\tilde{\Omega}^T\varepsilon\tilde{V}_f = 0$$

is correct. Thus, we can give the following theorem.

Theorem 4.2. *In a one-parameter dual spherical movement in dual Lorentz space D_1^3, the sliding velocity a point $\tilde{X}$ in time t crosses the instantaneous rotation axis at that time.*

Considering equation (16), the instantaneous rotation axis $\overrightarrow{\tilde{\Omega}} = \vec{\omega} + \mathscr{E}\vec{\omega}^* = (\omega_1, \omega_2, \omega_3) + \mathscr{E}(\omega_1^*, \omega_2^*, \omega_3^*)$ can be written in real and dual parts as

$$\begin{aligned}
\omega_1 &= \overset{\bullet}{\psi} \sinh\varphi \sin\theta - \overset{\bullet}{\theta} \cosh\varphi \\
\omega_2 &= -\overset{\bullet}{\psi} \cosh\varphi \sin\theta + \overset{\bullet}{\theta} \sinh\varphi \\
\omega_3 &= -\overset{\bullet}{\varphi} - \overset{\bullet}{\psi} \cos\theta
\end{aligned}$$

and

$$\begin{aligned}
\omega_1^* &= \overset{\bullet}{\psi}\theta^* \cos\theta \sinh\varphi + \overset{\bullet}{\psi}\varphi^* \sin\theta \cosh\varphi + \overset{\bullet}{\psi^*} \sinh\varphi \sin\theta \\
&\qquad - \varphi^* \overset{\bullet}{\theta} \sinh\varphi - \overset{\bullet}{\theta^*} \cosh\varphi \\
\omega_2^* &= -\overset{\bullet}{\psi}\theta^* \cosh\varphi \cos\theta - \overset{\bullet}{\psi}\varphi^* \sin\theta \sinh\varphi - \overset{\bullet}{\psi^*} \cosh\varphi \sin\theta + \varphi^* \overset{\bullet}{\theta} \cosh\varphi \\
&\qquad + \overset{\bullet}{\theta^*} \sinh\varphi \\
\omega_3^* &= -\overset{\bullet}{\varphi^*} + \overset{\bullet}{\psi}\theta^* \sin\theta - \overset{\bullet}{\psi^*} \cos\theta.
\end{aligned} \tag{17}$$

Therefore, the pitch of the movements becomes

$$k = \frac{\langle \vec{\omega}, \vec{\omega}^* \rangle}{\|\vec{\omega}\|^2} = \frac{\overset{\bullet}{\psi}\overset{\bullet}{\psi^*} - \overset{\bullet}{\theta}\overset{\bullet}{\theta^*} + \overset{\bullet}{\varphi}\overset{\bullet}{\varphi^*} - \overset{\bullet}{\varphi}\overset{\bullet}{\psi}\theta^* \sin\theta + \left(\overset{\bullet}{\varphi}\overset{\bullet}{\psi^*} + \overset{\bullet}{\psi}\overset{\bullet}{\varphi^*}\right)\cos\theta}{\overset{\bullet}{\psi}{}^2 - \overset{\bullet}{\theta}{}^2 + \overset{\bullet}{\varphi}{}^2 + 2\overset{\bullet}{\varphi}\overset{\bullet}{\psi}\cos\theta}. \tag{18}$$

Example 4.3. Let $\psi(t) = \varphi(t) = t, \theta(t) = \frac{\pi}{4}, \psi^*(t) = \varphi^*(t) = \theta^*(t) = 1$. Thus, form equations (17) and (18) the dual instantaneous axis and pitch are

$$\begin{aligned}
\overrightarrow{\tilde{\Omega}} = \vec{\omega} + \mathscr{E}\vec{\omega}^* &= \left(\frac{\sqrt{2}}{2}\sinh t, \frac{-\sqrt{2}}{2}\cosh t, \frac{-2-\sqrt{2}}{2}\right) + \\
&+ \mathscr{E}\left(\frac{\sqrt{2}}{2}\sinh t + \frac{\sqrt{2}}{2}\cosh t, \frac{-\sqrt{2}}{2}\sinh t - \frac{\sqrt{2}}{2}\cosh t, \frac{\sqrt{2}}{2}\right)
\end{aligned}$$

and

$$k = \frac{1-\sqrt{2}}{2}$$

respectively. Furthermore, from equation (14) we find the unit instantaneous rotation axis of the movement to be

$$\begin{aligned}
\overrightarrow{\tilde{\Omega}_0} = \vec{\omega}_0 + \mathscr{E}\vec{\omega}_0^* = \frac{1}{\left(2+\sqrt{2}\right)^{\frac{1}{2}}} &\left[\left(\tfrac{\sqrt{2}}{2}\sinh t, \tfrac{-\sqrt{2}}{2}\cosh t, \tfrac{-2-\sqrt{2}}{2}\right)\right. \\
&\left. + \mathscr{E}\left(\tfrac{\sqrt{2}+2}{4}\sinh t + \tfrac{\sqrt{2}}{2}\cosh t, \tfrac{-\sqrt{2}-2}{4}\cosh t - \tfrac{\sqrt{2}}{2}\sinh t, \tfrac{\sqrt{2}}{4}\right)\right].
\end{aligned}$$

Therefore, the spacelike ruled surface formed by the instantaneous rotation axis becomes (see figure IV)

$$\vec{y}(t,v) = \vec{\omega}_0^* \wedge \vec{\omega}_0 + v\vec{\omega}_0 =$$

$$\left(\frac{-1}{2}\cosh t - \frac{\sqrt{2}}{4}\sinh t, \frac{1}{2}\sinh t + \frac{\sqrt{2}}{4}\cosh t, \frac{\sqrt{2}-2}{4}\right) +$$

$$+v\left(\frac{\sqrt{2-\sqrt{2}}}{2}\sinh t, \frac{-\sqrt{2-\sqrt{2}}}{2}\cosh t, \frac{-\sqrt{2+\sqrt{2}}}{2}\right)$$

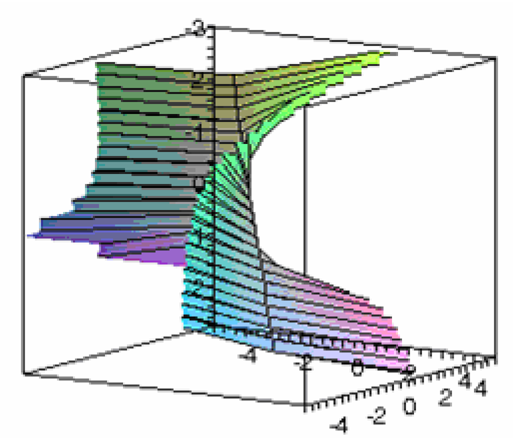

Figure 4: Spacelike ruled surfaces

Example 4.4. Let $\psi(t) = \varphi(t) = t, \theta(t) = \frac{\pi}{4}, \psi^*(t) = -t, \varphi^*(t) = \theta^*(t) = 1$. From the equations (17) and (18) the dual instantaneous rotation axis and the pitch read to be

$$\overrightarrow{\tilde{\Omega}} = \vec{\omega} + \mathscr{E}\vec{\omega}^* =$$

$$= \left(\frac{\sqrt{2}}{2}\sinh t, \frac{-\sqrt{2}}{2}\cosh t, \frac{-2-\sqrt{2}}{2}\right) + \mathscr{E}\left(\frac{\sqrt{2}}{2}\cosh t, \frac{-\sqrt{2}}{2}\sinh t, \sqrt{2}\right)$$

and

$$k = \frac{1-\sqrt{2}}{2}$$

respectively. In addition to that from equation (14) the unit instantaneous rotation axis of the movement is obtained as follows

$$\overrightarrow{\tilde{\Omega}_0} = \vec{\omega}_0 + \mathscr{E}\vec{\omega}_0^* =$$

$$= \frac{1}{\left(2+\sqrt{2}\right)^{\frac{1}{2}}}\left[\left(\frac{\sqrt{2}}{2}\sinh t, \frac{-\sqrt{2}}{2}\cosh t, \frac{-2-\sqrt{2}}{2}\right)+\right.$$

$$\left.+\mathscr{E}\left(\frac{\sqrt{2}}{2}\cosh t+\frac{1}{2}\sinh t, \frac{-\sqrt{2}}{2}\sinh t-\frac{1}{2}\cosh t, \frac{\sqrt{2}-1}{2}\right)\right].$$

Thus the timelike ruled surface formed by the instantaneous rotation axis becomes (see figure V)

$$\vec{y}(t,v) = \vec{\omega}_0^* \wedge \vec{\omega}_0 + v\vec{\omega}_0$$

$$= \left[\begin{array}{l} \left(\frac{-\sqrt{2}}{4}\sinh t + \frac{\sqrt{2}-2}{2}\cosh t, \frac{\sqrt{2}}{4}\cosh t + \frac{2-\sqrt{2}}{2}\sinh t, \frac{\sqrt{2}-2}{4}\right) \\ +v\left(\frac{\sqrt{2-\sqrt{2}}}{2}\sinh t, \frac{-\sqrt{2-\sqrt{2}}}{2}\cosh t, \frac{-\sqrt{2+\sqrt{2}}}{2}\right) \end{array}\right]$$

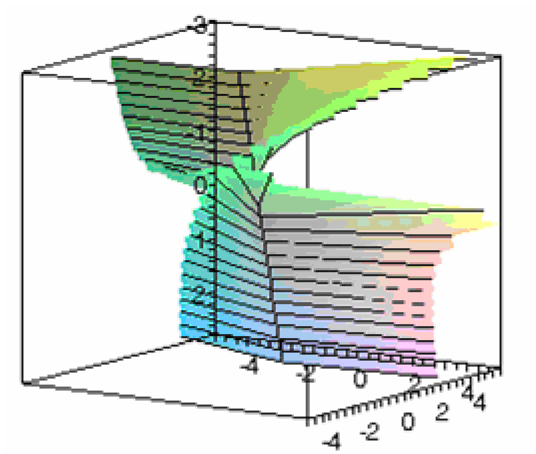

Figure 5: Timelike ruled surfaces

5. Accelerations and Acceleration Centre

If we consider equation (7) we see that

$$\ddot{\tilde{X}}' = \ddot{\tilde{A}}\tilde{X} + \ddot{\tilde{U}}' + 2\dot{\tilde{A}}\dot{\tilde{X}} + \tilde{A}\ddot{\tilde{X}} \tag{19}$$

where

$$\tilde{\gamma}_a' = \ddot{\tilde{X}}' \tag{20}$$

is absolute acceleration,

$$\tilde{\gamma}'_f = \ddot{\tilde{A}}\tilde{X} + \ddot{\tilde{U}}' \tag{21}$$

is called sliding acceleration and

$$\tilde{\gamma}'_c = 2\dot{\tilde{A}}\dot{\tilde{X}} \tag{22}$$

is named Coriolis acceleration where as

$$\tilde{\gamma}'_r = \tilde{A}\ddot{\tilde{X}} \tag{23}$$

is called relative acceleration. There exists the following relation between these accelerations

$$\tilde{\gamma}_a = \tilde{\gamma}_f + \tilde{\gamma}_c + \tilde{\gamma}_r.$$

Thus we give the following theorem.

Theorem 5.1. *The absolute acceleration of a point $\tilde{X}$ under one-parameter movements in dual Lorentz space D_1^3 is the sum of sliding, relative and Coriolis accelerations.*

Considering equation (22) we write

$$\tilde{\gamma}_c = \tilde{A}^{-1}\tilde{\gamma}'_c = -2\tilde{S}\dot{\tilde{X}} = 2\varepsilon\tilde{S}^T\varepsilon\dot{\tilde{X}}.$$

From the last equation together with equation (9) we see that

$$\overrightarrow{\tilde{\gamma}_c} = 2\left(\overrightarrow{\tilde{\Omega}} \wedge \overrightarrow{\tilde{V}_r}\right).$$

Therefore, we can give the following theorem.

Theorem 5.2. *The Coriolis acceleration vector related to the point $\tilde{X}$ at time t under one-parameter motion in dual Lorentz space D_1^3 is perpendicular to the relative velocity vector and instantaneous vector axis at that time.*

Now, we search the points for which the sliding acceleration is zero at time t. Thus, from equation (21) we find

$$\ddot{\tilde{A}}\tilde{X} + \ddot{\tilde{U}}' = 0 \tag{24}$$

or

$$\tilde{A}^{-1}\ddot{\tilde{A}}\tilde{X} + \tilde{A}^{-1}\ddot{\tilde{U}}' = 0. \tag{25}$$

If we consider $\tilde{A}\tilde{A}^{-1}=I$ we write for $\tilde{A}^{-1}\ddot{\tilde{A}}$ the following equation

$$\begin{aligned}\ddot{\tilde{A}}\,\tilde{A}^{-1} &= -\dot{\tilde{A}}\,(\dot{\tilde{A}^{-1}}) - \tilde{A}\,(\ddot{\tilde{A}^{-1}}) - \dot{\tilde{A}}\,(\dot{\tilde{A}^{-1}}) \\ &= (\dot{\tilde{A}^{-1}})\,\tilde{A}\,(\dot{\tilde{A}^{-1}})\,\tilde{A} - (\ddot{\tilde{A}^{-1}})\,\tilde{A} - (\dot{\tilde{A}^{-1}})\,\dot{\tilde{A}}.\end{aligned}$$

Therefore, since $\tilde{S}=(\dot{\tilde{A}^{-1}})\,\tilde{A}$ we get

$$\tilde{A}^{-1}\ddot{\tilde{A}}=\tilde{S}^2-\dot{\tilde{S}}\;,\;\; \det\left(\tilde{S}^2-\dot{\tilde{S}}\right)=\left\|\overrightarrow{\tilde{\Omega}}\wedge\dot{\overrightarrow{\tilde{\Omega}}}\right\|^2. \tag{26}$$

Considering equation (24), since $\tilde{U}'=-\tilde{A}\tilde{U}$ we find $\ddot{\tilde{U}}'=-\ddot{\tilde{A}}\,\tilde{U}-2\dot{\tilde{A}}\,\dot{\tilde{U}}-\tilde{A}\,\ddot{\tilde{U}}$. In this case, from equations (24) and (26) we reach

$$\left(\tilde{S}^2-\dot{\tilde{S}}\right)\tilde{X}=\left(\tilde{S}^2-\dot{\tilde{S}}\right)\tilde{U}-2\tilde{S}\dot{\tilde{U}}+\ddot{\tilde{U}}. \tag{27}$$

Thus, if

$$\left\|\overrightarrow{\tilde{\Omega}}\wedge\dot{\overrightarrow{\tilde{\Omega}}}\right\|^2\neq 0 \tag{28}$$

then the equation system (27) can be solved in one way only. So, from equation (24) we find the acceleration centre to be

$$\tilde{P}_1=-\left(\ddot{\tilde{A}}\right)^{-1}\ddot{\tilde{U}} \qquad \tilde{P}_1'=\tilde{U}'-\tilde{A}\left(\ddot{\tilde{A}}\right)^{-1}\ddot{\tilde{U}}'. \tag{29}$$

From these results we can write the sliding acceleration given by equation (21) terms of these coordinates. Since $\tilde{\gamma}_f'=\ddot{\tilde{A}}\,\tilde{X}+\ddot{\tilde{U}}'$ and $\tilde{\gamma}_f=\tilde{A}^{-1}\tilde{\gamma}_f'$ we write

$$\begin{aligned}\tilde{\gamma}_f &= \tilde{A}^{-1}\ddot{\tilde{A}}\,\tilde{X}+\tilde{A}^{-1}\ddot{\tilde{U}}' \\ &= \tilde{A}^{-1}\ddot{\tilde{A}}\left[\tilde{X}+\left(\ddot{\tilde{A}}\right)^{-1}\ddot{\tilde{U}}'\right] \\ &= \tilde{A}^{-1}\ddot{\tilde{A}}\left(\tilde{X}-\tilde{P}_1\right)\end{aligned} \tag{30}$$

Similarly, since $\tilde{\gamma}_f'=\ddot{\tilde{A}}\,\tilde{X}+\ddot{\tilde{U}}'$, substituting the values of $\tilde{X}=\tilde{A}^{-1}\tilde{X}'+\tilde{U}$ and $\tilde{U}=-\tilde{A}^{-1}\tilde{U}'$ we obtain

$$\tilde{A}\left(\ddot{\tilde{A}}\right)^{-1}\tilde{\gamma}_f'=\tilde{X}'-\tilde{U}'+\tilde{A}\left(\ddot{\tilde{A}}\right)^{-1}\ddot{\tilde{U}}'.$$

Considering the last equation we find

$$\tilde{\gamma}_f' = \ddot{\tilde{A}}\,\tilde{A}^{-1}\left(\tilde{X}' - \tilde{P}_1'\right). \tag{31}$$

We can give the following theorem.

Theorem 5.3. *The sliding acceleration of one-parameter dual spherical movement in 3-dimensional dual Lorentz space D_1^3 is expressed in terms of the coordinates $\tilde{P}_1$ and $\tilde{P}_1'$ to be*

$$\tilde{\gamma}_f = \tilde{A}^{-1}\,\ddot{\tilde{A}}\left(\tilde{X} - \tilde{P}_1\right) \quad , \quad \tilde{\gamma}_f' = \ddot{\tilde{A}}\,\tilde{A}^{-1}\left(\tilde{X}' - \tilde{P}_1'\right).$$

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