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THE UNIQUENESS PROBLEM FOR A MODEL OF AN INCOMPRESSIBLE FLUID MIXTURE

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Two uniqueness theorems for an isothermal mixture of two miscible fluids are proved. The mixture F is incompressible in a generilized sense and able of exerting Korteweg stresses.

1. Introduction.

Let F be a mixture of two miscible incompressible fluids A and B and let $\mathbf{U}, \Phi, \rho, \theta$, π denote velocity, concentration, temperature and pressure field, respectively. Recently, Joseph [2] has studied the case when F can be considered incompressible in a generalized sense. Such a generalization consists into assuming that the density ρ is not influenced by the pressure π, which has still the meaning of a dynamical variable, but, rather, that ρ is related through a constitutive equation to Φ and θ. As a consequence, the velocity field $\mathbf{U}$ is not supposed solenoidal. Furthermore, beside the usual stresses, Joseph considers also those due to density changes according to the theory of Korteweg of 1901. The evolution equation determined by Joseph when F is isothermal, under the hypothesis that ρ obeys the linear constitutive equation

$$\rho(\Phi) = \rho_A \Phi + \rho_B (1 - \Phi)$$

where ρ_A and ρ_B are constants, are the following ones

$$(\star)\begin{cases} \dfrac{d\Phi}{dt} + \Phi\,\mathrm{div}\mathbf{U} = \mathrm{div}(D\nabla\Phi) \\ \rho_\Phi\,\dfrac{d\Phi}{dt} + \rho\,\mathrm{div}\mathbf{U} = 0 \\ \rho\dfrac{d\mathbf{U}}{dt} = -\nabla\pi + 2\mathrm{div}(\mu\mathbf{D}[\mathbf{U}]) + \nabla(\lambda\,\mathrm{div}\mathbf{U}) + \mathrm{div}\mathbf{T}^{(2)} + \rho\,\mathbf{g} \end{cases}$$

with

$$T_{ij}^{(2)} = \hat{\delta}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j} + \hat{\gamma}\,\frac{\partial^2\Phi}{\partial x_i\partial x_j}$$

$$\hat{\delta} = \delta_1\,\rho_\Phi^2 + \delta_2 + 2\nu\rho_\Phi$$

$$\hat{\gamma} = \gamma_1\rho_\Phi + \gamma_2$$

where $\mathbf{D}[\mathbf{U}]$ is the deformation rate tensor, g is the gravity, μ and λ are the viscosity coefficients, D is the diffusion coefficient, while $\delta_1, \delta_2, \nu, \gamma_1$ and γ_2 are coefficients depending, in general, on ρ and Φ. Equations $(\star)$ are very interesting from both mathematical and physical point of view and several problems arise .

Here, we are interested into the uniqueness problem of the solutions to the equations we obtain from the $(\star)$ when the fluids of the mixture have *the same, or nearly the same density, but not the same viscosity.*

2. Preliminaries.

When the fluids of the mixture have the same, or nearly the same density, but not the same viscosity, Galdi, Joseph, Preziosi, Rionero [1] proved that equations $(\star)$, in dimensionless form can be written:

$$\mathrm{div}\mathbf{U} = 0 \tag{1}$$

$$\frac{\partial\Phi}{\partial t} + \mathbf{U}\cdot\nabla\Phi = \mathbf{\Delta}_2\Phi \tag{2}$$

$$\frac{1}{S}\left[\frac{\partial\mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U}\right] = -\nabla\pi + 2\,\mathrm{div}(\mu(\Phi)\,\mathbf{D}[\mathbf{U}]) + K_1\nabla\Phi\,\mathbf{\Delta}_2\Phi - R^2\Phi\mathbf{k} \tag{3}$$

where $\mathbf{U}, \Phi, \pi$ are respectively velocity, concentration and pressure fields; $\mathbf{D}[\mathbf{U}]$ is the symmetric part of $\nabla\mathbf{U}$; $\mu = \mu(\Phi)$ is the viscosity, where $\Phi \in [0,1]$; $\mathbf{k} = -\mathbf{g}/|\mathbf{g}|$; $\mathcal{S}$ and R are positive constants; K_1 is a real constant (Korteweg constant) arising from Korteweg stresses.

To (1), (2), (3) we add the initial boundary conditions :

$$\text{(4)} \quad \mathbf{U}(\mathbf{x},t) = \mathbf{U}_S(\mathbf{x},t) \qquad \Phi(\mathbf{x},t) = \Phi_S(\mathbf{x},t), \qquad \forall \mathbf{x} \in S = \partial\Omega, \quad \forall t \geq 0$$

$$\text{(5)} \qquad \mathbf{U}(\mathbf{x},0) = \mathbf{U}_0(\mathbf{x}) \qquad \Phi(\mathbf{x},0) = \Phi_0(\mathbf{x}), \qquad \forall \mathbf{x} \in \Omega.$$

where Ω is a compact time-independent domain of $\mathbf{R}^3$.

Assuming that $(\mathbf{U}, \Phi, \pi)$ and $(\mathbf{U}+\mathbf{u}, \Phi+\phi, \pi+p)$ are two solutions to the equations (1),...,(5) then easily follows that the equations governing the perturbation $\mathbf{u}, \phi, p$ are

$$\text{(6)} \qquad \operatorname{div}\mathbf{u} = 0.$$

$$\text{(7)} \qquad \frac{\partial \phi}{\partial t} + (\mathbf{U}+\mathbf{u}) \cdot \nabla\phi = -\mathbf{u} \cdot \nabla\Phi + \mathbf{\Delta}_2\phi.$$

$$\text{(8)} \qquad \begin{aligned} \frac{1}{\mathcal{S}}\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U}+\mathbf{u}) \cdot \nabla\mathbf{u}\right] &= -\frac{1}{\mathcal{S}}\mathbf{u} \cdot \nabla\mathbf{U} - \nabla p + 2\operatorname{div}\mu(\Phi+\phi)\mathbf{D}[\mathbf{u}] \\ &\quad + 2\operatorname{div}([\mu(\Phi+\phi) - \mu(\Phi)]\,\mathbf{D}[\mathbf{U}]) + K_1\nabla\Phi\mathbf{\Delta}_2\phi \\ &\quad + K_1\nabla\phi\mathbf{\Delta}_2\Phi + K_1\nabla\phi\mathbf{\Delta}_2\phi - R^2\phi\mathbf{k} \end{aligned}$$

$$\text{(9)} \qquad \mathbf{u}(\mathbf{x},t) = 0 \qquad \phi(\mathbf{x},t) = 0, \qquad \forall \mathbf{x} \in \partial\Omega, \quad \forall t \geq 0$$

$$\text{(10)} \qquad \mathbf{u}(\mathbf{x},0) = 0 \qquad \phi(\mathbf{x},0) = 0, \qquad \forall \mathbf{x} \in \Omega$$

where we assume that the motion $\mathbf{U}+\mathbf{u}$, $\Phi+\phi$ satisfy the same initial-boundary conditions of the basic flow $\mathbf{U}, \Phi$.

In order to obtain uniqueness theorems, we shall introduce the following generilized energy :

$$\text{(11)} \qquad \mathcal{E} = \frac{1}{2}\int_\Omega \left(\frac{1}{\mathcal{S}}|\mathbf{u}|^2 + \lambda\phi^2 + \sigma|\nabla\phi|^2\right) d\Omega,$$

with λ and σ real parameter to be chosen later suitably according to $\mathcal{E} \geq 0$. From (6),...,(9) -since Ω does not depend on time- we have

$$\begin{aligned}\frac{d\mathcal{E}}{dt} = \{ &-\frac{1}{\mathcal{S}} < \mathbf{u} \cdot \nabla \mathbf{U} \cdot \mathbf{u} > + K_1 < (\mathbf{u} \cdot \nabla \phi) \mathbf{\Delta}_2 \Phi > - R^2 < w\phi > - \\ &- \lambda < (\mathbf{u} \cdot \nabla \Phi) \phi > + 2 < \mathrm{div}([\mu(\Phi + \phi) - \mu(\Phi)]\, \mathbf{D}[\mathbf{U}]) \cdot \mathbf{u} > \} + \\ &+ \{ (K_1 + \sigma) < (\mathbf{u} \cdot \nabla \Phi) \mathbf{\Delta}_2 \phi > + \sigma < (\mathbf{U} \cdot \nabla \phi) \mathbf{\Delta}_2 \phi > \} - \\ &- \{ 2 < \mu(\Phi + \phi) |\mathbf{D}[\mathbf{u}]|^2 > + \lambda < |\nabla \phi|^2 > + \sigma < |\mathbf{\Delta}_2 \phi|^2 > \} + \\ &+ (K_1 + \sigma) < (\mathbf{u} \cdot \nabla \phi) \mathbf{\Delta}_2 \phi > \end{aligned} \tag{12}$$

where $< \cdot > = \int_\Omega \cdot \, d\Omega$ and $w = \mathbf{u} \cdot \mathbf{k}$.

3. A uniqueness theorem for negative Korteweg constants.

In the sequel we shall assume that the constitutive equation $\mu = \mu(\Phi)$ is such that

$$\begin{cases} \mu \in C^2([0,1]) \\ \exists\, \mu_0 = const > 0 : \quad \mu \geq \mu_0, \quad \forall \Phi \in [0,1]. \end{cases}$$

Furthermore, we shall denote by $\mathcal{I}(\Omega)$ the class of $(\mathbf{U}, \Phi)$ such that:

(i) $\mathbf{U}(\cdot, t), \Phi(\cdot, t) \in C^2(\Omega)$, $\forall t \geq 0$;

(ii) $\mathbf{U}(\mathbf{x}, \cdot), \Phi(\mathbf{x}, \cdot), \nabla\Phi(\mathbf{x}, \cdot) \in C^1([0,T])$, $\forall \mathbf{x} \in \Omega, \forall T > 0$;

(iii) $\nabla \mathbf{U}(\mathbf{x}, \cdot), \mathbf{\Delta}_2 \Phi(\mathbf{x}, \cdot) \in C^0([0,T])$, $\forall \mathbf{x} \in \Omega, \forall T > 0$.

The following theorem holds:

THEOREM 1. *If* $K_1 < 0$ *, then in* $\mathcal{I}(\Omega)$ *exists at most one solution* $(\mathbf{U}, \Phi)$ *to the initial-boundary value problem (1),...,(5) such that* $\mathbf{U}$ *and* Φ *are bounded with their first and second spatial derivatives.*

Proof. First of all, let us choose

$$\lambda = 1 \qquad \sigma = -K_1$$

in (11) and (12) . We get then $\mathcal{E}(t) \geq 0$, $\forall t \geq 0$, and

$$\frac{d\mathcal{E}}{dt} = \Big\{ -\frac{1}{\mathcal{S}} < \mathbf{u}\cdot\nabla\mathbf{U}\cdot\mathbf{u} > + K_1 < (\mathbf{u}\cdot\nabla\phi)\mathbf{\Delta}_2\Phi > - R^2 < w\phi > -$$

$$- < (\mathbf{u}\cdot\nabla\Phi)\phi > + 2 < \operatorname{div}([\mu(\Phi+\phi) - \mu(\Phi)]\,\mathbf{D}[\mathbf{U}])\cdot\mathbf{u} > \Big\} -$$

$$- K_1 < (\mathbf{U}\cdot\nabla\phi)\mathbf{\Delta}_2\phi > -$$

$$- \{2 < \mu(\Phi+\phi)|\mathbf{D}[\mathbf{u}]|^2 > + < |\nabla\phi|^2 > - K_1 < |\mathbf{\Delta}_2\phi|^2 > \}$$

Let us observe that:

$$< \operatorname{div}([\mu(\Phi+\phi) - \mu(\Phi)]\,\mathbf{D}[\mathbf{U}])\cdot\mathbf{u} > =$$

$$= < \mu''(\Phi+\xi\phi)\,\phi\,\nabla\Phi\cdot\mathbf{D}[\mathbf{U}]\cdot\mathbf{u} > + < \mu'(\Phi+\phi)\,\nabla\phi\cdot\mathbf{D}[\mathbf{U}]\cdot\mathbf{u} > +$$

$$+ < \mu'(\Phi+\eta\phi)\,\phi\,(\mathbf{\Delta}_2\mathbf{U}\cdot\mathbf{u}) >$$

with μ' and μ'' first and second derivatives of μ with respect to Φ ; and where ξ, η are functions whose values are in $(0,1)$.

Therefore by Schwarz inequality :

$$\frac{d\mathcal{E}}{dt} \leq h\mathcal{E} - K_1 < (\mathbf{U}\cdot\nabla\phi)\,\mathbf{\Delta}_2\phi > + K_1 < |\mathbf{\Delta}_2\phi|^2 >$$

where h is a positive constant. Since

$$< (\mathbf{U}\cdot\nabla\phi)\mathbf{\Delta}_2\phi > \leq \frac{M^2}{2} < |\nabla\phi|^2 > + \frac{1}{2} < |\mathbf{\Delta}_2\phi|^2 >,$$

with $M = \sup_{t\geq 0}\sup_{\Omega}|\mathbf{U}|$, then it follows

$$\frac{d\mathcal{E}}{dt} \leq \gamma\,\mathcal{E}$$

where γ is a suitable positive constant.

From the last inequality, since $\mathcal{E}(0)=0$, follows that $\mathcal{E}(t)=0$, $\forall t\geq 0$, and the uniqueness theorem is proved.

4. A uniqueness theorem for positive Korteweg constants.

Assume $K_1 > 0$. Moreover, let us suppose $\exists \mu^*, M$ positive constants such that

$$\mu^* \geq \sup_{0\leq\Phi\leq 1} \{\mu(\Phi), |\mu'(\Phi)|, |\mu''(\Phi)|\}$$

$$M \geq \sup_{t\geq 0} \sup_{\Omega} \{|\mathbf{U}|, |\nabla\mathbf{U}|, |\mathbf{\Delta}_2\mathbf{U}|, |\nabla\Phi|, |\mathbf{\Delta}_2\Phi|\} .$$

From (12), choosing $\lambda > 0$ and $\sigma > 0$, by virtue of Young inequality, we get

$$\frac{d\mathcal{E}}{dt} \leq \gamma\,\mathcal{E} - \mathcal{D} + \mathcal{N}, \tag{13}$$

where

$$\mathcal{D} = \mathcal{D}(\mathbf{u}, \phi) = \mu_0 < |\nabla\mathbf{u}|^2 > +\lambda < |\nabla\phi|^2 > +\frac{\sigma}{2} < |\mathbf{\Delta}_2\phi|^2 >, \tag{14}$$

$$\mathcal{N} = \mathcal{N}(\mathbf{u}, \phi) = (K_1 + \sigma) < (\mathbf{u}\cdot\nabla\phi)\mathbf{\Delta}_2\phi >, \tag{15}$$

$$\begin{aligned}\gamma = \gamma(\sigma, \lambda) = {} & 2\,M + M\,[2\,\mu^* + K_1 + M\,(K_1 + 2\,\sigma)]\left(\mathcal{S} + \frac{1}{\sigma}\right) + \\ & + M^2\,K_1\mathcal{S}\,\frac{K_1 + 2\,\sigma}{\sigma} + \\ & + [2\,\mu^* M\,(M+1) + R^2 + M\,\lambda]\left(\mathcal{S} + \frac{1}{\lambda}\right),\end{aligned} \tag{16}$$

where $\gamma > 0$ and $\mathcal{D} \geq 0$. Furthermore, from Poincaré inequality we have:

$$\mathcal{E} \leq h\,\mathcal{D}, \tag{17}$$

with $h = \alpha\,\max\left\{\frac{1}{2\,\mu_0\mathcal{S}}, 1\right\}$, α being the constant of Poincaré inequality in Ω.

Setting

$$\gamma_0 = h^{-1}(> 0),$$

the following theorem holds:

THEOREM 2. *If* $K_1 > 0$ *and*

$$\gamma < \gamma_0 ,$$

then in $\mathcal{I}(\Omega)$ *exists at most one solution* $(\mathbf{U}, \Phi)$ *to the initial-boundary value problem (1),...,(5) such that* $\mathbf{U}$ *and* Φ *are bounded with their first and second spatial derivatives.*

Proof. From (15), accounting for

$$< |\psi|^4 > \leq 4 \left(< |\psi|^2 >\right)^{\frac{1}{2}} \left(< |\nabla\psi|^2 >\right)^{\frac{3}{2}} ,$$

with ψ vector or scalar field (see [3]), by Schwarz inequality easily follows that

$$\exists\, c = const > 0 \ : \quad \mathcal{N} \leq c\mathcal{D}^{\frac{3}{2}} .$$

From (13), (17) and the latter inequality, then it follows

$$\frac{d\mathcal{E}}{dt} \leq \left[c\mathcal{D}^{\frac{1}{2}} - (1-\gamma\, h)\right] \mathcal{D} . \tag{18}$$

Now, since

$$\gamma < \gamma_0 ,$$

it can be shown - by (18)- that

$$\frac{d\mathcal{E}}{dt} \leq 0, \quad \forall t \geq 0.$$

Let us observe that at any fixed t

$$\mathcal{E}(t) = 0 \Longrightarrow \mathcal{D}(t) = 0 .$$

Since $\mathcal{E}(0) = 0$, we get

$$c\mathcal{D}^{\frac{1}{2}}(0) - (1-\gamma\, h) = -(1-\gamma\, h) < 0 .$$

Let

$$t^* = \sup\left\{t \geq 0 : \quad c\mathcal{D}^{\frac{1}{2}}(t) < 1-\gamma\, h\right\} .$$

By continuity $t^* > 0$. We shall prove that $t^* = +\infty$.

In fact, assume $t^* < +\infty$. In such hypothesis

$$\frac{d\mathcal{E}}{dt} \leq 0 , \quad \forall t \in [0, t^*) .$$

Hence

$$\mathcal{E}(t) \leq \mathcal{E}(0) = 0\,, \quad \forall t \in [0, t^*)\,,$$

and then $\mathcal{E} = 0$ in $[0, t^*)$. Because of this

$$\mathcal{D}(t) = 0\,, \quad \forall t \in [0, t^*)\,,$$

and then by continuity

$$c\mathcal{D}^{\frac{1}{2}}(t^*) - (1 - \gamma\, h) < 0\,.$$

So, there exists $\tau > t^*$:

$$c\mathcal{D}^{\frac{1}{2}}(\tau) - (1 - \gamma\, h) < 0\,,$$

which is a contradiction. Then $t^* = +\infty$. Therefore $\mathcal{E}(t) \leq \mathcal{E}(0) = 0$, $\forall t \geq 0$, and then

$$\mathcal{E} \equiv 0\,.$$

From this, easily, follows the uniqueness of $\mathbf{U}$ and Φ.

Remark. Since $\gamma = \gamma(\sigma, \lambda)$, concerning the choice of parameters λ and σ, it is useful - in order to allow μ^*, M, K_1, $\mathcal{S}$, R to be the largest possible - to choose λ and σ such that

$$\gamma(\sigma, \lambda) = \min_{\mathbf{R}^+ \times \mathbf{R}^+} \gamma(\cdot, \cdot)\,.$$

This happens when we choose:

$$\sigma = \sqrt{\frac{2\,\mu^* + K_1\,(1 + M + M\,K_1\,\mathcal{S})}{2\,M\,\mathcal{S}}}\,,$$

$$\lambda = \sqrt{\frac{2\,\mu^* M\,(M + 1) + R^2}{M\,\mathcal{S}}}\,.$$

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