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NONALIGNED SHOCKS FOR DISCRETE VELOCITY MODELS OF THE BOLTZMANN EQUATION

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1. Introduction.

At the conclusion of I. Bonzani's presentation on the existence of structured shock solutions to the six-velocity, planar, discrete Boltzmann equation (with binary and triple collisions), Greenberg asked whether such solutions were possible in directions $e(\alpha) = (\cos\alpha, \sin\alpha)$ when α was not one of the particle flow directions, namely when $\alpha \neq \frac{(i-1)\pi}{3}, i = 1, \ldots, 6$. This question generated a spirited discussion but the question was still open at the conclusion of the conference.

In this note the author will provide a partial resolution to the question raised above. Using formal perturbation arguments he will produce approximate solutions to the equations considered by Bonzani which represent traveling waves propagating in any direction $e(\alpha) = (\cos\alpha, \sin\alpha)$. These approximate solutions involve a small positive parameter ϵ which represents the strength of the shock

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wave and are of the form

$$\mathbf{n} = n_{-\infty}\left(\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} + \epsilon a_0^0(\xi_1)\left(\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} + 2^{1/2}\begin{pmatrix}\cos\alpha\\ \cos(\pi/3-\alpha)\\ \cos(2\pi/3-\alpha)\\ \cos(\pi-\alpha)\\ \cos(4\pi/3-\alpha)\\ \cos(5\pi/3-\alpha)\end{pmatrix}\right)\right) + O(\epsilon^2). \tag{1.1}$$

The wave speed c is related to $0 < \epsilon \ll 1$ by $c = (1-\epsilon)^{1/2}/2^{1/2}$, $\xi_1 = \epsilon(x\cos\alpha + y\sin\alpha - ct)$ is the slowly varying traveling wave variable, and a_0^0 satisfies

$$\frac{da_0^0}{d\xi_1} = 3\cdot 2^{1/2}k_1 n_{-\infty} a_0^0(4 + a_0^0), \quad \lim_{\xi_1\to-\infty} a_0^0(\xi_1) = 0, \quad \text{and} \quad \lim_{\xi_1\to\infty} a_0^0(\xi_1) = -4. \tag{1.2}$$

The parameter $n_{-\infty} > 0$ characterizes the uniform upstream particle densities and k_1 is the constant coefficient in front of the binary collision terms in the governing system; for details see equation (2.12). This result implies to leading order in $\epsilon > 0$ the amplitude of the shock is unaffected by the triple collision term.

In the special case when $\alpha = \pi/6$ one does not have to rely on a formal expansion. Symmetries reduce the number of unknowns from six to three, the triple collision term drops out of the governing equations, and the exact solution is of the form

$$\mathbf{n} = n_{-\infty}\left(\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} + \epsilon a_0^0(\xi_1)\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} + \right. \tag{1.3}$$

$$\left. \frac{3^{\frac{1}{2}}(1-\epsilon)^{\frac{1}{2}}\epsilon a_0^0(\xi_1)}{2^{\frac{1}{2}}}\begin{pmatrix}1\\1\\0\\-1\\-1\\0\end{pmatrix} - \epsilon^2 a_0^0(\xi_1)\begin{pmatrix}1\\1\\-2\\1\\1\\-2\end{pmatrix}\right)$$

where $0 < \epsilon$, c and ξ_1 are as before, and now a_0^0 satisfies

$$\frac{da_0^0}{d\xi_1} = \frac{3\cdot 2^{1/2}k_1 n_{-\infty}}{(1-\epsilon)^{1/2}(1+2\epsilon)}a_0^0(4 + (1 + 3\epsilon + 2\epsilon^2)a_0^0)$$

and

$$\lim_{\xi_1=-\infty} a_0^0(\xi_1) = 0 \quad \text{and} \quad \lim_{\xi_1\to\infty} a_0^0(\xi_1) = \frac{-4}{1+3\epsilon+2\epsilon^2}.$$

All that is required of $\epsilon > 0$ is that it is such that $n_i \geq 0$, $i = 1, \ldots, 6$.(1)

2. Problem Formulation and Perturbation Results.

As stated in the introduction our interest is in nonaligned traveling wave solutions to the six-velocity, planar, discrete Boltzmann equation with binary and triple collisions. The basic unknowns are the particle densities $n_i, i = 1, \ldots, 6$. Specifically, $n_i(x, y, t)$ represents

(1) The invariance of the governing equations to rotations by $\pi/3$ implies that similar results hold for $\alpha = \pi/6 + i\pi/3, i = 1, \ldots, 4$.

the number of particles per unit area at (x, y) at time t traveling with velocity $\mathbf{v}_i = \cos((i-1)\pi/3)\mathbf{e}_1 + \sin((i-1)\pi/3)\mathbf{e}_2, i = 1, \ldots, 6.$([2]) The evolution equations for the particle densities are

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_i \cdot \nabla\right) n_i = k_1 Q_i + k_2(-1)^{i-1}T, \quad i = 1, \ldots, 6, \tag{2.1}$$

where

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y}, \tag{2.2}$$

$$Q_1 = Q_4 = (n_2 n_5 + n_3 n_6 - 2n_1 n_4), \tag{2.3}$$

$$Q_2 = Q_5 = (n_3 n_6 + n_1 n_4 - 2n_2 n_5), \tag{2.4}$$

$$Q_3 = Q_4 = (n_1 n_4 + n_2 n_5 - 2n_3 n_6), \tag{2.5}$$

$$T = (n_2 n_4 n_6 - n_1 n_3 n_5), \tag{2.6}$$

and k_1 and k_2 are fixed positive constants with $\dim(k_1) = \dfrac{\text{Area}}{\text{(number of particles)(time)}}$ and $\dim(k_2) = \left(\dfrac{\text{Area}}{\text{number of particles}}\right)^2 \dfrac{1}{\text{time}}$.

For any $\alpha \in [0, 2\pi)$ we let

$$\mathbf{e}_1(\alpha) = \cos\alpha \mathbf{e}_1 + \sin\alpha \mathbf{e}_2 \text{ and } \mathbf{e}_2(\alpha) = -\sin\alpha \mathbf{e}_1 + \cos\alpha \mathbf{e}_2 \tag{2.7}$$

be the orthonormal basis obtained by rotating the fixed basis, $\mathbf{e}_1$ and $\mathbf{e}_2$, through an angle α. Noting that relative to this new basis the gradient admits the representation

$$\nabla = \mathbf{e}_1(\alpha)\frac{\partial}{\partial \phi} + \mathbf{e}_2(\alpha)\frac{\partial}{\partial \psi} \tag{2.8}$$

where ϕ and ψ are related to x and y by

$$\phi = x\cos\alpha + y\sin\alpha \text{ and } \psi = -x\sin\alpha + y\cos\alpha, \tag{2.9}$$

([2]) Throughout $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ will be a fixed orthonormal basis.

we find our basic equation (2.1) takes the form

$$\left(\frac{\partial}{\partial t} + \left(\cos\alpha\cos\left((i-1)\frac{\pi}{3}\right) + \sin\alpha\sin\left((i-1)\frac{\pi}{3}\right)\right)\frac{\partial}{\partial\phi} + \left(\cos\alpha\sin\left((i-1)\frac{\pi}{3}\right) - \sin\alpha\cos\left((i-1)\frac{\pi}{3}\right)\right)\frac{\partial}{\partial\psi}\right) n_i = k_1 Q_i + k_2(-1)^{i-1}T,\ i = 1,\dots,6. \tag{2.10}$$

Our interest is in the solutions to (2.10) which are functions of

$$\xi = \phi - ct. \tag{2.11}$$

Previous efforts on this problem have been limited to the case where $\alpha = 0$, that is to waves propagating in the particle direction $(1,0)$. Gatignol [1] gave an elegant analysis of traveling waves propagating in the direction $(1,0)$ for the six velocity gas under consideration here when only binary collisions were accounted for. Her system supports an additional conservation law not satisfied by our system and thus her results are not directly comparable to the ones presented here, even for waves propagating in the direction $(1,0)$. References [2]-[5] contain other efforts on shock wave propagation for waves moving in one of the particle directions. The solutions of interest to us must satisfy

$$\left(\cos\alpha\cos\left((i-1)\frac{\pi}{3}\right) + \sin\alpha\sin\left((i-1)\frac{\pi}{3}\right) - c\right)\frac{dn_i}{d\xi} = k_1 Q_i + k_2(-1)^{i-1}T, \qquad i = 1,\dots,6 \tag{2.12}$$

and at $\xi = \pm\infty$ the right hand side of (2.12) must vanish.

Following Bellomo and Longo [2] we shall restrict our attention to solutions which converge to a uniform state, $n_{-\infty}$, in the upstream direction; that is ones which satisfy

$$\lim_{\xi\to-\infty} n_i(\xi) = n_{-\infty} > 0 \quad i = 1,\dots,6. \tag{2.13}$$

An analysis similar to that found in [2] indicates that if a nontrivial solution to (2.12) and (2.13) exists, then the downstream state must be of the form

$$\lim_{\xi\to\infty} n_i(\xi) = n_i^{\infty} = a_\infty \exp(b_\infty \cos((i-1)\pi/3 - \alpha)),\ i = 1,\dots,6, \tag{2.14}$$

where $b_\infty < 0$,

$$\frac{n_{-\infty}}{a_\infty} = \frac{(2U_{,b_\infty b_\infty} + U + ((2U_{,b_\infty b_\infty} - U)^2 + 8(U_{,b_\infty})^2)^{1/2})}{12}, \tag{2.15}$$

$$c^2 = \frac{(((2U_{,b_\infty b_\infty} - U)^2 + 8(U_{,b_\infty})^2)^{1/2} + (U - 2U_{,b_\infty b_\infty}))}{2(((2U_{,b_\infty b_\infty} - U)^2 + 8(U_{,b_\infty})^2)^{1/2} + (2U_{,b_\infty b_\infty} - U))}, \tag{2.16}$$

and

$$U(b_\infty, \alpha) \stackrel{\text{def}}{=} \sum_{i=1}^{6} \exp(b_\infty \cos((i-1)\pi/3 - \alpha)). \tag{2.17}$$

The equations (2.15)-(2.17) are merely the Rankine-Hugoniot relations for the system (2.12). If one defines the moments

$$P_m = \sum_{i=1}^{6} (\cos((i-1)\pi/3 - \alpha))^m n_i(\xi), \quad m = 0, 1, \text{and } 2,$$

then a simple calculation shows that any solution of (2.12a) and (2.13) must satisfy

$$P_1(\xi) = c(P_0(\xi) - 6n_{-\infty}) \ \textit{and} \ P_2(\xi) - 3n_{-\infty} = cP_1(\xi).$$

These relations are respectively the mass and momentum balance for the system. The relations (2.15) and (2.16) are obtained from these balance laws by substituting the downstream Maxwellian, (2.14), into the defining relations for the P_m's. The results of this substitution are

$$P_0^\infty = a_\infty U(b_\infty, \alpha), \quad P_1^\infty = a_\infty \frac{\partial U}{\partial b_\infty}(b_\infty, \alpha), \quad \text{and } P_2^\infty = a_\infty \frac{\partial^2 U}{\partial b_\infty^2}(b_\infty, \alpha)$$

and the resulting relations

$$P_1^\infty = c(P_0^\infty - 6n_{-\infty}) \ \textit{and} \ P_2^\infty - 3n_{-\infty} = cP_1^\infty$$

are the Rankine-Hugoniot or shock relations for the system (2.12). The identity (2.15) follows from the equation $\dfrac{P_1^\infty}{(P_0^\infty - 6n_{-\infty})} = \dfrac{P_2^\infty - 3n_{-\infty}}{P_1^\infty}$ and (2.16) follows from $c^2 = \dfrac{P_2^\infty - 3n_{-\infty}}{P_0^\infty - 6n_{-\infty}}$ and (2.15).

The identities

$$U(0,\alpha)=6,\ \frac{\partial U}{\partial b_\infty}(0,\alpha)=0,\ \text{and}\ \frac{\partial^2 U}{\partial b_\infty^2}(0,\alpha)=3 \tag{2.18}$$

imply that

$$\mathcal{F}(b_\infty,\alpha)\stackrel{\text{def}}{=}2\frac{\partial^2 U}{\partial b_\infty^2}(b_\infty,\alpha)-U(b_\infty,\alpha) \tag{2.19}$$

satisfies

$$\mathcal{F}(0,\alpha)=\frac{\partial \mathcal{F}}{\partial b_\infty}(0,\alpha)=0\ \text{and}\ \frac{\partial^2 \mathcal{F}}{\partial b_\infty^2}(0,\alpha)=3/2$$

and thus that for $-1\ll b_\infty<0$

$$c^2<1/2,\ \ 0<6n_{-\infty}-a_\infty U(b_\infty,\alpha),\ \ \text{and}\ \frac{\partial U}{\partial b_\infty}(b_\infty,\alpha)<0. \tag{2.21}$$

The second inequality in (2.21) is equivalent to the assertion that the upstream density exceeds the downstream density while the third inequality in (2.21) implies that the momentum and flow velocity in the direction $e_1(\alpha)$ is negative at the downstream equilibrium state.

Motivated by these necessary conditions which must be satisfied by a nontrivial solution we look for a solution to (2.12) of the form

$$\mathbf{n}=n_{-\infty}\left(\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}+\epsilon\mathbf{N}(\xi_1)\right) \tag{2.22}$$

where $n_{-\infty}>0$ and $0<\epsilon\stackrel{\text{def}}{=}(1-2c^2)\ll 1$ are the basic parameters describing the system. With this choice, the downstream state of the system will be determined in terms of these parameters. The variable ξ_1 is defined by

$$\xi_1=\epsilon\xi=\epsilon(x\cos\alpha+y\sin\alpha-ct). \tag{2.23}$$

Anticipating our final results we introduce the vectors $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$, $\mathbf{s}_1$ and $\mathbf{s}_2$ which form an orthogonal basis for our state space $\mathbb{R}^6$:

$$\mathbf{c}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{c}_1 = \begin{pmatrix} 1 \\ 1/2 \\ -1/2 \\ -1 \\ -1/2 \\ 1/2 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \\ 1 \\ -1/2 \\ -1/2 \end{pmatrix}, \mathbf{c}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \tag{2.24}$$

$$\mathbf{s}_1 = \begin{pmatrix} 0 \\ 3^{1/2}/2 \\ 3^{1/2}/2 \\ 0 \\ -3^{1/2}/2 \\ -3^{1/2}/2 \end{pmatrix} \text{ and } \mathbf{s}_2 = \begin{pmatrix} 0 \\ 3^{1/2}/2 \\ -3^{1/2}/2 \\ 0 \\ 3^{1/2}/2 \\ -3^{1/2}/2 \end{pmatrix}, \tag{2.25}$$

let

$$\mathcal{C}_1 \stackrel{\text{def}}{=} \operatorname{diag}(\mathbf{c}_1), \quad \mathcal{S}_1 \stackrel{\text{def}}{=} \operatorname{diag}(\mathbf{s}_1), \tag{2.26}$$

and

$$\begin{aligned} M = k_1 n_{-\infty} &\begin{bmatrix} -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 1 & 1 & -2 & 1 & 1 & -2 \\ -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 1 & 1 & -2 & 1 & 1 & -2 \end{bmatrix} + \\ + k_2 n_{-\infty}^2 &\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}, \end{aligned} \tag{2.27}$$

and note that the following identities obtain:

$$\mathcal{C}_1 \mathbf{c}_0 = \mathbf{c}_1, \tag{2.28}$$

$$\mathcal{C}_1 \mathbf{c}_1 = (\mathbf{c}_0 + \mathbf{c}_2)/2, \tag{2.29}$$

$$\mathcal{C}_1\mathbf{s}_1 = \mathbf{s}_2/2, \tag{2.30}$$

$$\mathcal{C}_1\mathbf{c}_2 = (\mathbf{c}_1 + \mathbf{c}_3)/2, \tag{2.31}$$

$$\mathcal{C}_1\mathbf{s}_2 = \mathbf{s}_1/2, \tag{2.32}$$

$$\mathcal{C}_1\mathbf{c}_3 = \mathbf{c}_2, \tag{2.33}$$

$$\mathcal{S}_1\mathbf{c}_0 = \mathbf{s}_1, \tag{2.34}$$

$$\mathcal{S}_1\mathbf{c}_1 = \mathbf{s}_2/2, \tag{2.35}$$

$$\mathcal{S}_1\mathbf{s}_1 = (\mathbf{c}_0 - \mathbf{c}_2)/2, \tag{2.36}$$

$$\mathcal{S}_1\mathbf{c}_2 = -\mathbf{s}_1/2, \tag{2.37}$$

$$\mathcal{S}_1\mathbf{s}_2 = (\mathbf{c}_1 - \mathbf{c}_3)/2, \tag{2.38}$$

$$\mathcal{S}_1\mathbf{c}_3 = -\mathbf{s}_2, \tag{2.39}$$

$$M\mathbf{c}_0 = M\mathbf{c}_1 = M\mathbf{s}_1 = 0 \tag{2.40}$$

$$M\mathbf{c}_2 = -6k_1 n_{-\infty}\mathbf{c}_2 \tag{2.41}$$

$$M\mathbf{s}_2 = -6k_1 n_{-\infty}\mathbf{s}_2 \tag{2.42}$$

$$M\mathbf{c}_3 = 6k_2 n_{-\infty}^2\mathbf{c}_3. \tag{2.43}$$

It is a relatively simple matter to verify that if $\mathbf{n}(\xi)$ satisfies (2.12a), then $\mathbf{N}(\xi_1)$ must satisfy

$$\begin{aligned} M\mathbf{N} = {} & \epsilon(\cos\alpha\mathcal{C}_1 + \sin\alpha\mathcal{S}_1 - 1/2^{1/2})\mathbf{N}_{,\xi_1} + \frac{\epsilon^2}{2^{1/2}(1+(1-\epsilon)^{1/2})}\mathbf{N}_{,\xi_1} \\ & - k_1 n_{-\infty}\epsilon((N_2N_5 + N_3N_6 - 2N_1N_4)\mathbf{c}_2 + \\ & + 3^{1/2}(N_3N_6 - N_2N_5)\mathbf{s}_2) - k_2 n_{-\infty}^2(\epsilon T_2 + \epsilon^2 T_3)\mathbf{c}_3, \end{aligned} \tag{2.44}$$

where

$$T_2 \stackrel{\text{def}}{=} (N_2N_4 + N_2N_6 + N_4N_6 - N_1N_3 - N_1N_5 - N_3N_5),$$

and

$$T_3 \stackrel{\text{def}}{=} (N_2N_4N_6 - N_1N_3N_5).$$

We seek an asymptotic solution of (2.44) of the form

$$\mathbf{N} \sim \mathbf{N}^0 + \epsilon \mathbf{N}^1 + \epsilon^2 \mathbf{N}^2 + \dots \tag{2.47}$$

and shall content ourselves with the first term $\mathbf{N}^0$. Insertion of the series (2.47) into (2.44) yields

(2.48)
(2.49)

$$\begin{aligned} M\mathbf{N}^0 &= 0, \\ M\mathbf{N}^1 &= (\cos\alpha \mathcal{C}_1 + \sin\alpha \mathcal{S}_1 - 1/2^{1/2})\mathbf{N}^0_{,\xi_1} \\ &\quad - k_1 n_{-\infty}((N_2^0 N_5^0 + N_3^0 N_6^0 - 2N_1^0 N_4^0)\mathbf{c}_2 + \\ &\quad + 3^{1/2}(N_3^0 N_6^0 - N_2^0 N_5^0)\mathbf{s}_2) - k_2 n_{-\infty}^2 T_2^0 \mathbf{c}_3, \end{aligned}$$

and

$$\begin{aligned} M\mathbf{N}^2 &= (\cos\alpha \mathcal{C}_1 + \sin\alpha \mathcal{S}_1 - 1/2^{1/2})\mathbf{N}^1_{,\xi_1} + (1/2^{3/2})\mathbf{N}^0_{,\xi_1} \\ &\quad + (\text{vectors in the span of } \mathbf{c}_2, \mathbf{s}_2 \text{ and } \mathbf{c}_3). \end{aligned} \tag{2.50}$$

Equations (2.40) and (2.48) imply that

$$\mathbf{N}^0 = a_0^0 \mathbf{c}_0 + a_1^0 \mathbf{c}_1 + b_1^0 \mathbf{s}_1 \tag{2.51}$$

while (2.28)-(2.43), (2.49) and (2.51) imply that

$$\frac{\cos\alpha}{2^{1/2}} a^0_{1,\xi_1} + \frac{\sin\alpha}{2^{1/2}} b^0_{1,\xi_1} - a^0_{0,\xi_1} = 0, \tag{2.52}$$

$$-\frac{1}{2^{1/2}} a^0_{1,\xi_1} + \cos\alpha\, a^0_{0,\xi_1} = 0, \tag{2.53}$$

$$-\frac{1}{2^{1/2}} b^0_{1,\xi_1} + \sin\alpha\, a^0_{0,\xi_1} = 0, \tag{2.54}$$

and

(2.55)
$$\begin{aligned}\mathbf{N}^1 &= a_0^1\mathbf{c}_0 + a_1^1\mathbf{c}_1 + b_1^1\mathbf{s}_1 \\ &+ 1/6\left(N_2^0N_5^0 + N_3^0N_6^0 - 2N_1^0N_4^0 - \frac{\cos 2\alpha}{2^{1/2}k_1n_{-\infty}}a^0_{0,\xi_1}\right)\mathbf{c}_2 \\ &+ 1/6\left(3^{1/2}(N_3^0N_5^0 - N_2^0N_6^0) - \frac{\sin 2\alpha}{2^{1/2}k_1n_{-\infty}}a^0_{0,\xi_1}\right)\mathbf{s}_2 \\ &+ \frac{1}{6k_1n_{-\infty}^2}(N_2^0N_4^0 + N_2^0N_6^0 + N_4^0N_6^0 - N_1^0N_3^0 - N_1^0N_5^0 - N_3^0N_5^0)\mathbf{c}_3.\end{aligned}$$

The relations (2.52)-(2.54) follow from the fact that in order for (2.49) to have a solution, the projection of the right hand side of (2.49) onto $\mathbf{c}_0$, $\mathbf{c}_1$ and $\mathbf{s}_1$ must vanish. These equations, together with

$$\lim_{\xi_1\to-\infty}(a_0^0, a_1^0, b_1^0) = (0,0,0), \tag{2.56}$$

imply that

$$a_1^0 = 2^{1/2}\cos\alpha a_0^0 \text{ and } b_1^0 = 2^{1/2}\sin\alpha a_0^0 \tag{2.57}$$

and thus that

$$\begin{aligned}\mathbf{N}^0 &= a_0^0(\mathbf{c}_0 + 2^{1/2}(\cos\alpha\mathbf{c}_1 + \sin\alpha\mathbf{s}_1)) = \\ &= a_0^0\left(\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}\cos(\alpha)\\ \cos(\pi/3-\alpha)\\ \cos(2\pi/3-\alpha)\\ \cos(\pi-\alpha)\\ \cos(4\pi/3-\alpha)\\ \cos(5\pi/3-\alpha)\end{pmatrix}\right).\end{aligned} \tag{2.58}$$

The identity (2.55) is a simple consequence of (2.58) and the formula's (2.28)–(2.43) which give the action of the operators $\mathcal{C}_1$, $\mathcal{S}_1$ and M on the basis vectors $\mathbf{c}_0$, $\mathbf{c}_1$, $\mathbf{s}_1$, $\mathbf{c}_2$, $\mathbf{s}_2$ and $\mathbf{c}_3$.

To obtain the governing equation for a_0^0 we substitute (2.55) and (2.58) into the right hand side of (2.50) and note that in order for

(2.50) to have a solution the projection of the right hand side of (2.50) onto c_0, c_1 and s_1 must vanish. The vanishing of these projections implies that

$$a_0^1 - \frac{\cos\alpha}{2^{1/2}} a_1^1 - \frac{\sin\alpha}{2^{1/2}} b_1^1 - \frac{a_0^0}{2} = 0, \tag{2.59}$$

$$\begin{aligned} &\cos\alpha a_0^1 - \frac{1}{2^{1/2}} a_1^1 + \frac{\cos\alpha}{2} a_0^0 + \frac{\cos\alpha}{12}(N_2^0 N_5^0 + \\ &+ N_3^0 N_6^0 - 2N_1^0 N_4^0 - \frac{\cos 2\alpha}{2^{1/2} k_1 n_{-\infty}} a^0_{0,\xi_1}) \\ &+ \frac{\sin\alpha}{12}\left(3^{1/2}(N_3^0 N_6^0 - N_2^0 N_5^0) - \frac{\sin 2\alpha}{2^{1/2} k_1 n_{-\infty}} a^0_{0,\xi_1}\right) = 0, \end{aligned} \tag{2.60}$$

and

$$\begin{aligned} &\sin\alpha a_0^1 + \frac{b_1^1}{2^{1/2}} + \frac{\sin\alpha}{2} a_0^0 - \frac{\sin\alpha}{12}(N_2^0 N_5^0 + \\ &+ N_3^0 N_6^0 - 2N_1^0 N_4^0 - \frac{\cos 2\alpha}{2^{1/2} k_1 n_{-\infty}} a^0_{0,\xi_1}) \\ &+ \frac{\cos\alpha}{12}(3^{1/2}(N_3^0 N_6^0 - N_2^0 N_5^0) - \frac{\sin 2\alpha}{2^{1/2} k_1 n_{-\infty}} a^0_{0,\xi_1}) = 0. \end{aligned} \tag{2.61}$$

If we now multiply (2.60) by $\cos\alpha$, (2.61) by $\sin\alpha$, add the results, and utilize the identity (2.59) we obtain the following equation for a_0^0:

$$\begin{aligned} \frac{1}{12 \cdot 2^{1/2} k_1 n_{-\infty}} \frac{da_0^0}{d\xi_1} = a_0^0 + \frac{\cos 2\alpha}{12}(N_2^0 N_5^0 + N_3^0 N_6^0 - 2N_1^0 N_4^0) + \\ + \frac{\sin 2\alpha}{4 \cdot 3^{1/2}}(N_3^0 N_6^0 - N_2^0 N_5^0). \end{aligned} \tag{2.62}$$

If we then exploit (2.58), we find that

$$N_2^0 N_5^0 + N_3^0 N_6^0 - 2N_1^0 N_4^0 = 3\cos 2\alpha (a_0^0)^2 \tag{2.63}$$

and

$$N_3^0 N_6^0 - N_2^0 N_5^0 = 3^{1/2} \sin 2\alpha (a_0^0)^2 \tag{2.64}$$

and (2.63) and (2.64), when combined with (2.62), imply that a_0^0 satisfies

$$\frac{da_0^0}{d\xi_1} = 3 \cdot 2^{1/2} k_1 n_{-\infty} a_0^0 (4 + a_0^0). \tag{2.65}$$

Equation (2.65) suffices for the determination of a_0^0 and thus for the 0th order term $\mathbf{N}^0$ in the expansion (2.47). The same methodology leading to $\mathbf{N}^0$ could, in principle, be used to obtain additional coefficients in the expansion. We shall not perform these calculations since the algebra involved is horrendous. Instead we shall focus our attention on the special case when $\alpha = \pi/6$ where one can obtain exact results. These results appear to be new and confirm the validity of our formal expansion in the $\epsilon = 0^+$ limit for the specific plane wave associated with the direction $\alpha = \pi/6$.

3. The Case $\alpha = \pi/6$.

We conclude by analyzing the case where $\alpha = \pi/6$. The form of the downstream Maxwellian (see (2.14) with $\alpha = \pi/6$) suggests looking for a solution of the form

$$\mathbf{n}(\xi) = N_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + N_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + N_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}. \tag{3.1}$$

This form is compatible with (2.12) and (2.13) provided

$$\mathbf{N} \stackrel{\text{def}}{=} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \tag{3.2}$$

satisfies

$$\left(\frac{3^{1/2}}{2}\operatorname{diag}(1,0,-1) - c\right)\mathbf{N}_{,\xi} = k_1(N_2^2 - N_1N_3)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.3}$$

and

$$\lim_{\xi \to -\infty} \mathbf{N}(\xi) = n_{-\infty} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{3.4}$$

If we introduce the orthogonal basis

$$\mathbf{a}_o = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \text{and } \mathbf{a}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \tag{3.5}$$

rewrite N as

$$\mathbf{N} = n_{-\infty}(A_0\mathbf{a}_0 + A_1\mathbf{a}_1 + A_2\mathbf{a}_2), \tag{3.6}$$

and observe that

$$\begin{gathered} \operatorname{diag}(1,0,-1)\mathbf{a}_0 = \mathbf{a}_1, \quad \operatorname{diag}(1,0,-1)\mathbf{a}_1 = \frac{(2\mathbf{a}_0 + \mathbf{a}_2)}{3} \\ \text{and } \operatorname{diag}(1,0,-1)\mathbf{a}_2 = \mathbf{a}_1, \end{gathered} \tag{3.7}$$

we find that A_0, A_1 and A_2 must satisfy

$$\frac{dA_1}{d\xi} = 3^{1/2} c \frac{dA_0}{d\xi}, \tag{3.8}$$

$$\frac{dA_2}{d\xi} = \frac{2c}{3^{1/2}} \frac{dA_1}{d\xi} - \frac{dA_0}{d\xi} = (2c^2 - 1)\frac{dA_0}{d\xi}, \tag{3.9}$$

$$\frac{cn_{-\infty}}{2}(1 + 2(1 - 2c^2))\frac{dA_0}{d\xi} = k_1(N_2^2 - N_1N_3), \tag{3.10}$$

and

$$\lim_{\xi \to -\infty} (A_0, A_1, A_2) = (1, 0, 0). \tag{3.11}$$

Equations (3.8), (3.9) and (3.11) imply that

$$A_1 = 3^{1/2}c(A_0 - 1) \text{ and } A_2 = (2c^2 - 1)(A_0 - 1) \tag{3.12}$$

and these, when combined with (3.6), yield

$$N_1 = n_{-\infty}[1 + (1 + (2c^2 - 1) + 3^{1/2}c)(A_0 - 1)], \tag{3.13}$$

$$N_2 = n_{-\infty}[1 + (1 + 2(1 - 2c^2))(A_0 - 1)], \tag{3.14}$$

$$N_3 = n_{-\infty}[1 + (1 + (2c^2 - 1) - 3^{1/2}c)(A_0 - 1)], \tag{3.15}$$

and

$$\begin{aligned} N_2^2 - N_1 N_3 = n_{-\infty}(6(1 - 2c^2)(A_0 - 1) + 3(c^2 + \\ + (1 - 2c^2)(2 + (1 - 2c^2)))(A_0 - 1)^2). \end{aligned} \tag{3.16}$$

If we now make the substitution

$$0 < \epsilon = (1 - 2c^2), \quad A_0 - 1 = \epsilon a_0^0, \quad \text{and} \quad \xi_1 = \epsilon \xi, \tag{3.17}$$

we find that a_0^0 satisfies

$$\frac{da_0^0}{d\xi_1} = \frac{3 \cdot 2^{1/2} k_1 n_{-\infty}}{(1-\epsilon)^{1/2}(1 + 2\epsilon)} a_0^0 (4 + (1 + 3\epsilon + 2\epsilon^2) a_0^0) \tag{3.18}$$

and (3.1) be easily integrated to obtain solutions satisfying

$$\lim_{\xi_1 \to \infty} a_0^0(\xi_1) = 0 \quad \text{and} \quad \lim_{\xi_1 \to \infty} a_0^0(\xi_1) = \frac{-4}{1 + 3\epsilon + 2\epsilon^2}. \tag{3.19}$$

That (1.3) is valid is a immediate consequence of (3.1), (3.13)-(3.15), and (3.17).

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